SPACE: A CONTRIBUTION TO THE THEORY OF SCIENCE

By Rudolf Carnap

Translation by Michael Friedman and Peter Heath

With the Assistance of Alan Richardson

CONTENTS

Introduction

It is well known that the last century's answers to questions about the cognitive source, the nature of the object, and the domain of validity of the theory of space conspicuously contradict one another. These questions have attracted special attention, as much from a philosophical as from a mathematical and physical point of view; for, on the one hand, the general problem of cognition is intimately connected with them, and, on the other, the construction of the mathematical sciences of space on a secure foundation unavoidably requires that they be answered.

That these contradictions are found precisely in the conceptions of the most prominent representatives of the sciences in question suggests that in this case the truth does *not* "lie in the mean"--which strictly speaking implies the falsity of all the competing views. In fact, closer investigation of the question teaches us that the appearance of contradiction has only arisen because very different objects are being talked about from the different points of view.

In order to clarify the situation, we shall therefore present here a survey of the various meanings of space and the types of space that emerge in connection with each meaning--not, certainly, according to their historical, but according to their factual interconnections.

Certain structures are designated as "space" in three different domains, not, indeed, by an accident in linguistic usage, but in virture of their intimate kinship, as will emerge later. In particular, then, we distinguish between *formal*, *intuitive*, and *physical* space. If by a *general order-structure* we understand a structure of relations, not between determinate objects of a sensible or non-sensible domain, but rather between completely undetermined relational terms of which we know only that from a connection of a certain kind we may infer a connection of another kind in the same domain, then *formal space* is a general order-structure of a particular kind. Hence it does not deal with the forms that are usually designated as spatial: triangles, circles, and so on, but rather with meaningless relational terms for which one may substitute the most diverse kinds of things (numbers, colors, degrees of kinship, circles, judgements, people, etc.) in so far as there are relations between them satisfying the particular formal conditions. By *intuitive space*, on the other hand, we understand the structure of relations between "spatial" forms in the customary sense, namely the linear, planar, and spatial elements whose particular individuality we grasp on an occasion of sense perception or even mere imagination. However, we are not thereby yet dealing with the spatial facts present in experiential reality, but only with the "essence" of these forms themselves, which can be discerned in any representative of the type whatsoever.

Actual spatial facts, by contrast, such as the experiential finding that this edge of this body stands in this particular spatial relation to that edge of another body, constitute the structure of *physical space*. It presupposes the cognition of intuitive space for its cognition, and the latter, in turn, finds the pure form of its structure prefigured in formal space and therefore has this as its conceptual presupposition.

From our presentation of these three different meanings of space and their interconnections it will also be possible to recognize the ground of our spatial cognition and, in particular, whether and to what degree it is dependent on experience.

I Formal Space

Since Euclid geometry has striven to be a purely deductive science: the proof of any proposition is supposed to rest only on the axioms forming the basis of the system and the general laws of logic, not on intuition, experience, or propositions tacitly taken for granted. Euclid set this goal for geometry and also advanced a significant part of the way towards it, but he did not himself attain it. Only the investigations into the foundations of geometry of the last several decades have succeeded in laying down all the required axioms.[1] It has thereby become clear that it is not necessary to give definitions for the basic elements: points, lines, planes (Euclid, indeed, had employed such definitions at the beginning of his system, but he did not later use them in any proof). Rather, only certain *relations* between the basic elements (a point lying on a line or in a plane, the equality of two line segments, and so on) are specified by the axioms: e.g., "there is always one and only one line through any two points," "any three points always lie together on one and only one plane," etc. Theorems are then derived from the axioms without any regard for whatever intuitive meaning these elements and relations possess. Therefore, it is by no means the entire meaning-content which the axioms possess for anyone already acquainted with the concepts point, line, plane, lying on . . . that is also logically operative for the scientific structure to be erected upon them. All that is operative is their logical *form*, if by this we understand that part of their meaning that is preserved when they are transformed into something like the following general form: The elements of three classes p, L, P and the relation **i** satisfy the following conditions--"there is always one and only one element of the class L bearing relation **i** to any two things of class p," "any three elements of class p always bear relation **i** to one and only one element of class P," and similarly for the other axioms. If we also imagine all theorems to be formulated in this more general form, then we have replaced geometry proper--the geometry of points, lines, and planes--with a "*pure theory of relations*" or "*theory of order,*" that is, a science of undetermined elements and equally undetermined relations that hold between them, for which only a few axioms are presupposed on the basis of which an unlimited number of theorems are derived. Thus, in place of space--the structure of points, lines, and planes determined by properly *geometrical* axioms (which will henceforth be called "intuitive space" for distinctness)--the object of our science will be a "relational structure" or "order-structure" determined by these *formal* axioms. Since this structure presents the formal design of that spatial structure and can, moreover, again be transformed into such by inserting spatial forms for the undetermined relational terms, it will also be called space, namely, "formal space."

