Three-Dimensionality of Space and Causality: An Investigation of the Logical Connection between two Fictions*

Rudolf Carnap

(The sections in *small print* can be *omitted* without interruption of the argument. They serve for introducing examples, for closer justification or explanation, for clarification of objections, or for more precise formulations which would complicate the text unnecessarily.)

Introduction

From Hume's critique of the concept of causality to the As-If doctrine of Vaihinger, the recognition has grown ever clearer that causality, conceived as an active relation, represents a fiction, based on the lived relation of an active will to its act. We do not find this causal relation in experience, at least not in "experience of the first level," where we understand by this the immediately given in its original ordering. This experience of the first level, according to the contemporary conception, exhibits only "unalterable succession"—law-governedness such that certain processes follow certain other processes in accordance with a rule.[1]

By contrast, it will be shown in the following that fictitiousness goes still further. Our *first thesis* is: *the course of what happens in experience* (of the first level*) exhibits no law-governedness; even this is already a fiction*.

---

[1] Vaihinger, pp. 310, 317f.

Our conception of fictitiousness also extends beyond the current one on another point. That the construction of a space of four or more dimensions is a fiction no longer needs to be demonstrated. Our *second thesis* goes further: *even three-dimensional space is already a fictitious enlargement of the two-dimensional space of* (primary) *experience*.

However, the most important aim of the following considerations does not consist in the demonstration of these two fictions, which are not difficult to recognize as such and have even already been presented as such here and there, hidden by other modes of expression. It lies, rather, in the connection between the two. The dimension number of space and the law-governedness of what happens have hardly ever been brought into relation with one another until now (Weyl's reference to the simplest integral-invariants of the four-dimensional manifold should perhaps be mentioned). Our *third thesis* is: *The fiction of three-dimensional space* (equivalent to the four-dimensionality of world-happenings) *is the logical consequence of the law-governedness of what happens*.

I. Experience of the first and second levels. The primary world of sense impressions and the fictitious secondary worlds of things and of physics.

The critique that has been made of the Kantian concept of experience, especially from the side of positivism, has taught us that not all form-factors in experience to which Kant ascribes necessity possess it. To be sure, (sensible) experience necessarily exhibits a certain spatial and temporal ordering, and also certain qualitative relations of equality and inequality. By contrast, the grouping together of certain elements in experience as "things" with "properties," and also the coordination of certain elements to others as their "causes," is not necessary—i.e., not a condition of every possible experience. It is, rather, a matter of free choice whether this elaboration takes place, and also, to a large extent, how it takes places. We designate experience bearing only necessary formation as "*experience of the first level*," that which is further elaborated as "*experience of the second level*."

It follows from the freedom of choice in question that *different* types of experience of the second level can be generated from the single experience of the first level, in accordance with the further re-formations that are undertaken. We wish to distinguish primarily between two different types of such re-formation: the "ordinary" and the "physical." To these correspond the experience of the second level to which we are accustomed in daily life and that of (theoretical) physics, respectively. However, this distinction is only a very rough one and can therefore only be characterized in broad outlines; in both cases there are manifold subspecies.

The "*ordinary*" *re-formation* applies principally the categories of substantiality and causality. Here experience of the second level is involved with things and their properties, and, in fact, with properties of the type of sensible qualities: color, hardness, etc. The things do and suffer. They exert forces on one another. Happenings are conceived as effects and causes.

The "*physical*" *re-formation*, by contrast, is not acquainted with the causal relation in the sense of an action—nor, in its purest form, with substantiality. It constructs a world free of sensible qualities in which there are only spatial and temporal magnitudes, together with certain non-sensible state-magnitudes. In its purest form, moreover, these three types of magnitudes have no character comparable with spatiality, temporality, or sensible qualities, but are rather mere numerical determinations, i.e., relational terms. In spite of this, the designations space, time, processes, alterations, etc., will be retained for the sake of intuitiveness. The relationship between this type of experience of the second level and that of the first level is established by means of a coordination. For example, a certain periodic form of distribution of a state-magnitude (physically designated as electrical vibrations of a certain frequency) at a certain location in the physical world corresponds to the color green I sense at a certain moment and a certain location in the visual field.[2] The processes in the physical world do not act on one another; rather, they are governed by a dependency that is to be conceived as a pure mathematical-functional relation, the nature of which will be further discussed in section IIIb.

As is evident, the "ordinary" re-formation employs a large number of fictions, whereas the physical re-formation is properly to be designated as a single enormous, systematic fiction. The boldness of the two fictions will first become truly clear in section IIc, where it is shown that both undertake a raising of the dimension number.

We will designate the content of experience of the first level as the "*primary world*." This consists, therefore, in the content of sense impressions that is not yet interpreted in any way. It corresponds on the whole to that which is called in epistemology the given (Rehmke) or the gignomen (Ziehen)—even though, as will be shown later, it in part has quite different properties from those these theories ascribe to it. The neo-Kantian philosophy is not acquainted with the primary world, since its conception that the forms of experience of the second level are necessary and univocal hinders it from recognizing the distinction between the primary and secondary worlds. However, its genuine achievement—namely, the demonstration of the object-generating

---

[2] For more on this relation of coordination and its significance for the system of the world of physics see Carnap, "Über die Aufgabe der Physik," *Kantstudien* 28 (1923).

function of thought—remains and also lies at the basis of our conception of the secondary world. The question whether primary experience is not further to be analyzed into two components—namely, into the original chaos of sense impressions and certain synthetic factors that transform the chaos into an ordering—will not be treated here. For we are not here concerned with the question of the *origin* of experience, but rather with the consideration of the properties it has when it is present as "experience," i.e., as cognitive content. The former is a question of epistemology or, properly speaking, metaphysics; the latter belongs to what is best designated by the Rehmkean expression "fundamental science." The elements of such an experience already stand in certain relations to one another (e.g., spatial contact of two simultaneous color sensations in the visual field). The minimum aggregate of these relations, and thus the set of all those that are never lacking as soon as experience in this sense is present, constitutes the ordering of experience of the first level.

