Proper and Improper Concepts

By Rudolf Carnap

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I. Proper Concepts

A concept is (in Kantian terminology) the predicate of a possible judgment, or (in the terminology of modern logic) a propositional function. The essential feature of a concept is that it holds for certain objects and does not hold for others. A third case is excluded. (We shall find exceptions to this for the improper concepts below.) The question of whether or not a particular object falls under a concept (or several objects, in the case of a relational concept) is thus uniquely determined; whether it is also possible in practice to decide the question is irrelevant.

The concepts of any field, such as Geometry or Economics, can be organized in such a way that certain concepts are taken at the start as undefined, and the other ones are then defined with the help of those "*basic concepts*". Thus for example in Jurisprudence one can take concepts like object, person, intent, action, and so on as the basic concepts, and with their help derive all of the other concepts of the field, either directly or with the help of intermediate steps. Such a *derivation* occurs through an *explicit definition*,[1] i.e. through the stipulation that a particular new concept word is to mean the same as an

[1] The expression "explicit definition" is meant here in its wider sense, as opposed to implicit definition. It is thus intended to include not only explicit definitions in the narrow sense, but also "definitions in use", e.g. the definition of a class by means of its propositional function and, as a special case of this, definitions by means of the Principle of Abstraction.

expression consisting of old words, i.e. ones that either have already been defined or that stand for basic concepts. If a concept has such a derivation, then we say that the concept is "*constituted*" on the basis of the basic concepts of the field. In this way, the concepts of any field can be organized into a "constitution system".

A few brief remarks on the relations between the different kinds of "real" and "formal" concepts should suffice, since these "proper" concepts are already familiar enough. We then turn to a closer examination of improper concepts, which have not been investigated as thoroughly.

A. Real Concepts

The most important kind of concepts, for the sake of which all science is conducted, are the *real concepts*, i.e. the concepts of real things (for example: mammal, Paris, the year 1927). (Strictly speaking, the real concepts also include the concepts of things that are not real, but are similar to real things, like the concept of the unicorn; this problem will not be pursued here.) All concepts of other kinds are merely aids for representing knowledge of real concepts.

The real concepts can be naturally divided first of all into the various fields of the natural and cultural sciences. Within each field, they can be organized, as already stated, into a constitution system. The concepts of different fields, by contrast, do not initially seem to be reducible to each other. Some of them do, of course, concern the same things of the external world (e.g. the concept of a cow in Zoology and in Economics), but from such different points of view that they appear to be incomparable (however closely the zoologist may examine a cow from his point of view, he will never discover its price). Nonetheless, the concepts of the various fields are not only related to each other in the obvious ways, but also in a genuine *system of derivations*; they can be derived from each other by definitions and thereby organized into a single constitution system for the whole of science. (The proof of this is given in "constitution theory", which is to be presented in the book "The Logical Construction of the World".) Only a brief indication will be given here of how to find the derivation for some cases, suggesting at least its possibility for the others.

Let us take as given the concepts of physical perception (e.g. spatial form, size, and position; color, hardness, etc.). Then it can be shown that the concepts of certain other fields can be constituted from these, e.g. those of Biology or Physiology, which do not concern perception alone. For we require that *criteria* be stated for the concepts of these fields (at least for the basic concepts from which the others are to be derived). These criteria must be perceptible and thus lead to a definition of the corresponding concept on the basis of the concepts of physical perception. For example, the biological concept of "organism" is not a physical concept; nor is it constituted from physical concepts, but from other biological ones like the concepts of metabolism, reproduction, and such. Now in order to be legitimate concepts of empirical science at all, these concepts (or even simpler biological ones through which they are defined) must have criteria of a perceptible kind, and so they can be defined by means of physical concepts (e.g. "metabolism" is a process having such and such perceptible criteria). Similarly, the psychologist must specify, for every mental process or state not reduced to more simple psychological concepts, the events available to sensory perception (e.g. expressive movements or verbal utterances) by which it can be recognized whether or not the mental state is occurring in a subject.