The advantage of this formal structure lies, on the one hand, in its logical completeness and rigor--since it is free of non-logical (intuitive or experiential) components--and in its great fruitfulness precisely for properly geometrical investigation as well--since the indeterminacy of its relational terms makes it applicable not only to points, lines, and planes, but also to the most varied kinds of basic forms. The study of structures that can only be developed very laboriously on the basis of points and lines is thereby often greatly simplified. This diversity of translation, as well as the general relation of formal space to intuitive space, will be exhibited more precisely later.

The construction of formal space can not only be undertaken in the above way, by laying down certain axioms about classes and relations, but also in another way: from *formal logic*, the general theory of classes and relations, we develop the [class of] (ordered) series and the [class of] continuous series as a special case. Within the [class of] continuous series of higher order (series of series) we then arrive at the most general case of formal space of several (in particular three) dimensions, from which we obtain, by particular specifications, (formal) projective space and the various types of (formal) metrical space. This method alone can lead to the complete construction of the formal space which comprises all subordinate types. We shall therefore briefly sketch this construction below. However, since the first method, which leads immediately to a particular case of (formal) space, is so far the only one to have been completely developed in science, we shall also briefly allude to it later.

We begin the construction of formal logic with the undefined basic concepts "true" and "false." We call anything that is either true or false a *judgement.*[2] A concatenation of signs, in particular written signs, that designates a judgement is a (complete) *sentence.* If we remove from such a concatenation a constituent with independent meaning, leaving a marker for the empty place, then such an "incomplete sentence" no longer designates a judgement.

Nevertheless, it is of particular importance for logic; many complete sentences can arise from it when other signs are put into the empty place. This place is therefore called the *argument-place;* what is designated by the inserted sign is called the *argument.* The complete sentences thereby constructed can then designate true or false judgements. It is therefore clear that, although the incomplete sentence does not designate a judgement, it does contain judgements according to their possibility as it were (potentially)--in fact, depending on the argument--and is therefore not meaningless: we say that it designates a "*concept*."[3] Depending on whether the sentence resulting from insertion of an argument designates a true or a false judgement, we say that the argument falls under the concept or does not fall under the concept respectively. For brevity we shall also say of the sentence (in a secondary sense) that it is true or false.

1. *Example.* From the complete sentence "2 + 3 = 5" we form the incomplete sentence "2 + ( ) = 5"; this designates the concept "that which added to 2 results in 5." Only the number three falls under this concept. True sentences are formed only by insertion of signs for this number ("3" or "2 + 1" or "6/2" etc.); for other insertions ("4", "+", "house") false sentences result (given sufficiently precise definitions of the arithmetical signs such sentences are not non-sensical).

2. *Example.* From "2 + 3 = 5" we can also form the incomplete sentence "2 + 3 ( ) 5" which designates the concept "relation between 2 + 3 and 5." The relations of equality, common membership in the same (arbitrary) class (e.g., the class of positive whole numbers) fall under this concept. For true sentences result from insertion of "=" or "is a positive whole number, as is". By contrast, false sentences arise through insertion of : $\neq$, <, 5, etc.