It must here be clearly emphasized that we are not involved with an abstraction in the case of the primary world (as in the case of the Kantian "material of intuition," which is never given in itself). Rather, entirely non-thing-like, indeed, even entirely uninterpreted sense impressions actually appear. In the case of the most important sense for cognizing things, sight, one should recall that many painters, for example, see not things but distributions of colors—and one should further recall a similar type of vision in cases of deflected attention, cases of non-recognition of that which is seen at great distances or weak illumination, the vision of operated people born blind, and the presumably analogous type of vision of the youngest children. However, we should also remind ourselves that, even if all these cases did not occur, the distinction between the two levels of experience would still be justified and significant with respect to the necessity of the forms of the first level and the freely chosen character of the forms of the second, which is manifested by the presence of different types of secondary worlds.

By the "*secondary world*" we understand the content of experience of the second level. In the following argument we mostly choose the "*world of physics*" as its representative; for, as a consequence of its methodical generation, it is more unified and more easily conceptually grasped than the "*ordinary world*," with its many fictions and anthropomorphisms and its manifold variations.

But which is now the "*actual*" *world*, the primary or the secondary world? According to the conception agreed upon by both idealistic and realistic philosophy, and also with the view that is customary in both physical investigation and in everyday life, the construction of the secondary world leads to the construction of "actuality." Positivistic philosophy, by contrast, recognizes only the reality-value of the primary world; the secondary world is only an arbitrary reorganization of the former, carried out on the basis of economy. We leave aside this properly transcendent question of metaphysics; our immanent discussion involves only the constitution of experience itself—in particular, the distinction of form-factors into necessary and freely chosen, which we call primary and secondary, and with the relations between the two types. Similarly, the expression "fiction" carries no metaphysically negative value character, but signifies that, in the case of the latter construction, certain form-factors are newly added: the construction proceeds "as if" the former factors belonged to experience necessarily, and thus primarily.

II. Dimension number (DN)

a) *Concept of the DN of a domain*

The question of the DN of a domain is posed in the most different areas, sensible as well as non-sensible. But it is not immediately univocal. The answer always depends on the stipulation of a class and a relation for which it holds.

First, one must indicate the class of those objects that are supposed to be "elements." In this regard there are often several possibilities, even in the same domain.

Example. Our usual space has the DN 3 with respect to the class of points, 4 with respect to the class of lines, 9 with respect to the class of ellipsoids. We therefore say: The class of points of space has the DN 3, etc.

The class is often defined by indicating in which case two objects of the domain are to be regarded as identical.

Example. The question of the DN of the domain of tones is made univocal by stipulating, for example, that two tones are to be identical if they have the same pitch (DN 1), or that two tones are to be identical if they have the same pitch and the same loudness (DN 2).

In addition, it must also be stipulated which neighborhood relation is to hold for the elements.

Example. If the domain is a mosaic, and if the class is so determined that the individual stones are to be elements, then, for example, the relation of lying next to one another can be stipulated as neighborhood relation, or also similarity in hue. It follows from the definition of dimension number (to be given below) that the class has two dimensions in the first case, three in the second.

For more precise geometrical investigations the neighborhood relation would have to be more precisely analyzed—in set-theory, for example, by means of axioms for neighborhoods; but here the general concept suffices.

The DN of a domain is customarily specified or investigated without explicitly specifying the class of elements and the neighborhood relation it refers to. That is permissible when it is immediately clear which class and which relation are intended. The given examples, however, show that this is not always case.

For the definition of DN some auxiliary concepts are necessary. If in a certain case the class k and the relation B are stipulated, then we say of a subclass of k that it is a "*B-series*" if its elements can be so ordered that each one stands in the B-relation to the following one. Of two elements x and y belonging to the subclass a, we say that they are "separated in b" by the subclass a if there is no B-series in b containing x and y that contains no element of a.

Example. K: Europe; B: spatial neighborhood. Berlin and Munich are "separated by the Elbe in Germany" (not in Europe!); for there is no curve between them in Germany containing no point of the Elbe.

Definition of the DN of a domain, with respect to class of elements k and the neighborhood relation B. We distinguish between the DN of a class "in one of its elements" and the DN of the class in general.

a) The (proper or improper) subset b of k has the DN zero "*in its element* x" if there is no neighboring element for x in b.

b) b has (in general) the DN zero if b has the DN zero in all of its elements.

c) b has in x the DN n+1 if for every element y of b not neighboring x there is always a subclass separating x and y in b with DN n, but not a separating subclass with DN n-1.

d) b has (in general) the DN m if m is the greatest DN for b in any of its elements.

For $m = 0$ (d) is in agreement with (b). Greater DNs than zero are regressively defined through (c) and (d).

The example of space can serve to make this intuitive. The class of points is taken as k, the spatial neighborhood relation as B. A B-series is then a connecting curve. Two points x and y are said to be "separated in b" by a class of points a if there is no curve between them containing no point of a. The point-set c_0 has the DN zero in x if x has no neighboring point in c_0. c_0 has the DN zero if only such *isolated* points belong to it. A curve c_1, which may also have many multiplex points and consist of many unconnected segments, has the DN 1 in every point and thus in general. For, in the case of any other point y of c_1 not neighboring x, it is always possible to specify one or more non-neighboring points separating x and y, i.e., points with the omission of which x and y cannot be connected by a segment of c_1. Further, a surface c_2 has in every point, and thus in general, the DN 2; since for every other point y of c_2 not neighboring x curves can always be specified that separate x and y in c_2—thus classes of the type c_1 and DN 1. The surface of a cube or a sphere can serve as an example for c_2; since for any two points of the cubic or spherical surface a (closed) curve can always be specified that separates them. The example clarifies the occasionally occurring misunderstanding according to which only the plane is two-dimensional, whereas a surface protruding out of the plane is three-dimensional. If c_2 is a finite class of surfaces, curves, and isolated points with arbitrary relations of connection, then its DN is likewise 2; in some of its points c_2 has the DN 0, in others 1, and in others 2. And, in a similar way, it can also be shown that the entire space k, as well as a finite class c_3 of bodies, surfaces, curves, and points, has the DN 3.