Of course, such a definition by criteria or "constitution" of a concept by no means exhausts the concept. It only specifies its location in the system of concepts, the way a location on the surface of the Earth is specified by its geographical latitude and longitude. Its other properties must be determined by empirical investigation and presented in the theory of the respective field. But in order for this presentation to relate to something in particular, its constitution (the geographic coordinates in the analogy) must first be specified.

The concepts of the higher disciplines, e.g. History of Religion or Sociology, can also be constituted from the physical ones; and the physical concepts that we took above as basic can in turn be reduced to more fundamental ones. Ultimately, it can be shown that the constitution system of all scientific concepts is constructible on the basis of just a very few basic concepts.

B. Formal Concepts

To give a derivation of a real concept from other ones, or a proposition about real concepts, we need in addition to the words for those concepts other intermediate signs which themselves do not stand for real concepts (in applications of word languages e.g. the words "and", "or", "all", "not", "if --- then ---", "equals", and similar). They do of course contribute to expressing something about reality, but nothing in reality actually corresponds to them; they only form the proposition. Although they have no independent meaning, it is still customary to speak of the "concepts" that they stand for; these "logical concepts" or "formal concepts" are, however, (to the extent that we recognize them as "concepts" at all) of a completely different kind than the real concepts.

In addition to the basic logical concepts, the formal concepts also include all those concepts that can be derived from the basic ones. These include not only the concepts of Logic in the narrow sense, but also the numbers, and all further concepts of Arithmetic and Analysis. The proof of this fact was given by Whitehead and Russell, who set up a complete constitut ion system for all formal concepts in which all mathematical concepts (except, as of yet, the geometrical ones) are derived from a few basic logical ones.[2] The required *basic concepts* are:

1. proposition: p
2. assertion or "true": $\vdash p$
3. incompatibility: $p|q$
4. propositional function: φx
5. "all": $(x).\varphi x$

Like all other formal concepts, the numbers, too, are derived by successive explicit definitions from these. The numbers defined in this way will be called the "*Russell numbers*", to distinguish them from the implicitly defined "Peano numbers" to be discussed below.

II. Improper Concepts

A. Implicit Definition

In addition to the introduction of new concepts by means of explicit definition, as already discussed, there is yet another way of determining concepts: definition by means of an axiom system (henceforth: AS), or so-called implicit definition. This method of definition often proves to be fruitful, particularly in various mathematical disciplines.[3]

Just as we can organize the concepts of a field so that basic concepts are taken at the start, from which all the other ones can be derived by definitions, we can also organize the propositions comprising the theory of some field so that we take at the start as "*axioms*"

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[2] Russell and Whitehead, *Principia Mathematica*. I^2 1925, II 1912, III 1913.
[3] Cf. Schlick, *Allgemeine Erkenntnislehre*. 1925^2, p. 29ff.

or "basic propositions", ones from which all the other propositions can be derived by deductions.

Example. The following five propositions form an AS for Arithmetic; i.e. from these propositions all arithmetical laws can be derived. The concepts of (natural) *number* (0, 1, 2, ...) and (immediate) *successor* occur in the axioms.

1. There is one and only one first number (i.e. a number that is not the successor of another one). (Namely, 0.)
2. Every number that is not the first is the successor of one and only one other number.
3. Every number has one and only one successor.
4. There are no repetitions in the sequence of numbers (i.e. starting from any number and proceeding to its successor, and thence to its successor, and so on, never returns one to the starting number). (From (3) and (4) it follows that the number sequence is infinite.)
5. Every number can be reached from the first one in finitely many steps. (The apparent circularity in the use of "one" in (1), (2), and (3), and "finitely many" in (5), results from the brief mode of expression, and vanishes in the precise formulation.)

This Peano AS (it originated with Peano, but appears here in the simplified form given it by Russell[4]), is initially intended as an answer to the question "what do we know about the numbers?". It is thereby assumed that the meaning of word "number" is already determined, since otherwise the question would make no sense. The answer is then given by stating certain propositions about numbers, and indeed enough such so that all other such propositions can be derived from these.