3. *Example.* From "Hamburg is a city" we form the incomplete sentence "( ) is a city." This designates the concept "that which is a city," briefly: the concept "city."

Just as an incomplete sentence with *one* argument place designates a concept, an incomplete sentence with *two* argument places designates a *relation.*[4] The two places must be distinguished; we therefore mark them with ($_1$) and ($_2$) and say that the first argument stands to the second in the relation in question.

1. *Example.* From "2 + 3 = 5" we form "2 + ($_1$) = ($_2$)." 3 and 5 (but not 5 and 3) stand to one another in the relation thereby designated; so do 4 and 6, and so on.

2. *Example.* From "Odysseus is the father of Telemachus" we form "($_1$) is father of ($_2$)." This incomplete sentence designates the fatherhood relation.

3. *Example.* Since natural language for the most part takes into account the requirement of conciseness rather than that of analysis, the linguistic expression must occasionally be transformed. Thus, in the above example, to transform the designation of fatherhood into an argument place, the sentence would have to be expressed something like this: "O stands in the kinship relation father to T", from which we can then form: (a) the incomplete *conceptual*

sentence "O stands in the kinship relation ( ) to T" which designates the concept "kinship relation between O and T"; (b) the incomplete *relational* sentence "O stands in the kinship relation ($_1$) to ($_2$)" which designates the relation between kinds of kinship and the person thereby connected to O.

4. *Example.* All *mathematical functions* (of a *single* real independent variable) are thus to be regarded as relations between the value of the variable and the corresponding value (or values) of the function. Thus, in order to display both argument places, the sine function, for example, would have to be expressed in the form "($_1$) = sin($_2$)."

We must now explain some distinguishing properties of relations; in order to be able to speak more generally about them we introduce the designation: R($_1$, $_2$), or more briefly R. If *a, b, c, d* designate particular objects, then R(*a,b*), R(*c,d*) are complete sentences. If we speak of several relations we distinguish them with subscripts: R_1(*a, b*) and R_2 (*a,b*) are therefore two different complete sentences.

By the *converse* of a relation R_1($_1$, $_2$) we understand the relation R_2($_1$, $_2$) whose argument places are reversed with respect to R_1($_1$, $_2$); therefore, whenever R_1(*a,b*) is true then R_2(*b,a*) is true also, whatever *a* and *b* may be. Example: If *a* is a descendent of *b* then *b* is always an anscestor of *a*; therefore the anscestor relation is the converse of the descendent relation.

A relation is called *symmetric* if it is identical to its converse; hence if R(*b,a*) follows from R(*a,b*) and vice versa. Example: being of the same age.

A relation is called *transitive* if R(*a,c*) always follows from R(*a,b*) and R(*b,c*). Example: In arithmetic $a > c$ always follows from $a > b$ and $b > c$; simlarly $a = c$ always follws from $a = b$ and $a = c$; therefore the relations ">'" and "=" are transitive.

A relation is called *single-valued* if there is only one second argument for each first argument; in this case $b = c$ always follws from R(*a,b*) and R(*a,c*), whatever *a* may be. Example: "the father of ($_1$) is ($_2$)" (in linguistic expression, the relation *to* the father). In all other cases the relation is called *many-valued*. Example: "($_1$) is father of ($_2$)" (in linguistic expression, the relation of being the father *of*).

The converse of a relation may be many-valued while the relation itself is single-valued: such a relation is called *many-one* (e.g., the relation to the father); the converse type is called a *one-many* relation (e.g., the relation of being the father of).

A relation is called *one-one* if both it and its converse are single-valued. Example: the relation "($_1$) + 1 = ($_2$)" is one-one.

