The Problem of Relations in Inductive Logic

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The concepts of state-descriptions and ranges seem useful means for the definitions of basic concepts both in deductive logic[1] (e.g., 'L-truth' and 'L-implication') and in inductive logic[2] (e.g., 'degree of confirmation'). A state-description is defined as a conjunction or class of basic sentences (i.e., atomic sentences and negations of such) which for every atomic sentence *S* contains either *S* or non-*S* but not both and no other sentences. A state-description is intended to represent a possible state of affairs of the universe of discourse. In order to assure that this purpose is fulfilled, the atomic sentences of the language-system in question must be logically independent of each other. This is called the requirement of *logical independence* (*Logical Foundations of Probability*, § 18B).

Now Dr. Yehoshua Bar-Hillel[3] has pointed out a peculiar difficulty which arises in connection with this requirement, when the primitive predicates, of the language system designate not only properties but also relations. For example, let '*W*' be a primitive predicate such that '*Wxy*' means '*x* is warmer than *y*.' Since the relation Warmer is asymmetric, the basic sentences '*Wab*' and '*Wba*' are incompatible, and hence any state-description containing both would not represent a possible case; and, since Warmer is transitive, '*Wab*,' '*Wbc*,' and '~ *Wac*' are incompatible and any state-description containing all three would not represent a possible case. Furthermore, since Warmer is irreflexive, the atomic sentence '*Waa*' is self-contradictory and hence should be banned from any description of a possible case. According to customary conceptions, the relation Warmer possesses the structural properties mentioned in virtue of its very meaning and hence with logical necessity; therefore the cases mentioned are *logically* impossible. It seems that, in a similar way, most of the other relations which might be taken into consideration as primitive concepts also possess certain structural properties by their very meanings and thus make the fulfillment of the requirement of independence impossible.

Thus Bar-Hillel's remarks throw light on an important new aspect of the situation concerning relations as primitive concepts. Our problem is now to find ways of adapting the method of state-descriptions to the situation as we see it now.

It should first be noted that the difficulty under discussion does not arise in the case of a completely quantitative language as, for instance, the language of physics. Here relations are not needed as primitive concepts. The individuals are space-time points, characterized by their coordinates and hence designated by ordered quadruples of real number expressions. The physical state of a point is described, not by qualitative predicates but by numerical functors. For example, a sentence of the form '$te\ (x_1, x_2, x_3, t) = r$' may say that the temperature at the place x_1, x_2, x_3 at the time t is r. The relation Warmer is now definable; 'x is warmer than y' is defined by 'the temperature at x is higher than that at y.'

However, we often wish to work with language-systems of a simpler form even in deductive logic. And in inductive logic we are compelled to do so at the present time, because the methods of inductive logic have so far been developed only for simple languages and their extension for a language with continuous coordinates and continuous scales for magnitudes seems to involve serious difficulties. Now the simplest language contains as primitive concepts only properties, not relations; here our difficulty does, of course, not yet arise. This language is sufficient for many purposes; in particular, in inductive logic, i.e., the logic of probability, the great majority of the traditional problems are expressible in a language of this kind. On the other hand, from the point of view of scientific problems the restrictions of this language seem very narrow. Our problem concerns languages of an intermediate status, rich enough to express relations, but not as rich as the full quantitative language. We shall now discuss three ways of solving the problem of relations, referring to three intermediate languages L_1, L_2, and L_3.

The first way uses a language L_1 which still has the customary form including relational primitive predicates but avoids the difficulty by means of a modified interpretation. The meaning assigned to the relational predicates in L_1 must be so weak that it does not include any structural properties. Thus there might, for example, be a primitive predicate 'W'' somehow corresponding to the earlier 'W,' but such that asymmetry and transitivity are not part of its meaning but hold, if at all, only contingently.

An interpretation of this kind might perhaps be based on Wilfrid Sellars' conception.[4] According to this conception the law of the transitivity of W′, for example, would not hold in all state-description ("Histories") but only in those of certain families; it would thus not be analytic but a synthetic "material invariance" within these families; in each family, the laws holding there for W′ would be constitutive of its meaning in this family.