[4] Russell, *Introduction to Mathematical Philosophy*. 1923, p. 5ff.

However, the AS can also be regarded in a completely different way: we take the words "number" and "successor"[5] as new terms that have not yet been given a meaning, and we stipulate that they are to refer to those concepts with the character specified by the AS. Thus here the AS makes no initial assumptions, but rather only through it is a class determined, which will then be called "the numbers", and a relation, which will be called "successor". In contrast to the determination of a concept by explicit definition, as discussed earlier, here the new concepts are not connected to old ones, but are specified by the formal characteristics they inherently possess; hence the terminology "*implicit definition*" for the determination of a concept by an AS.

As a consequence, the words "the number class", or more precisely "the number sequence" (since it is a class that is *ordered* in a certain way), then mean nothing other than "that which behaves as specified by the Peano AS". The numbers implicitly defined in this way will be called the "*Peano numbers*", in contrast to the "Russell numbers" already discussed, which are explicitly defined from basic logical concepts.

B. The Interpretations of an AS

The implicit definition of the numbers just mentioned is of course a legitimate method of introducing the number concept; it would be used where for some reason the explicit definition of number seems to be less appropriate or even impossible. But it has the disadvantage that there is then not just *one* class of numbers, but many different ones, since there are many different *interpretations*[6] of the AS. For the AS is, of course, satisfied by any arbitrary sequence of objects having the required properties: it must be infinite, but without repetitions, have a first but no last element, and every element must be reachable from the first one in finitely many steps. There are interpretations among

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[5] [Translator's note: the original has "*Vorgänger*" ("predecessor"), which is surely a slip.]

[6] [Translator's note: the original has "*Interpretationen oder 'Anwendungsfälle'* ", literally: "interpretations or 'cases of application' ".]

the real concepts, "*realizations*" of the AS, and also among the formal concepts, "*formal models*". One realization of our AS is, for instance, the following sequence of points of physical space: the right corner of the edge of this table, the midpoint between that corner and the left corner, the midpoint between that point and the left corner, etc. (i.e. the points along the edge with the coordinates 1, 1/2, 1/4, etc.). Sequences of time points, spheres, etc. can also be realizations of the AS. Interpretations in the domain of logical (and arithmetical) concepts, i.e. "formal models", are e.g.:

1. the sequence of cardinal numbers (as defined by Russell);
2. the sequence of cardinal numbers beginning with 5;
3. the sequence of functions a, ax, ax^2, etc.

The first model, the sequence of cardinal numbers, is that for the sake of which the AS was set up. As we see, however, the AS, and therefore the implicit definition it expresses, applies not only to that case, but also to infinitely many others, namely all those that agree with it with respect to the specified formal properties, i.e. the structure. In the theory of relations the sequences with these properties are called "progressions". The realizations and formal models mentioned are examples of progressions. The implicit definition of the sequence of numbers therefore does not uniquely determine the number sequence, but only the unique class of all progressions, a particular element of which is the number sequence (in the proper sense), and each element of which can be regarded as the number sequence (in the improper sense).

Does every AS have, like the one we just considered, various different interpretations, both different realizations, an also different formal models? We can exclude the case of a contradictory AS; such a one clearly has neither a realization nor a formal model. The question of whether a consistent AS has any realizations, exactly one, or several, is usually considered to be an empirical question. We shall leave open the question whether this is so, or whether the existence of arbitrarily many realizations can be asserted *a* *priori* (i.e. as a tautology). (This is related to the problem of whether the axioms of Choice and Infinity are propositions about reality, as Russell and Wittgenstein believe, or are also just tautologies like the other propositions of Logic.) Be that as it may, a consistent AS has infinitely many formal models. (This thesis does not hold from all standpoints either; in this generality, it only does from the usual mathematical standpoint, according to which logical existence means the same thing as consistency. According to *intuitionism* (Weyl and Brouwer), on the other hand, logical existence can only be asserted for that which has been constructed, or for which a method has at least been stated for the construction in finitely many steps. From this point of view "there is" a formal model as soon as one can be constructed. When one such has been given, a method can be stated to derive arbitrarily many from it. Thus, if there are any at all, then there are infinitely many formal models.)