We now proceed to a connection between concept and relation. If between the objects of one concept and those of another there holds a one-one relation, a "*coordinating*-relation", such that every object of the first concept stands in this relation to one of the second concept, and to very object of the second one of the first, then we say that the two concepts have the *same power* (are *equivalent*). Equivalence is therefore a symmetric, transitive relation between concepts. On the basis of this relation we can construct the theory of classes or concept-extensions, as well as a part of mathematical set theory fundamentally in agreement with this: the theory of powers--although the latter has been developed from a mathematical point of view proceeding mainly from the treatment of infinite aggregates, and therefore under a different mode of designation.[5] From this same conceptual source arises the theory of *numbers* (arithmetic of cardinal numbers), where by number we understand a concept of mutually equivalent concepts.[6]

If an asymmetric, transitive relation (e.g., "older than") holds among the objects of a concept (e.g., the students in a class), such that any two of these objects stand either in this relation or its converse, then we say that the objects form a *series* on the basis of this "series-forming relation."[7]

Let there be given another concept of the same power as the first (e.g., the coat-hooks in a schoolroom; one-one coordinating-relation: "hook ($_1$) belongs to student ($_2$)"). The objects of the second class may also form a series (e.g., series-forming relation: "to the right of"). Then, if the two series-forming relations and the coordinating-relation are such that whenever any two objects in the first series stand to one another in the first series-forming relation, the coordinated objects in the second series stand to one another in the second series-forming relation also, then the two series are called *similar* . (Example: If for any two students it is always the case that the older student's hook is to the right, then the series of hooks is similar to the series of students.) Similarity is therfore a symmetric, transitive relation (based on a coordinating-relation) between two series. In such a case we also call the series-forming relations similar. (In the example, the relation "to the right of in the series of hooks" stands to the relation "older in this class of students" in the relation of similarity.)

We call the concept of those relations similar to a given relation its *order-number* (ordinal number)--the concept, not the relations falling under it! On this basis we construct the theory of *order-types* as the second main part of set theory.[8] The determinations of the most important order-types will now briefly be given, since they take us further towards our goal: the construction of formal space.

All those series (in other words: their series-forming relations) are similar to one another that fulfill the following conditions: there is a first object ("initial term") with respect to the series-forming relation; for every object there is one that follows it and, except for the initial term, another that precedes it; thus, in the whole series there is no last object. Such series are called *progressions* (in set theory: order-type ω). In order to express more briefly anything that holds for these mutually similar series, we can assert it of one formal representative which we construct for this purpose. This formal representative of the progressions we call "*the series of natural (order-) numbers*." Strictly speaking, this representative of the progressions is nothing else but their concept (in our sense of the word).

Further, all series are similar to one another that satisfy the following condition: the series has the same power as a progression; with respect to the series-forming relation there is no first and no last term; for any two objects of the series there is always (at least) a third which stands to the second, and to which the first stands, in the series-forming relation. (Order-type ρ.) The formal representative of these series we call "the series of *fractional numbers* " (rational numbers).

In a fashion similar but too intricate to be briefly explained here, the conditions for series can be given whose formal representative we call "the series of *real numbers*" (order-type λ). These series are *continuous*. In this way the *continuum* is constructed by a purely formal process, without reference to intuition.[9]

The objects of a concept can also be ordered in series of series ("series of the second level") instead of, as heretofore, in *one* series. For example, the students of a school can be ordered according to classes which form a series from first to last, and, within each class, according to size; or the possible tones of a piano can be ordered by pitch, and all tones of the same pitch can be ordered by loudness. These series of the second level constitute the subject matter of the theory of number-pairs (arithmetic of complex numbers with two units), which can be developed purely formally from the preceding.

Similarly, series of the third and arbitrarily higher levels--in general, series of the n^{th} level--can be constructed and treated in the theory of number-triples or n-tuples.[10] A continuous series of the third (or n^{th}) level is called a *formal space* of 3 (or n) dimensions, although there has been no mention, as yet, of spatial elements. It will become clear later than there is an intimate kinship between this "space" and that which is usually called such. For this reason, we will now place further conditions (specializations) on this formal space of n (or 3) dimensions--designated by S_{nt} (S_{3t})--which receive their proper sense only through their later application to properly spatial structures. For we are here still dealing with merely formal relations, without making any assumptions about what sort of objects stand in these relations to one another. The different S's are therefore also called structures of order-relations (systems of ordinal-relations), briefly, *order-structures*.