In the domain of sense impressions there are three types of neighborhood relation B: it refers either to spatial or temporal or other types of properties. The corresponding dimensions with reference to B we call *spatial*, *temporal*, or *qualitative dimensions*.

Example: The realm of tones. 1. Two tones shall count as identical if they have the same pitch; for B we take neighboring pitch. DN 1; a qualitative dimension. 2. Two tones shall count as identical if they have the same pitch and the same loudness and are simultaneous; for B we take neighboring pitch or neighboring loudness or temporal neighborhood (the "or" is to be understood inclusively). The result is: DN 3; two qualitative dimensions, one temporal dimension.

From the given definition of DN it follows that the concept of dimension does not, according to its proper meaning, say, refer only to space, and it can then be carried over to others (time or sense qualities) only by spatial symbolization of these qualities. The definition is completely independent of the type of domain of the qualities and neighborhood relation in question.

The psychological question whether the representation of neighborhood and series for any non-spatial quality may perhaps always be of spatial type remains entirely untouched here and is not under consideration.

For the following investigation we make the following stipulation. This concerns only spatial and temporal dimensions, not qualitative dimensions. If we designate a domain as "(n+m)-dimensional," then it is always to be understood by this that it has n spatial and m temporal dimensions. In the following m is always equal to 1. That is taken to be identical which occupies the same space and the same time. The neighborhood relation B is spatial or temporal neighborhood.

Thus, for example, the ordinary world, and also the world of physics, is to be designated as (3+1)-dimensional.

b) *The primary world (of sense impressions) is* (2+1)-*dimensional*

In order to examine the main part of the thesis of this section—namely, that the primary world has only two spatial dimensions—the individual sensory modalities (beginning with those of an individual subject) are first to be investigated separately. It will then be investigated whether a further dimension for this domain can perhaps arise from the cooperative action of several senses, and, finally, whether this might be possible by the addition of "other people."

1. *The sense of sight*. The totality of visual sensations are first organized into a temporal series of momentary experiences. Each momentary experience consists of two classes of spatially organized color sensations, namely, the two visual fields. Each visual field has approximately the form of a mosaic—like a surface. Its DN is therefore easily determined from the given definition as two.

To be sure, the surface-like character of the visual field is not necessarily present to consciousness from the beginning. However, when two color spots of a momentary visual field appear in consciousness in general, it is also given with them whether they are in contact or not. The class *k* of these spots and a neighborhood relation *B* between them is thereby also given, on the basis of which the DN can be investigated. That closer consideration then results in the number two, and that the color spots can therefore be ordered on a surface on the basis of *B*, does not need to be also given with this original experience. The meaning of our assertion is only that *k* and *B* always have that peculiarity which is to be characterized by the DN 2 or by the expression "surface-like."

"Bodily seeing," the perception of the depth of things, does not belong to experience of the first level; it is rather an interpretation, and, in fact, one of a rather developed nature.

Perception of depth rests to begin with on the interpretation of very small deviations between the two approximately congruent visual fields (cross-disparation). Moreover, the cooperative action of other senses is also relevant: the sensation of tension of the muscles of the lens, sensations of touch in the case of simultaneously seen and touched bodies, and muscular sensations one customarily interprets as the continued motion of one's own body. That the perception of depth is in fact an interpretation and not a primary sense impression is shown most clearly by the effect of the stereoscope: Two surface-like images, which have precisely that approximately congruent but very small deviation from one another as is manifested in the two visual fields in the consideration of certain bodily objects, are interpreted in perception as just as bodily as these objects themselves.

We will speak of the cooperative action of other senses later (sensations of touch, muscular sensations in lens-accommodation and bodily motion). But we here have the result that the class of simultaneous visual sensations consists of two two-dimensional subclasses, and thus as a whole is two-dimensional.

2. *The haptic senses*. For the class of sensations of pressure the DN could at first appear to be doubtful. Can one not recognize the bodily nature, the three-dimensionality, of a stone held in one's hand? Certainly one infers this bodily nature from certain sensations of pressure: more precisely, one so interprets certain sensations which,

however, are themselves two-dimensional; they are in fact extended only over the surface of the stone (compare the example of the surface of a cube in section IIa).

The consideration that the location of a sensation of pressure depends on the place on the skin receiving the stimulation leads to the same result. These places on the skin together constitute the surface-like structure of the skin (as a whole), and thus constitute a two-dimensional class. Matters become otherwise only through the addition of muscular sensations, which will be discussed later.

However, no special significance is to be attached to this argument, which is based on the DN 2 of the skin (as a whole), because considerations proceeding from the point of view of experience of the first level must be carried out without bringing in physiological knowledge, which in fact always already contains the interpretations of experience of the second level.

The remaining haptic senses (sensations of heat, cold, and pain) are, with respect to localization, either less distinct than the sensations of pressure or very closely connected with the latter. One will therefore assume of none of them that the DN of the class of their sensations may be greater than the class of sensations of pressure.