Others might interpret the predicate 'W″ as referring to or testable by an immediately given experiential quality *sui generis*, not involving a comparison. The observer would experience this quality, e.g., when he touches a warm body a with his right hand and simultaneously a lukewarm body *b* with his left hand. If, furthermore, this quality occurs similarly in connection with bodies *b* and *c*, it does not logically follow that it will also occur with *a* and *c*. Whether or not it does so is a matter of a third experience. If it does, transitivity has been empirically confirmed for this instance; even if it holds in all instances, it does so contingently, not with logical necessity.

It is not my intention either to defend or to reject the conceptions just indicated. I merely wish to point out that anybody who finds a way of interpreting a relational predicate in such a weak form that no structural property is implied by the meaning assigned to the predicate, is free to take this predicate as primitive (provided, of course, that it is logically independent of the other predicates he has chosen as primitive).

While the first way keeps the customary language form and changes only the interpretation, the other two ways involve a change to stronger language forms. The second procedure reduces qualitative relations by definitions to primitive qualities which exhibit a certain order. The language L_2 here used—and that holds likewise for L_1 and L_3—is of first order (i.e., all predicates and functors are of first level and all variables are of zero level, that is to say, their values are individuals). Therefore the order of the qualities cannot be expressed in L_2 by a relational predicate of second level (which would involve us again in the difficulty we wish to avoid). The order is instead indicated by subscripts attached to the predicates. For example, let the following five predicates be taken as primitive: 'P_1' for 'cold,' 'P_2' for 'cool,' 'P_3' for 'lukewarm,' 'P_4' for 'warm,' 'P_5' for 'hot.'

These five predicates constitute a family of related predicates,[5] i.e., they are such that for every individual one and only one of them holds. (Atomic sentences with the same individual constant and predicates of the same family are thus logically dependent. The requirement of logical independence refers in L_2 of course, only to atomic sentences which have either different individual constants or predicates of different families (see *Logical Foundations of Probability*, p. 77). The five subscripts indicate an order; but this is meant here merely as a comparative order (rank order), not a metrical order. (In technical terms, this means that the concept of degree of confirmation would be defined with respect to L_2 in such a manner that its value is influenced by the relations Greater and Equal among subscript numbers but not by their differences.)

On this basis, the predicate '*W*' for 'warmer' can now be introduced by the following definition: '*Wxy*' for '$(P_5x \bullet P_4y)$ v $(P_5x \bullet P_3y)$ v ... v $(P_2x \bullet P_1y)$.' (The definiens is a disjunction of ten components; each component contains two predicates, the first having a higher subscript than the second.) The asymmetry and transitivity of W hold here in virtue of the definition, hence with logical necessity. Other comparative relations, e.g, Darker, Louder, etc., can be defined in a similar way on the basis of other families of related primitive predicates. Furthermore, the symmetry and transitivity of equivalence relations follow from their definition; for example, 'equally warm' may be defined by '$(P_1x \bullet P_1y)$ v ... v $(P_5x \bullet P_5y)$, and similarly 'equicolored' on the basis of the family of color predicates.

Similarity relations, e.g., 'similar in color,' can also be defined here. This language form L_2, in which only qualities occur as primitive while relations are defined, would presumably be preferred by those who regard qualities as prior to relations either in an ontological sense (whatever that may mean) or in an epistemological sense. The latter view would mean that relations are not directly apprehended but only derived from the perception of qualities; for example, it would be held that, when we touch two bodies and judge the one to be warmer than the other, there is no immediate relational experience but only two experiences of separate qualities on which the recognition of the relation is based. However, it seems to me that L_2 is in itself of interest from the point of view of logic, quite aside from the question of the priority of qualities or relations.