Through narrower conditions on the series-forming relations in these structures, there arises from S_{nt} (which is then called topological space for distinctness) first projective space S_{np} and then metrical space S_{nm}, which are therefore related to S_{nt} as species and subspecies to a genus (not as individuals to a species). Similarly, from topological space with three dimensions S_{3t} arise projective space S_{3p} and metrical space S_{3m}, as well as still further subspecies.[11] (See the accompanying overview.)

OVERVIEW OF SPACE-TYPES

(The classification is the same for *formal space* S, *intuitive space* S', and *physical space* S".)

Three Dimensional Space:	**Space with Arbitrarily Many Dimensions**
(continuum of the third level)	(continuum of the n^{th} level)
topological space S_{3t}	topological space S_{nt}
projective space S_{3p}	projective space S_{np}
metrical space S_{3m} (characterized by the measure of curvature k)	metrical space S_{nm}

Subspecies of metrical space:

isotropic spaces: (3 equal values of k at each point)	*non-isotropic spaces:*
non-homogeneous spaces: S_i	S_u

(most general case: all values of k unequal)

Subspecies:

$S_{u\leq}$,	$S_{u\geq}$,	$S_{u\lesseqgtr}$
$(k \leq 0)$	$(k \geq 0)$	$(k \lesseqgtr 0)$
		(Einstein)

homogeneous spaces:
(the same 3 values of k at all points)

S_{ih}	S_h

(spaces of constant curvature; congruence spaces)

Subspecies:

$S_{ih<}$	$S_{ih=}$	$S_{ih>}$	
$(k < 0)$	$(k = 0)$	$(k > 0)$	
(Lobatchevsky)	(Euclid)	(Riemann)	
hyperbolic	*parabolic*	*elliptical*	(special case: spherical space)

[< 180° = 180° > 180° -- sum of angles in a triangle]

It has now become clear that the resulting order-structures (e.g., S_{3p}), if they are to be investigated on their own (i.e., without reference to S_{3t} or S_{nt}), are simpler to construct if they are presented directly as structures of certain simple relations whose formal properties are given--rather than taking the circuitous route by way of continuous series of the first, and then of the third level subject to certain limiting conditions. This will not be shown here for all the aforementioned space-types but only for S_{3p}, since that will suffice to make the principle clear.[12]

The structure restricted by the following conditions (which are presented here with a view to brevity and comprehensibility rather than precision and completeness) is of the same type as the structure S_{3p} to be developed through specialization from S_{3t}.

Let a concept P, under which the objects P_1, P_2, . . . fall, fulfill the following conditions: (1) there is a concept L, under which fall not objects but concepts L_1, L_2, . . . , such that under each L-concept only P-objects fall--in fact, at least three such--but no L-concept comprises all P-objects; (2) for any two P-objects there is always one and only one L-concept under which both fall (their "common" L-concept); (3) no matter which P-objects may be chosen the following holds in general: if P_1, P_3, P'_2 fall under the L-concept L_1, P_2, P_3, P'_1 under L_2, then firstly there is an object P_4 that falls under both the common L-concept of P_1 and P'_1 and that of P_2 and P'_2, and secondly there is an L-concept L_3 comprising P_1 but no object falling under L_2. (See figure 1 below.)

The structure so-defined is the *formal projective space* S_{3p}. As an example of the theorems valid therein which can be derived from the above conditions we present the following. In its unintuitive form we will hardly be able to recognize *Desargues's Theorem*, which is so central to projective geometry. The characterization of the structure and the theorem will only be presented intuitively, and then by means of figures as well, in the following examples of applications.