3. *Hearing*. The class of simultaneous sensations of sound is mostly 0-dimensional. When a determination of location in the perception of sound occurs, it is limited to the mere perception of direction and is also very imprecise. But the class of directions proceeding from a location is only two-dimensional, even though three-dimensional space is taken as the basis. The frequently supervening perception of distance is certainly an interpretation, and thus belongs to experience of the second level. And perhaps this even holds for any perception of direction such that what it is based on is not yet fully clarified. Therefore, the class of sensations of sound has at most 2 spatial dimensions, and perhaps only 0.

4. *The muscular sense*. The muscular sense brings to consciousness the relations of tension and pressure of particular muscles, tendons, and joints. Here the class of simultaneous sensations has primarily no spatial determinacy in itself; localization proceeds by an empirical reduction to the localization effected by the senses of sight and pressure.

5. *The statical sense.* In so far as self-sufficient sensations of this sense can be spoken of at all, they do not stand to one another in a spatial order; their class is therefore to be conceived as (0+1)-dimensional. Hence, if a special significance attaches to this sense for the genesis of the representation of space, in so far as the situation and motion of the head is thereby recognized (probably via a type of sensation of pressure), then this proceeds only by the cooperative action of other senses—especially the visual and muscular senses. Neither a characteristic sensory quality nor a relation of spatial proximity belongs to the statical sense itself.

6. *The remaining senses*. In the cases of sensations of smell or taste or organic sensations, either no spatial determinations at all are connected with them or spatial determinations so indistinct that there it is clear that no class of more than two spatial dimensions is to be found here.

7. *The cooperative action of several senses*. As we found in the case of the sense of sight and the cooperative action of the two visual fields, the perception of spatial three-dimensionality often arises by the cooperative action of several senses (senses of sight and touch, sight and muscular sense, touch and muscular sense). That we are here involved, not with experience of the first level, but rather with the re-formation of the second level, is easy to see. For, when the sensations of two different senses occur simultaneously and act together, nothing is contained in either of the two sensory domains that would not be there without the cooperative action of the other sense. Therefore, if each of the two classes in itself is at most two-dimensional, then nothing other than two at most two-dimensional classes results from their simultaneous occurrence—thus, as a whole, an at most two-dimensional subclass of the primary world.

The *connection* of two different sensory domains into an experience of the second level is, moreover, of a very developed type. It proceeds in such a way that an element of the one is regarded as identical with a simultaneous element of the other. But this identity cannot, due to the complete disparity of the (individual) sensory domains, immediately come to light in sensation. Rather, it is inferred from the fact that simultaneously in the two domains a discontinuous process always occurs at discontinuous places (for example, the collision of two edges simultaneously in the senses of sight and touch). And it also depends on this kind of developed connection when the perception of three-dimensionality arises by the cooperative action of two senses. This perception is therefore quite distant from that which belongs to experience of the first level.

It follows from our definition of DN that a class consisting of finitely many at most two-dimensional individual subclasses cannot be three-dimensional. This can easily be concluded intuitively and abstractly inferred geometrically. Hence, the primary world, which consists of individual sensory domains, can also only be (2+1)-dimensional.

8. The same conclusion holds also for the addition of the sensations of "other people." We wish to leave entirely to one side the question whether it has any sense at all, when the primary world is at issue, to consider anything other than the sensations of a single subject. In any case, however, the DN for any (such) total domain, no matter how it is thought, can be no higher than that of the individual domains of (countably many) people—namely, the class of their sensations, which are to be united by communication. This does not subvert the fact that communication of the sensations of another is an important means for ordering one's own sensations three-dimensionally. But we are

thereby involved precisely not with experience of the first level, but rather with a further procedure of ordering, which can take place in different ways. In short, we are involved precisely with that which we have called experience of the second level.

c) *The secondary (physical) world is* (3+1)-*dimensional*

No one doubts that the secondary world, both the ordinary world and the world of physics, is (3+1)-dimensional.

The question of homogeneity of the four dimensions, which plays a large role in relativity theory, has no significance for our considerations. In the former theory we are certainly not involved with our question about a smaller or larger DN, but rather with the relation to one another of dimensions whose number 4 is not at all in question.

People have sometimes attempted to derive the number 3 of the spatial dimensions a priori. Sometimes this number has been conceived as an empirical finding, but one of a higher degree of certainty than other empirical facts. However, it is cognized neither a priori nor a posteriori, because it is not cognized at all. It is rather decided, chosen: the primary world (as the preceding section has shown) has a lower DN. That this choice has been made instinctively in the course of experience so far, without consciousness of freedom of choice, cannot shake the fact of this freedom nor the possibility of now making the choice consciously. To be sure, conscious choice, at least so far as can be envisioned at the moment, will determine the same DN as the instinctive one: both types of secondary world, the ordinary and the physical, are constructed (3+1)-dimensionally.[3] In principle, however, the insight that this DN is based on choice is of particular importance in view of the circumstance that the (degree of) causality or determinacy governing the secondary world closely depends on this choice, as will be discussed in section IV.

III. Determinacy

a) *Concept of law-governedness; determining and constraining laws*

If any element of a class depends on other elements in such a way that it is univocally determined as soon as a certain subclass of the remainder is fixed, then we call

[3] Compare, however, the sketch of a five-dimensional physical world by Kaluza, *Berl. Adad.*, 54 (1922), p. 966.

the relation of dependency a "determining law" and the class "determined." If a class contains a determined subclass then it itself is determined.

Examples. 1. The numbers of an additive calculation. Each individual number (summand or sum) can be univocally determined when all the others are given. 2. The pitches of the strings of a piano. Each individual one is determined as soon as even one other is fixed. These are therefore examples of the two extreme cases of determining laws; since in the first case all the remaining elements must be determined for the sake of the univocal determinacy of a single element, in the second case only an arbitrary one of them.

We call laws of dependency that do not result in univocal determinacy for any element, even if all the rest are determined, but still limit the possibilities for this element, "*constraining laws*."