C. Monomorphism

For some AS the various formal models --- and occasionally also the realizations --- exhibit notable differences among themselves. For some others this is not the case. This will be illustrated with an example.

We imagine a family with three members as a realization of the following AS. The relation "father of" is irreflexive, as are the fatherhood chains (i.e. no one is his own father or his own ancestor), and the fatherhood relation is intransitive (no one is the father of his grandchild); thus it can be taken as an interpretation of *R*.

AS I.
1. The field of *R* has three elements.
2. *R* and the *R*-chains are irreflexive.
3. *R* is intransitive.

(In symbols: $C'R \; \varepsilon \; 3 \, . \, R_{po} \subset J \, . \, R^2 \subset -\!\!\!-R$.)

The realizations (and the formal models as well) can now have at least two different forms: e.g. 1. a man with son and grandchild; 2. a man with two children. These forms are formally distinct; e.g. one difference is that in the first case *R* is one-to-one, but not in the second case. The difference in form of the interpretations thus means the same as that there are propositions --- e.g. "*R* is one-to-one" --- that can neither be derived nor refuted from the AS. If we now understand the AS I as an implicit definition of the concept *R* occurring in it, then this of course means this concept has all and only the properties stated by AS I. Now, since for *R* it neither holds that it is one-to-one, nor that it is not one-to-one, we see that *the Law of Excluded Middle does not hold for this concept*.

Now let us form a new AS by adding a new axiom.

AS IIa. 1., 2., 3. as before.

4. there is just one element in the domain of *R* ($D'R\ \varepsilon\ 1$).

Only the second of the two forms of family can now occur. (We could instead add the negation of (4) to obtain the AS IIb; then only the first form of family can occur.) All of the realizations and formal models of the AS IIa are "*isomorphic*" to each other, i.e. any two of these interpretations can be mapped one-to-one into each other in such a way that the relations *R* consist of corresponding pairs. We therefore call the AS "*monomorphic*", i.e. it determines just one form, and with respect to AS IIa every proposition containing only *R* (and logical symbols) is then either true or false. We say that the AS is "*decidable*".

By passing from AS I to AS IIa the AS thus became both monomorphic and decidable at once. And this holds in general: *an AS that is decidable is also monomorphic, and conversely*. That can be easily seen in the case of our example. For if a consistent AS, such as AS I, is not decidable, that means there is a proposition *s* about the concept or concepts of AS I that can neither be deduced as true nor as false from this AS. Therefore one can consistently add to AS I on the one hand *s*, and on the other hand its negation *s′*, as new axioms, resulting in the consistent systems AS IIa and AS IIb. Now since any consistent AS has at least one formal model (we only need this more modest thesis here, in contrast to the earlier one that there are infinitely many), there is then at least one model for each of AS IIa and AS IIb. The axiom *s* holds for the model of AS IIa, its negation *s′* for the model of AS IIb. The models thus differ by a formal property, so they cannot be isomorphic. But now the two models are also models of AS I, since the axioms of AS I also belong to AS IIa and AS IIb, and so are satisfied in both models. AS I therefore has two non-isomorphic models, so it is not monomorphic, but rather "*polymorphic*".

If, conversely, an AS is not monomorphic, that means there are two models of it that are not isomorphic. Thus there is a formal property that holds of one model and not the other, and therefore a proposition containing only concepts of the AS, and applying to the one model but not the other. Neither this proposition nor its negation can then be deducible from the AS, since in the former case the second model would be impossible, and in the latter, the first one. The AS is therefore not decidable. Thus our thesis is proven, that *the concepts "decidable" and "monomorphic" have the same extension*. An AS that has these coextensive properties will now also be called "*complete*". This designation should express the fact that is not possible to add to such an AS a further axiom that is independent and consistent (unless the axiom contains new concepts). For since the AS is decidable, every proposition containing only concepts of the AS is either deducible, and thus dependent on the old axioms, or in contradiction to the AS. It follows from this that it is impossible to reduce the number of applications of a concept that is implicitly defined by a complete AS without the aid of new concepts.