Theorem. If, in the aforementioned structure, there are nine objects P_1, P_2, P_3, P'_1, P'_2, P'_3, $P_{1,2}$, $P_{2,3}$, $P_{3,1}$, and seven L-concepts $L_{1,2}$, $L'_{1,2}$, $L_{2,3}$, $L'_{2,3}$, $L_{3,1}$, $L'_{3,1}$, L_4 such that

P_1, P_2, $P_{1,2}$ fall under $L_{1,2}$
P'_1, P'_2, $P_{1,2}$ fall under $L'_{1,2}$
P_2, P_3, $P_{2,3}$ fall under $L_{2,3}$
P'_2, P'_3, $P_{2,3}$ fall under $L'_{2,3}$
P_3, P_1, $P_{3,1}$ fall under $L_{3,1}$
P'_3, P'_1, $P_{3,1}$ fall under $L'_{3,1}$
$P_{1,2}$,$P_{2,3}$,$P_{3,1}$ fall under L_4

but neither P_1, P_2, P_3 nor P'_1, P'_2, P'_3 have a common L-concept, then there is an object ($P_{1,2,3}$) in the structure which falls under the common L-concept of P_1 and P'_1, the common L-concept of P_2 and P'_2, and the common L-concept of P_3 and P'_3. (See figure 2 below.)

In order to show clearly that such a formal structure is not limited to things of any particular kind, we shall now present certain aggregates of the most different kinds of objects,of which, if we assume the above characterization of S_{3p}, all the theorems of projective geometry--e.g., the above proposition of Desargues's--therefore hold as well. These examples are not *subspecies* of the space-type S_{3p} (like S_{3m}), but individual *instances*; each one presents a particular three dimensional projective space (in the formal sense).[13]

1. *Example*. Let a structure $\mathbf{c}_{3p}$ of *colors* fulfill the following conditions: the colors are found in certain configurations called *color-strips*; each strip bears at least three different colors, but no strip bears all the colors that appear on the remaining strips. If we choose any two of the colors then there is always one and only one strip that bears both; it is called their common carrier. Moreover, the following is to hold generally: if we imagine any three colors c_1, c_3, c'_2 of a strip s_1 and c_2, c_3 (identical with the former c_3), c'_1 of a strip s_2, then firstly the common carrier of c_1 and c'_1 and the common carrier of c_2 and c'_2 have a color c_4 in common; and secondly there is then a strip s_3 which bears the color c_1 but none of the colors borne by the strip s_2.

Now all propositions governing S_{3p} hold for this color-structure, including our *theorem:*

If there are nine colors c_1, c_2, c_3, c'_1, c'_2, c'_3, $c_{1,2}$, $c_{2,3}$, $c_{3,1}$ such that the following color triples each occur on a common strip: (c_1, c_2, $c_{1,2}$), (c_2, c_3, $c_{2,3}$), (c_3, c_1, $c_{3,1}$), (c'_1, c'_2, $c_{1,2}$), (c'_2, c'_3, $c_{2,3}$), (c'_3, c'_1, $c_{3,1}$), ($c_{1,2}$, $c_{2,3}$, $c_{3,1}$), whereas there is no common carrier either for (c_1, c_2, c_3), or for (c'_1, c'_2, c'_3), then there is a color ($c_{1,2,3}$) that has a common strip with c_1 and c'_1, with c_2 and c'_2, and with c_3 and c'_3.

2. *Example*. In order to pick some entirely different objects we shall choose a structure J_{3p} of *judgements* for our next example, and indeed judgements for which only certain formal relations are presupposed: whose objects are still undetermined. This formal projective three dimensional judgement-structure J_{3p} should not be confused with formal projective geometry--which is of course also a formal structure of judgements--nor with a structure derived from projective geometry by any sort of transformation (e.g., specialization or generalization). J_{3p} does not arise from projective geometry, but from its *object* S_{3p}: it arises from S_{3p} by instantiation. And S_{3p} is a structure not of judgements, but of undetermined things P (terms); for these P judgements are now inserted (substituted), and J_{3p} is thereby constructed. So here not only the *theory* of space, but the "space" itself consists of judgements! What the objects of these judgements are or might be is not at issue.