Example. In general, constraining laws hold for the words in a book. For, it is not the case that every word is univocally determined, even if as many of the rest as you like are known; however, if sufficiently many of the nearby words are fixed, then all the possibilities for it itself are no longer open.

The concept of constraining laws will be applied later; first we will treat only the determining laws.

We ascribe "law-governedness" to a domain if either determining, or at least constraining, laws hold for the class of its elements. We hereby think of the dependency expressed in the laws purely functionally, with no ontological secondary meeting—such as that of action.

A subclass `f` of a determined calls is called a "free class" if no element of `f` is determined by the other elements of `f`. Either no laws at all, or at most only constraining laws, hold between the elements of a "free class." Every subclass of a free class is itself a free class.

If a determined class k has the DN n in its element E, and if those subclasses of k that are free classes and contain E have the DN p_1, p_2, etc., and if p is the largest of these numbers, then we designate the difference $n - p$ as the "*degree of determinacy*" of k in E. If k has the same degree of determinacy q in all of its elements, then we say that k has a *homogeneous determinacy* of the q-th degree. Every free class is a class with homogeneous determinacy of the 0-th degree, but not conversely.

In the above examples, the first can be conceived as a class with the DN 1 and homogeneous determinacy of the 0-th degree but not as a free class; the second can be conceived as a class with the DN 1 and homogeneous determinacy of the first degree. A third example: In a table of numbers ordered in rows and columns the only law holding is that every row of the table presents an arithmetical progression. Then, consistently with this law, any chosen column can be put arbitrarily at any place; indeed, even arbitrary pairs of columns constitute a free class, since any row is only determined by two determinations of number (analogously to the case of physical causality to be discussed later). The class is homogeneously determined of the first degree.

If an element *E* of the class *k* is determined by the determining laws of *k* as soon as the elements of a certain subclass *b* of *k* not containing *E* are specified, then *b* is called a "*condition class*" of *E*. Every class containing a condition class of *E* is itself a condition class of *E*.

In the third example, a subclass is a condition class of an element *E* if it contains at least two elements belonging to the same row as *E*, otherwise it is arbitrary.

b) *The determinacy of the physical world*

We here take only the world of physics as representative of the secondary world, because we are here involved with conceptually much clearer relationships than in the case of the ordinary world precisely with respect law-governedness. The validity of causality in the sense of physics means that the physical world is governed by *determining laws, and, in fact, all processes are univocally determined if the totality of processes of an arbitrarily small temporal interval are fixed*. The concepts "to effect," "bring about," and the like have therefore nothing to do with the physical concept of causality. This becomes especially clear through the circumstance that not only the later, but also all earlier processes, are univocally determined by the course of events in the arbitrary temporal interval.

The expression "arbitrarily small temporal interval" is inexact. For, if the temporal interval has a finite length, we introduce over-determination. Thus, instead of speaking of an infinitely small temporal interval, we say more precisely: The spatial distribution of certain state magnitudes and their first temporal derivatives must be fixed; this distribution, conceived as the class of spatial points to which the magnitudes are coordinated, we will call for brevity the "world-state" at the time in question; we then understand by the physical world the class uniting all of these states, and thus the class of space-time points.

This "world-state" is not yet the most general and precise expression for the subclasses of the physical world such that all the rest is determined by their determinacy. For the distribution of the state magnitudes and their derivatives at an arbitrary moment is equivalent to a distribution of the state magnitudes themselves, without their derivatives, at two neighboring points of time. And this, in turn, is logically equivalent to the same distribution at two arbitrary points of time (the latter is determined by the former and conversely). Therefore, if we were to understand by a world-state only the distribution of the state magnitudes themselves, which would logically be more correct, then we would have to require two arbitrary world-states for the univocal determinacy of the entire physical world. For the sake of simplicity, however, we will understand the derivatives to be included in a world-state, and we will similarly require only one world-state for determination. For, in the first place, this is more intuitive and closer to the usual conception, and, in the second place, the DN for two world-states is the same as for one—the simplification thus has no effect on our investigation of DN and degree of determinacy.

The world-state at an arbitrary moment—by which all the rest are determined—is itself arbitrary, in the sense that there are no physical laws that determine the state in a part of space, or even limit it in its possibilities, even if as much as you like of the state of the rest of the world at the same moment is fixed. To be sure, not all possible word-states (i.e., those arbitrarily composed of actual states of partial domains of different places and different times) also occur in reality at some time or another; but there is no physical law that distinguishes the actual from the possible. Therefore, every world-state is a free class.

However, the *condition class* of any element *E* is already a sufficiently large, finite subclass of a world-state (state of a part of space). If this world-state has the temporal distance *t* from *E*, and if *c* designates the maximal velocity of propagation for energy (the velocity of light according to contemporary knowledge), then a subclass of the world-state is a condition class of *E* if it contains the sphere around the "place of *E*" described by (radius) *t* x *c*. (Otherwise expressed, every cross-section through the Minkowskian "forward and backward cone" whose apex lies in *E* is a condition class of *E*.) The justification for this (in everyday language) is that, if an element of the world-state outside this sphere is supposed to exert an action on *E*, then this action would have to propagate in the time *t* over an interval exceeding the radius *t* x *c*, and thus with a greater velocity than *c*, which would contradict the definition of *c*.