The third way uses a language L_3, which in the form of its expressions is similar to the quantitative language but in its logical structure (i.e., the contents of its sentences and the implication relations among them) is quite similar to L_2. There is in L_2 a sentence 'P_5a_7' saying that the seventh individual (or position) has the fifth heat-quality, i.e., is hot. In L_3 the same is expressed by the sentence '*te* (7) = 5' with '*te*' as the numerical functor for temperature. (Full sentences of the same functor for the same argument but with different values (e.g., '*te* (7) = 5' and '*te* (7) = 3') are incompatible; the requirement of logical independence must be stated for L_3 in a form not referring to this case.) In contradistinction to a genuinely quantitative language, the values of numerical functors (e.g., the value 5 of the functor '*te*' in the example above) are meant in L_3 in merely comparative sense, not a genuinely quantitative sense; they determine not a metrical order but a comparative (rank) order, just as the subscripts of the predicates in L_2.

In one point L_3 is stronger and more convenient than L_2: the individual variables in L_3 refer both to individuals (positions) and to numbers as values of functions,[6] while in L_2 there are no variables for the subscripts of predicates. The relation Warmer can be defined in L_3 in a much simpler form than in L_2: '*Wxy*' for '*te*(*x*) > *te* (*y*). Here again, the asymmetry and the transitivity of *W* hold on the basis of the definition. Furthermore, other kinds of relations, among them relations of comparison, equivalence, ad similarity, can here be defined on the basis of functors. The only nonlogical primitive signs of L_3 are one-place functors. Therefore the difficulty connected with primitive relations (and analogously with functors of more than one argument) does not arise here. There is a stronger version L'_3 of the third language form in which the function values are interpreted in a genuinely quantitative way (that is to say, the logical rules of L'_3 and, in particular, those for the degree of confirmation, involve also differences of function values).

If one of the three ways just described is used, it is still true that "the requirement of independence concerns only the *interpretation* of the nonlogical signs . . . so that for the purely logical work both in deductive logic . . . and in inductive logic . . . we need not consider any particular interpretation of the nonlogical signs" (*Logical Foundations of Probability*, p. 73). As long as we deal with logical systems as such, aside from their application, we simply presuppose that the requirement is fulfilled. Only when a particular interpretation of the nonlogical signs is specified is it necessary to examine whether the requirement is fulfilled.

Besides the three language forms here discussed, other related forms come, of course, into consideration. Thus there are many ways of avoiding the difficulty connected with primitive relations. Which of them is chosen and which particular language form is used for some investigation in deductive or inductive logic depends upon the purpose of the investigation and also upon subjective preferences of the investigator. I personally feel some inclination, in view of the new situation, to avoid primitive predicates for relations; it seems to me that the use of a quantitative language like L_3 or L'_3 has great advantages. It is helpful to make a distinction between positional (i.e., spatio-temporal) relations and qualitative relations. (Only the latter ones have been discussed in this paper.)

If the individual expressions are arithmetized, i.e., given the form of numerical expressions (in a so-called coordinate language[7]), then positional relations can be defined as logical (arithmetical) relations. If qualities are arithmetized, i.e., represented by numerical functors, then qualitative relations can be defined; they are, in general, of a nonlogical nature. In a quantitative language of the simplest form, like L_3 or L'_3, only integers are taken as coordinates and as function values. It is true that inductive logic has so far been developed only for nonquantitative languages. But its extension for a quantitative language of the form L_3 or even the stronger form L'_3 does not seem to involve great difficulties.

NOTES

[1] See *Meaning and Necessity* (Chicago: University of Chicago Press, 1947), p. 9. and *Logical Foundations of Probability* (Chicago: University of Chicago Press, 1950), §§ 1.8A,D and 20.

[2] *Logical Foundations of Probability*, §§,55A, 110A.

[3] "A Note on State-Descriptions," in this issue.

[4] "Concepts as Involving Laws and Inconceivable without Them," *Philosophy of Science*, 15:287-315 (1948).

[5] See *Logical Foundations of Probability*, § 18C.

[6] Concerning this double use of individual numerical variables, see *Logical Syntax of Language* (London: Kegan Paul, Trench, Truebner & Co., 1937), § 3 and *Logical Foundations of Probability*, pp. 62f.

[7] On coordinate languages, see *Logical Syntax*, § 3; *Meaning and Necessity*, Chap. 2 and *Logical Foundations of Probability*, pp. 62f.