From the *intuitionistic standpoint*, decidability is a problematic concept, since it is in general not provable; in this case, "completeness" simply means "monomorphism".[7]

On *terminology*: "decidable" ["entscheidungsdefinit"] (O. Becker[8]) or "saturated" (B. Merten). "Monomorphic -- polymorphic" (H. Feigl) or "categorical -- disjunct" (Veblin[9]) or "individual -- general" (Couturat[10]) or "sufficient" (Huntington, cf. Couturat and Fraenkel[11]) or "complete" (Fraenkel, Weyl).

On the problem of *decidability* also see Behmann[12] and Fraenkel[13].

The concept of *completeness* in Hilbert's Completeness Axiom[14] does not refer to the AS but to the system of points occurring therein. There are general relations between this completeness and that of the corresponding AS, which however have not yet been thoroughly investigated.

The coincidence of decidability and monomorphism was stated by Dubislav.[15]

We extend the terms "*monomorphic*" and "*polymorphic*" from ASs to implicitly defined *concepts*. The concepts defined by a monomorphic AS are also monomorphic. The concepts defined by a polymorphic AS are generally also polymorphic; such a concept (in an AS that defines several concepts at once) can also be monomorphic, however, if despite the polymorphism of the models of the AS itself, the parts of the models corresponding to this concept are isomorphic.

D. The Indeterminacy of Improper Concepts

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[7] Weyl, "Philosophie der Mathematik und Naturwissenschaften", in: *Handbuch der Philosophie*, Bäumler and Schröter (ed.s), IIA, 1926, pp. 20ff.
[8] Becker, "Beiträge zur phänomenologischen Begründung der Geometrie". In: *Jahrb. f. Philos. u. phänom. F.* VI, 385, 1923, pp. 404ff.
[9] Veblin (*Trans. Amer. Math. Soc.* V, 1904, p. 346), following H. J. Dewey.
[10] Couturat, *Die philosophische Prinzipien der Mathematik*, 1908, p. 179.
[11] Fraenkel, *Einleitung in die Mengenlehre*, 1923[2], p. 227.
[12] Behmann, "Beiträge zur Algebra der Logic, insbesondere zum Entscheidungsproblem". *Math. Ann*. 86, 163, 1922.
[13] Fraenkel *ibid*., pp. 169ff., and further references there.
[14] Hilbert, Grundlagen der Geometrie, 1922[5], pp. 22, 240.
[15] Dubislav, "Über das Verhältnis der Logik zur Mathematik". *Ann. d. Philos*. V 193, 1926, p. 202; a proof is deferred to "the forthcoming work of Dörge and Dubislav*, Zum sog. Satz vom ausgeschlossenen Dritten*".

Logically, the implicitly defined concepts are essentially different from the proper ones, so that one hesitates to even call them "concepts". We retain this terminology nonetheless, in accordance with the usual mode of expression, above all in mathematics, e.g. in geometry (specifically, in pure geometry, which is not concerned with the real concepts of figures in physical space, but rather with the concepts implicitly defined by a geometric AS). In geometry one usually expresses oneself as though one were concerned with concepts "point", "line", "between", etc. satisfying all the requirements of a legitimate concept. Since this is not really the case, however, we restrict our terminological concession by calling the implicitly defined concepts "*improper concepts*".

We have already encountered one *difference between proper and improper concepts*: the Law of Excluded Middle, which applies without restriction to proper concepts, does not hold for all improper ones, specifically those that are polymorphic.