This conditioning relation (and not only that referred to the entire, infinite world-state) is also a purely logical relation, which cannot be applied to the practice of physical science in this form. For in each finite part of space there are infinitely many arguments and thus functional values of the state magnitudes. Even if we disregard the circumstance that fixing infinitely many such values is practically impossible, it is also in principle impossible to specify them—since we are not here involved with a law-governed function. Determinacy itself is not thereby put into question, for one must certainly distinguish between the logical property of univocal determinacy and the practical property of calculability. But for the sake of practical applicability the relation of dependence must be expressed in something like the following manner: The values of the determining state magnitudes at an arbitrary space-time point are univocally determined (more precisely, are expressed by a probability function) when the values of these magnitudes for an arbitrary space-time point (more precisely, for two such points) are given in a sufficiently large, finite part of space—and, indeed, are specified either for the points of a grid, or in their average values for finite portions of the part of space, or in the form of an interpolated law-governed function constructed from finitely many such specifications.

Yet we will not take account of this modification in the expression of the relation of dependence required for application in what follows, since it has no principled effect on our investigation.

We will call the peculiarity of physical causality considered so far its "*general character*," whereby the particular constitution of the natural laws expressing the relation of dependence, the DN of the physical world, and so on, are not supposed to be included. Less precisely but more intuitively put, the general character of physical causality consists in the circumstance that the past and future are univocally determined by the present, but the relations in the present itself are not subject to law. And it now can be shown that a law-governedness having the general character of physical causality is a homogeneous determinacy of the first degree, independently of the DN of the world.

That the *laws* of this general character are determining—and not, say, merely constraining—follows from the univocal determinacy in question.

Let the physical world have the DN (q+1). For the following consideration we suppose that q is not known, in order to show that the degree of determinacy we want to derive, in which q no longer appears, is independent of the DN. It must first be shown that *any subclass* (a) *of the same* DN *as the world, thus* (q+1)*, cannot be a free class*. If we understand by a p-dimensional spherical class around the center C with radius r, the subclass of all elements of a p-dimensional class whose distance from the element C is equal to or less than r, then the following holds: Every continuous p-dimensional subclass of a p-dimensional class contains p-dimensional spherical classes of radius r, provided only that r is chosen to be sufficiently small. Therefore, every subclass a with DN (q+1) also contains (q+1)-dimensional spherical classes. Let k_1 be one these, C_1 its center, r_1 its radius. We take that diameter of k_1 having the direction of the temporal dimension as the axis. The equatorial cross-section referred to this axis, a q-dimensional spherical class k_2 around C_1 with r_1, is then the subclass of the world-state of C_1 lying within k_1. According to what was said above concerning condition classes, k_2 is the condition class of every element lying on the axis at a distance of less than $t = r_1/c$ from C_1. Since at least part of this element must belong to k_1, and thus to a, it follows that a contains elements of which it also contains the condition class. Therefore, a is not a free class.

For every element E of the physical world the following holds. E belongs to a world-state; call this e. e has in E the DN (q+0), thus q, and is, like every world-state, a free class. The greatest DN that any free class in E has is therefore at least q. But it is also exactly equal to q, since, as has just been proved, no subclass of the world with DN (q+1) can be a free class. And since the DN of the world is (q+1), then, according to the definition of degree of determinacy, the latter in E is equal to (q+1) – q, thus q. Since this consideration holds for every element, it follows that the *determinacy is homogenous of the first degree*.

c) *The primary world exhibits no determinacy*

It is easy to see that the course of uninterpreted sense impressions is not regulated by any determining laws, although this proposition contradicts a widespread opinion. As in our investigation of the DN, we first consider the individual sensory domains of a single subject, next their cooperative action, and then the additional effect of the sense impressions of other subjects. Afterwards, it will be shown that an inference back from physical law-governedness is also not possible.

1. *The sense of sight*. The most important sub-domain of the primary world is the temporal series of visual fields passing from one into another. It is clear that the elements of a single visual field, the simultaneous sensations, do not condition one another. Any give element, i.e., the color of a certain place in the visual field, remains undetermined no matter how much of the rest of the field may be given. The dependency that is often maintained is customarily not concerned with such simultaneous sensations, but rather with those that follow one another. Now, is an element perhaps univocally determined when the temporally immediately preceding visual field is fixed—or perhaps by an entire series of visual fields? Even this is not the case. Otherwise, surprising visual sensations would only be the consequence of deficient memory and the circumstance that the functional dependence is unknown. However, the visual sensations of a stone in a previously never visited desert, for example, or those of a newly luminous star, are certainly not conditioned by the preceding visual sensations, and they therefore cannot be inferred from the latter even if there is prefect memory and perfect knowledge of any supposedly present determining laws.

But not only is determinacy denied of this domain; not even constraining laws hold here. No color is in principle excluded from any place in a visual field, even after the whole remainder and arbitrarily many preceding and succeeding visual fields are fixed.

To be sure, in the case of two spatially or temporally neighboring elements of the visual field, one has the color of the other more frequently than any other given color; but it can also have any other color. Therefore, we have neither determining nor constraining laws, but we do have frequency functions both for the spatial distribution of simultaneous elements and for the temporal series. The possibility of predictions rest on a developed form of such frequency functions.

2. *The other senses*. For every other sensory domain a corresponding consideration arrives at the same result. But these can be omitted here, since fewer doubts are encountered in these cases than for the sense of sight.

3. *The totality of sensory domains*. Even when there is no determinacy in each individual sensory domain, an element of one sense might still be univocally determined by that of another sense.

Example: When those visual sensations occur that would be interpreted at the second level as the double striking of a bell, together with the auditory sensation simultaneous with the first blow, must the same auditory sensation also be given simultaneously with the second? Is it in no way possible to produce an experience of the first level in which this second auditory impressions is missing? Certainly; it is only necessary to call upon that which one calls a sensory illusion in the language of experience of the second level—which we are in fact almost always used to speaking. The blow must be struck, for example, in such a way that the bell does not ring, while the first ringing proceeds from a bell that is not seen.