One says that two judgements (or two classes of judgements) are *equivalent* if one holds under the same conditions as the other, so that an inference from one to the other (and vice versa) is permitted. For example, the judgements "this triangle is equilateral" and "this triangle is equiangular" are equivalent; each can be inferred from the other; they are either both true or both false. Similarly, the class of axioms of a particular geometry (e.g., Euclidean geometry) is equivalent to the class of theorems; if one is valid so is the other; and each follows from the other.

We shall now (for this example only) call a class of three or more judgements *associated* if any pair of judgements from the class is always equivalent to the whole class. (Thus, for example, all those mutually independent linear equations in x and *y* that are satisfied for x = 3 and y = 4 are *associated*; for if any two of them are taken to be true if of course follows that x = 3 and y = 4 and thus that all the rest are also true.) Further, let a judgement (here only) be called "compatible" with two or more judgement pairs (J_1, J_2; J'_1, J'_2) if it is associated with each individual pair.

Desargues's Theorem now runs thus: If there are six judgements J_1, J_2, J_3, J'_1, J'_2, J'_3 of J_{3p} such that the three judgements that are compatible with (J_1, J_2; J'_1, J'_2), (J_2, J_3; J'_2, J'_3), (J_3, J_1; J'_3, J'_1) respectively are associated, then there is a judgement that is compatible with the three pairs (J_1, J'_1; J_2, J'_2; J_3, J'_3).

3. *Example*. If we take as out next example a structure of *points* and *lines* in space--in the proper, intuitive sense of the word--then this example is especially significant for us, since it allows us to recognize the relationship between formal space and the intuitive space that is due for discussion later.

We assume the following axioms for points and lines in space: If an arbitrary line is given, then there are at least three points on the line and at least one point not on the line. Through two points there is always one and only one line. The following holds generally: if P_1, P_3, P'_2 lie on one line, P_2, P_3, P'_1 on another, there there is firstly a point P_4 that lies on both the line through P_1 and P'_1 and the line through P_2 and P'_2, and secondly a line through P_1 that has no point in common with the line through P_2 and P'_2.

In this example we can finally make our presuppositions intuitive in a figure (figure 1), which can also serve as a symbolic representation for the other examples and for S_{3p} itself. Similarly, we can make the following theorem intuitive in figure 2.

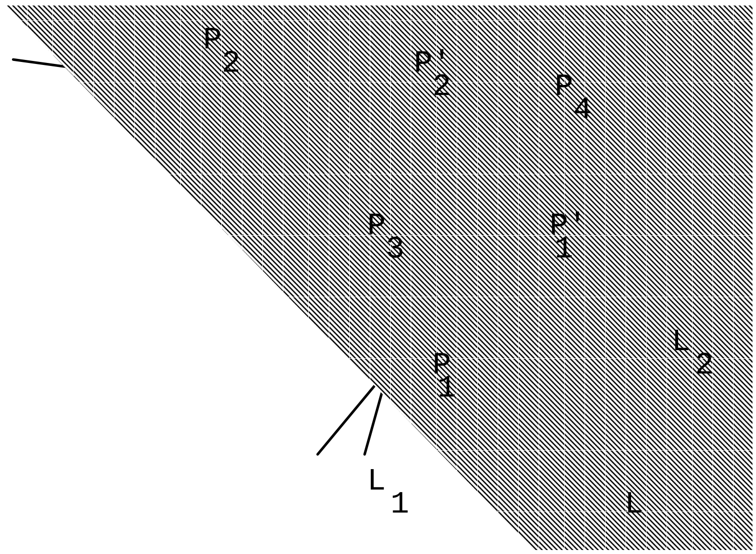

Figure 1

Desargues's Theorem: If the points of intersection of any two of the corresponding sides of the two triangles P_1, P_2, P_3 and P'_1, P'_2, P'_3 (which need not lie in the same plane) lie on a line, then the three lines connecting any two of the corresponding angles meet in a point.