A second difference applies to all improper concepts. It is essential to a proper concept that for any object it is in principle decidable whether the object falls under that concept or not; and the decision can be made in practice given sufficient knowledge of the object. For instance, for the real concept horse and any given object, if the concept is sharply enough defined, and the object sufficiently known, then it is uniquely decidable whether the object satisfies the concept, i.e. whether or not the object is a horse. However, for an improper concept, the question whether a particular object falls under it is not decidable, and thus has no sense, regardless of how much is known about the object. Consider for example the Peano number concept. We have already considered as realizations of this concept certain sequences of spatial points (in physical space), points in time, spheres, etc. But the question whether a particular given sphere is a number makes no sense, and is not uniquely determined. The given sphere is a number (i.e. an element of the sequence of numbers, and then a particular one, e.g. zero or seven) if as realizations for the other elements, other spheres are taken in a suitable way; the sphere is not a number, however, if e.g. points in time are taken. Thus we see that for a whole sequence of spheres the question does makes sense whether the AS is satisfied and the sequence can therefore be treated as a number sequence; in other words, whether it is a progression. The Peano number concept is an improper one, but the concept of a progression is of course proper. This concept is defined by the Peano AS, not implicitly, but explicitly (namely, as the class determined by the propositional function that is the logical product of the axioms in the AS). In this way, every AS not only introduces one or more improper concepts, i.e. by implicit definition, but also a certain proper concept, i.e. by explicit definition. But this cannot be used in place of the implicit concepts. For unlike those, it does not occur in the AS, and thus also not in the theorems of the theory based on that AS (in the Peano AS and in the theorems of arithmetic there occur "numbers" but not "progressions"). The proper concept is always one level higher than the highest level improper concept of the AS.

> *The indeterminacy of improper concepts* is a different one than the familiar indeterminacy affecting all general concepts whatsoever. For instance the concept horse is not determinate with respect to color, since some horses are brown and others are not. Just the same holds for an improper concept, e.g. that of (Peano) number: since some numbers are prime and others not, the concept itself is not determinate with respect to this property. But in this case there is also a further indeterminacy of another kind. For the concept horse, the realization is at least uniquely determined, i.e. the class of actual objects for which the concept holds, namely the unique, determinate class of horses. For the concept number, on the other hand, the realization is also undetermined; there is more than one class of actual objects that can be regarded as the numbers. (Moreover, each of the many realizations is to be regarded as the class of *all* numbers, not as a subclass of the class of numbers; whereas the class of white horses cannot be said to be the class of all horses, but only a subclass of the horses.)

The indeterminacy of improper concepts is even worse in those cases where several concepts are implicitly defined at once by the AS, rather than just a single concept. The Peano AS can be set up in such a way that it introduces only one concept (and strictly speaking, it is the relation "successor number" rather than the concept of number, to which we have often referred in this discussion for the sake of simplicity). A well-known example of an AS with several basic concepts is Hilbert's AS for Geometry. The basic concepts include three classes: points, lines, and planes; and three relations: incidence, betweenness, and (segment) congruence. In the case of the number concept we saw that, even though its occurrence in a particular instance was not decidable, as would be a proper concept, at least for an entire sequence it can indeed be decided whether the sequence is a realization, or formal model, of the number concept. In the case of (Hilbert's) concept "point", by contrast, there is not even an entire class for which it can be decided whether or not it can be treated as a class of points. For that will always depend on the interpretation being given at the same time to the other five basic concepts of the AS.

> *Example:* Let us suppose that physical space is Euclidean and infinite. Then there results a realization of the Hilbertean AS by taking as (Hilbertean) "points" the physical points, as "lines" the physical lines, etc. This is just the realization for which the Euclidean AS was originally formulated. But now there are also many other realizations of the AS that refer to figures in physical space.[16] The AS is satisfied e.g. by taking as the class of "points", the class of all physical points except for a single point *P* ; as the class of "lines", the class of physical circles that pass through *P* ; as the class of "planes", the class of physical spheres through *P* ; and as incidence, betweenness, and congruence, certain suitably chosen relations. If it is now asked whether the class of physical points except for a particular point *P* satisfies the (Hilbertean) concept of point, then this question is not determinate. For that class satisfies the point concept if the class of "lines" is taken to be the circles through *P*, and the other concepts are realized as already stated; but it does not satisfy the concept point if the class of "lines" is taken to be the class of physical lines (although there is another realization in which this class satisfies the concept of line).
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> Like any other AS, the Hilbert AS, also explicitly defines a certain *proper concept*. If we denote the three basic classes of the AS with p, q, r, and the three basic relations with A, B, C, then this proper concept is the six-place relation H, the arguments of which may be denoted by these six basic concept variables:
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> $$H = \hat{}p\,\hat{}q\,\hat{}r\,\hat{}A\,\hat{}B\,\hat{}C \text{ [... (logical product of the axioms) ...] Df.}$$