Examples of sensory illusions show that no univocal determination subsists here. To be sure, cases can be imagined in which the required sensory illusion cannot be artificially produced with our technical means. But a more precise consideration then shows that even under such circumstances the corresponding experience of the first level does not appear to be impossible in principle. The customary judgements of expectation, which infer from one sense to another, always presuppose a "normal" constitution of the environment.

Strictly speaking, the visual fields of the two eyes (in order again to adduce as an example the most important sense even for this question) are not determined by the distance and direction of the object to be pictured, but rather only by the directions that the two pencils of rays have when meeting the eyes. The usually assumed conception, which is certainly almost always confirmed practically, that the distance and direction of the object has a univocal effect and can thus also be univocally inferred, holds only under the presupposition, which can in principle never be checked, that our present environment has precisely that optical state we customarily call "normal."

We should bear in mind, moreover, that experience of the first level (in contradistinction to that of the second level) can be very strongly influenced by changes of state of the sense organs and nervous system. That we will frequently trace back a "non-normal" constitution of experience of the first level to a "non-normal" state of the organs or nerves, does not change in the least the circumstance that experience of the first level then has precisely the former character—and thus that the course of the primary world proves itself to be an undetermined one.

4. *The cooperative effect of the sensations of others*. If we now add the sensations of other subjects, we must first once again enter the proviso, as we did in the discussion of the DN, that it will here remain undecided whether this addition is permissible or even in general meaningful in considering the primary world. Yet, just as in the case of putting together the different sensory domains, we come to the very same result of

indeterminacy when we put together the sensations of several subjects. We take as our basis the preceding facts appealed to in the example of the sense of sight: the non-univocal relation between the place (and, we can add, the constitution) of an object and the light-bundle entering the eyes, together with the non-univocal relation, due to the intervening organs, between this light-bundle and the sensation.

5. *The inference back from the determinacy of the physical world*. (This section serves to ward off an objection and can be passed over.) We have already been involved with bodies, light-bundles, the retina, nerves, and so on in our last considerations. Although here the primary world is in question, these object of the secondary world must be called upon as aids, because the objections against our assertion of the indeterminacy of the primary world are made from the standpoint of the secondary world—and they must thus be rebutted from the very same standpoint. For, even when someone only indicates to us his experience of the first level (i.e., the sensations he has had, without putting them together into things, interpretations, and so on), we are still always accustomed to attempting to explain these specified sensations: that is, to insert them in a law-governed way into the determinate secondary world. And so the assertion of any character at all for the primary world (here its indeterminacy) must always justify itself at the forum of experience of the second level, where the possibilities for its character are already known to some extent.

Another twist on the reservations about our thesis proceeding from the secondary world, which is in principle closely connected with the former objections, goes as follows: The world of physics is governed by determining laws. But now there is a relation of coordination, already mentioned, between it and the primary world, in virtue of which certain elements of sensation can be univocally substituted for certain subclasses of the physical world. Must it then not be possible, by this substitution, to derive from the determining laws of the physical world precisely such laws of the primary world? For two different reasons this is not possible.

In the first place, the secondary-primary coordination is in fact univocal in one direction but not the other (thus not one-one but many-one): to each primary element or complex of elements a large number of different physical complexes are coordinated. This does not only rest on the ambiguity of localization and on the threshold of stimulation, but it is also true above all for the sensory qualities. For example, to a given color sensation are coordinated infinitely many wave-forms, which are distinguished by the phase differences of their components and the direction of vibration. And something similar holds for the sensations of sound. The obstacle this ambiguity of coordination raises for the substitution in question is due to the circumstance that determining laws express conditioning relations. Univocality of coordination in the direction from the secondary to the primary world suffices substitution in place of the conditioned, but the coordination must also be univocal in the reverse direction for substitution of the conditioning. This situation can be best expressed in the language of the theory of relations (according to Russell): If P designates the asymmetric conditioning relation of the physical laws, Z the secondary-primary coordination, and Z' its converse, then the relational product $Z' \mid P \mid Z$ could be advanced as the sought for conditioning relation within the primary world. However, this relation is non-univocal: although Z and P are in fact univocal, Z' is not. It can therefore not yield determining laws, but only constraining ones.

The *second reason* lies in the circumstance that these substitutions are only possible in a certain subclass (g) of the physical world.

If we consider only a single subject, then g comprises those parts of the surfaces of physical bodies that are precisely objects of its sense impressions; we mainly think in this connection of impressions of the sense of sight. (We shall not go into the question here whether, in order to make this coordination univocal without exception, we do not have to take certain processes in the sensory sphere of the cerebrum as the range of the coordination.)

That *not even constraining laws result* is to be ascribed to this second reason. For every condition class of a physical element that is sufficiently temporally distant from it to belong to other primary elements is, for the most part, not contained in g, and thus not subject to these substitutions.

In order to make this state of affairs *intuitive*, we assume that the distance between the condition class and E must amount to at least 0.001 sec. in order to belong to different primary elements. Then (cf. section IIIb) the condition classes contain a sphere of radius 0.001 c, thus 300 km. It is obvious that such a subclass of the physical world always (and indeed for the most part) contains elements not belonging to g. Therefore, the substitution for the condition class cannot be carried out.

We have to forsake the more precise derivation in the language of the theory of relations, since, despite its fruitfulness for these kinds of investigations, this theory can unfortunately not yet be presupposed as well-known today.

IV. The connection between the two fictions

In our considerations so far two aspects of the constructed physical world have been recognized as fictions, i.e., as properties that are ascribed to it in virtue of the construction, without holding in the primary world constituting the point of departure for the construction: the DN (3+1) as opposed to that of the primary world (2+1), and determinacy of the first degree as opposed to the indeterminacy of the primary world. It shall now be shown what conditioning connection subsists between the two fictions.

Let us assume that the DN of the secondary world (in particular the physical world) is not yet known. We designate it by DN_S and that of the primary world by DN_P. DN_P is known as (2+1). We shall attempt to determine DN_S on the basis of the known character of physical causality. We designated as g those subclasses of the secondary world comprising all and only those elements to which elements of the primary world are coordinated. We designate its DN by DN_g and the degree of determinacy of the two classes by DG_S and DG_g respectively.

Now it can be shown that g is a free class, i.e., no determining laws hold between the elements of g. For, if there were such laws, then we would thereby have relations of dependence between those secondary elements to which primary elements are coordinated. And if we were to replace the secondary elements in these relations of dependence by the primary elements coordinated to them, we would not then obtain univocal relations of dependence between the primary elements (as a consequence of the ambiguity of coordination), but we would obtain non-univocal relations of dependence, and thus constraining laws in the primary world. And this would contradict our findings about this world.

The derivation of constraining laws of the primary world from determining laws of the secondary world precisely corresponds to our discussion in the first part of section $IIIc_5$. But now the objection presents itself that, corresponding to our discussion in the second part of $IIIc_5$, one would also have to conclude here that not even constraining laws for the primary world can be derived from the determining laws of the secondary world; and the just given indirect proof that g is a free class would then be weak. However, the previous discussion cannot be applied here. There it was a matter of determining laws for the entire world, and it was shown that the condition class of an element under consideration is, for the most part, not contained in g at all. But here we are considering determining laws within g. Here, therefore, the condition class of an element with respect to such a law would be entirely contained in g, and thereby subject to the substitutions of the secondary-primary coordination.

Since DN_P is equal to (2+1), or, if we neglect the analysis into a sum, equal to 3, then $DN_g \geq 3$, since an element of g corresponds to every primary element.

It follows from a theorem of point-set theory (cf. the Peano curve) that this conclusion holds only under the following presupposition: There are equi-dimensional subdomains in the primary world in which the neighborhood relation referred to in the definition of DN—here, therefore, the spatio-temporal neighborhood relation—always corresponds to the spatio-temporal neighborhood relation in the secondary world. An example to make this intuitive: Two neighboring elements of the visual field frequently correspond to two separated elements of the secondary world, namely, such that lie, seen from the eye, in almost the same direction but at very different distances. But there are also surface elements in the visual field to which connected surface elements of the physical world are coordinated, e.g., a piece of a body's surface surveyable at a glance. The presupposition in question is therefore satisfied. The converse does not follow from this; therefore we have the inequality $DN_g \geq 3$.

Let E be an element of g. g is a free class, and $DN_g \geq 3$. If p_1, p_2, etc. are the DN of the free classes to which E belongs, in E, and if p is the largest of these numbers, then $p \geq 3$. DG_S in E is then equal to $DN_S - p$, and thus less than or equal to $DN_S - 3$.

If we now wish to undertake the construction of the secondary world so that, in contradistinction to the primary world, a determining law-governedness holds, then there are very many different possibilities for this. We cannot go here into the discussion of the point of view that must have guided the choice, or the basis for the fact that the forms of the secondary world familiar to us, that of everyday life and the physical world, exhibit precisely one particular type of law-governedness. We simply assume that we wished to introduce into the secondary world a law-governedness of the type we earlier designated as the general character of physical causality (section IIIb). This character was independent from the DN of the domain and also from the special peculiarity of the individual laws of dependence. This wish (to which we are certainly not necessitated) then forces us, by what has been derived above, to give the secondary world homogeneous determinacy of the first degree. It follows from homogeneity that DG_S is equal to the value we found for DG_S in E, and therefore $DG_S \leq DN_S - 3$. But since DG_S is now supposed to be equal to 1, we obtain the result: $1 \leq DN_S - 3$, and thus $DN_S \geq 4$. Therefore, the DN *of the secondary world is at least equal to* 4 *or* (3+1).

If one attributes to the dimensions of the secondary world meanings taken from the primary world (which, as the general theory of relativity shows, is not somehow unavoidable), then it is customary, both in the everyday and the physical world, to effect variations always in the number of spatial dimensions. To all appearances there is no reason for disputing the singular number of the temporal dimension. We shall also restrict ourselves, therefore, to always tacitly presupposing this singular number, especially since the analysis of the DN into a sum of spatial and temporal dimensions has little significance for our investigation.

Thus, the general character of physical causality has forced us into raising the DN. *The fiction of the three-dimensionality of space is the logical consequence of the fiction of physical causality*. And, in fact, it is a precondition for this character of causality that space has no less than three dimensions; the circumstance that we do not attribute to space more than three dimensions has the consequence that the determinacy can be no higher than the first degree.

Summary of results

I. In *experience* we should distinguish between two levels: The primary world consists of sense impressions, not yet interpreted in terms of things, in their simplest ordering by distinctions in time, space, and quality. All ordering and processing of experience of such a kind that it can also be omitted is counted as belonging to the second level. Its content is the secondary world. Examples are the ordinary world of daily life and the world of physics.

II. The concept of *dimension number* (DN) is defined. Whereas the secondary world (both ordinary and physical) has the DN (3+1) (i.e., 3 spatial dimensions and 1 temporal dimension), the investigation of the primary world (the domain of uninterpreted sense impressions) result only in the DN (2+1). The construction of the secondary world therefore involves a raising of the DN by 1.

III. According to a definition of the concept of determining and constraining *laws*, it is shown that in the secondary (physical) world determining laws of a specific kind (of the first degree) hold. In the primary world there are neither determining nor even constraining laws. Thus, the construction of the secondary world introduces determinacy for the first time.

IV. The *two fictions* built into the secondary world—three-dimensionality of space (equivalent to the four-dimensionality of the course of world-happenings) and determinacy or physical causality—stand in a *relation of logical dependence* with one another. The former is conditioned by the latter.