[16] Cf. Wellstein, "Grundlagen der Geometry", in: Weber and Wellstein, *Enzyklopädie der Elementarmathematik*, vol. II. 1907.

Based on these considerations, we shall call an improper concept that is introduced by an AS together with other concepts a *dependent* (improper) *concept*, since the question is not determinate, whether it applies (not only to a particular given case, like any improper concept, but also) to a given extension as a whole. Only the question whether a given system is a simultaneous realization (or formal model) of the system of concepts of the AS is decidable. An improper concept that is the only concept implicitly defined by its AS is called an *independent* (improper) *concept*.

Overview of Kinds of Concepts

I. Proper concepts:

1. Real concepts
2. Formal concepts

II. Improper concepts:

1a. independent, monomorphic concepts

1b. dependent, monomorphic concepts

2a. independent, polymorphic concepts

2b. dependent, polymorphic concepts

Overview of Decidability

(to be read as a table)

Instances of the concept are	Proper Concepts example: horse	Improper, Independent Concepts example: (Peano) number	Improper, Dependent Concepts example: (Hilbert) point
undecidable for:		a single number	a single point, a complete class of points

decidable for:	a single horse	a complete sequence of numbers	a system of 6 basic concepts

E. Improper Concepts are Variables

The two above mentioned differences between improper and proper concepts do not really get to the heart of the matter, however, but are merely symptoms. The essential difference consists in the fact that *improper concepts are variables, while proper concepts are constants*. The symbol for a constant has a determinate meaning; a sentence-like complex of symbols in which only such symbols occur has a determinate truth value (truth or falsehood). The symbol for a variable, on the other hand, has no determinate meaning, but rather indicates an open space ("argument position"), into which symbols for constants can be put. Sentence-like forms with one or more symbols for variables, i.e. with open spaces, are not sentences at all (but rather symbols for "propositional functions"); only with the insertion of constant symbols do they become sentences.

Are the propositions of (Peano's) arithmetic or (Hilbert's) geometry then not sentences? After all, they contain symbols for improper concepts, thus variables. As they stand, indeed, they are not sentences, but rather functional expressions. But they serve as very effective abbreviations for proper sentences on the basis of an implicit convention. A sentence-like expression of this kind, in which variable symbols of a given AS occur, is to be taken as short for the sentence that looks like this (see the example below): first comes a universal prefix containing all the variables of the AS and applying to the entire implication, then comes the symbol for the logical product of the axioms of the AS as antecedent, and finally comes the sentence-like expression at issue as the consequent. The variables thus occur here only as apparent variables.

As an *example*, we take the Peano AS in the form in which the class *n* of numbers and the relation *S* of successor (number) occur as the two improper concepts of the AS, together with the arithmetical theorem "there are infinitely many numbers" or "*n* is an infinite class" (i.e. a class whose elements can be mapped one to one into those of some proper subclass). As it stands, this expression is not strictly speaking a sentence, but the expression for a propositional function with the variable *n*. As an abbreviation, however, it stands for the sentence:

$$(n, S): \ldots n \ldots S \ldots \text{(logical product of the axioms)}. \supset . \; n \; \varepsilon \; \text{clsrefl}$$

In words: "for any class *n* and any relation *S*, if *n* and *S* have such and such properties and relations (i.e. satisfying the axioms of the AS), then *n* is an infinite class".

Thus we have seen: *an improper concept is a variable that makes reference to a certain AS*. More precisely: the symbol for an improper concept is the symbol for a variable that refers to a certain AS, in the sense that the sentence-like expressions in which it occurs are to be completed in a specific way to proper sentences by the axioms of the AS.

In addition to logical concepts and the variables that are its implicitly defined improper concepts, an AS can also contain real concepts that are assumed as already known. This then imposes restrictions on the possible values of the variables: for a concept that is implicitly defined by such an AS there is sometimes no formal model, but only realizations; the number of such realizations (none, one, or several) then depends in certain cases on empirical factors, while in other cases it is logically deducible. Such improper concepts, which are implicitly defined by an AS with real concepts, have not yet been investigated.

III. The Relation between Real and Improper Concepts in the System of Knowledge

The real concepts are constituted step by step in the systematic construction of knowledge of the real world. As a link in this construction, each real concept has a direct relation to the real world. On the other hand, the improper concepts at first hang in the air, as it were. They are introduced by an AS, which, however, does not relate directly to the real world. The axioms of this AS, and the theorems deduced from them, do not constitute a proper theory (since they don't concern anything in particular), but only a theory schema, the empty form of a possible theory.[17] But if in the system of knowledge a real concept occurs that empirically turns out to have the formal characteristics of the improper concept specified by the AS, then the AS has found a realization; in place of the improper concept, which is after all a variable, the real concept at issue can now be substituted. Thus e.g. the figures of physical space (points, lines, etc.) empirically exhibit the characteristics that the axioms of geometry specify for the "points" (in the improper sense), etc.; the class of physical points can then be substituted for the class *p*, etc. Through the contact between the real concept and the axioms (the former satisfying the latter), in a single stroke, a connection is also established to the entire theory schema resting on the AS. The blood of empirical reality streams in through this point of contact and flows to the most remote capillaries of the hitherto empty schema, which is thereby transformed into a genuine theory. (In the example, abstract geometry is transformed into the real theory of physical space.) Setting up improper concepts and deriving their valid theorems thus represents the production of empty theories in reserve, for later application. The fruitfulness of this procedure rests on the fact that the theory schemata produced can be used more than once; the same schema can be connected to different places in the system of real concepts. Moreover, since there is some flexibility in the formation of real concepts, it is sometimes possible to form a concept in such a way that a connection to an (already constructed or particularly simple) AS results.

Summary

The *real concepts* comprise the proper object of science. They can be organized into a uniform system by the reduction of each one to others, and ultimately of all such to a small foundation of basic concepts. (The demonstration is not carried out here.) The formal concepts (logical and arithmetical concepts) serve merely to aid in the

[17] Cf. "doctrinal function" in C. J. Keyser, *Mathematical Philosophy*, 1924[2]; also Weyl, *loc. cit.*, p. 21.

representation of knowledge of real concepts; so-called knowledge of formal concepts (e.g. mathematical knowledge) consists of tautologies.

Distinct from these two kinds of concepts, the "*proper concepts*", are the "*improper concepts*", which are implicitly defined by an AS (axiom system). They are called "concepts" only for the sake of preserving normal terminology, but are actually variables. The values of such a variable may be formal concepts as well as real concepts, since in general an AS has both "*formal models*" and "*realizations*" among its applications. If the formal models are all isomorphic, then the AS and the improper concept it defines are said to be "*monomorphic*", and otherwise "*polymorphic*". The Law of Excluded Middle does not apply to polymorphic concepts. If every proposition about the concepts of an AS can be established to be true or false, then the AS is called "decidable". If an AS is decidable, then it is also monomorphic, and conversely; it is then also called "*complete*". If an AS introduces only a single improper concept, then that concept is called "*independent*"; if there are several at once, they are called "*dependent*". The occurrence of a single instance of a proper concept is decidable; for an independent improper concept, a single instance is not decidable, but an instance of the entire extension is; for a dependent improper concept, even this is not, but only an instance of the extensions of the entire system of interrelated concepts.

The investigation of an improper concept on the basis of its AS is a mere theory schema which, however, becomes a proper (real) theory once a realization of the concept has been empirically determined. The method of formation of improper concepts is fruitful because such a concept may have several different applications.