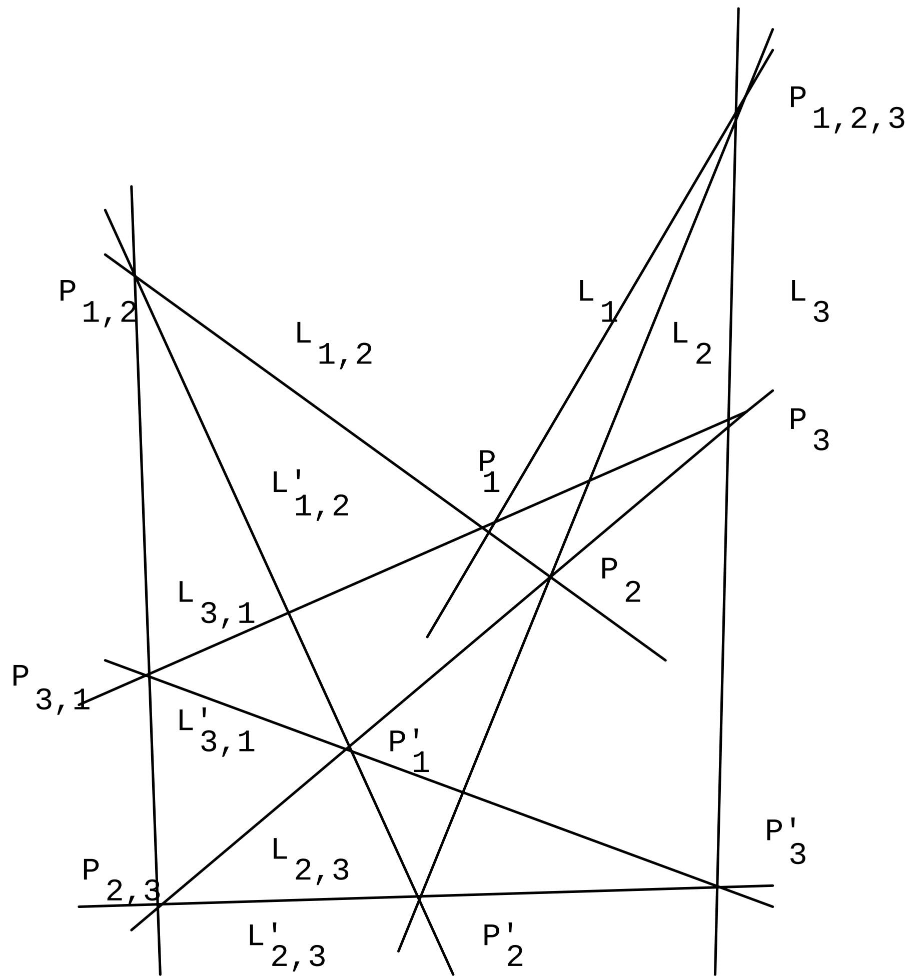

Figure 2

4. *Example*. In order to make clear the fruitfulness of formal space in its *multiple* applicability to intuitive space, *circles* and *pencils of circles* in a plane may be taken as terms of our structure in place of points and lines in space.

Assumptions precisely corresponding to our earlier ones now hold for these new objects: For any given pencil there are at least three circles in it and at least one not in it. Two circles always have one and only one common pencil. The following holds generally: if C_1, C_3, C'_2 belong to one pencil, C_2, C_3, C'_1 to another, then there is firstly a circle C_4 belonging to both the common pencil of C_1 and C'_1 and the common pencil of C_2 and C'_2, and secondly a pencil to which C_1 belongs, but none of the circles in the common pencil of C_2 and C'_2.

This composite structure of circles and pencils of circles would require an intricate treatment on its own. Now, however, we can simply transfer all the *theorems* governing our formal space to this structure with no special proofs. Thus Desargues's Theorem now runs as follows:

If there are six circles C_1, C_2, C_3; C'_1, C'_2, C'_3 lying in a plane, such that neither the three first nor the three last have a common pencil, and such that the pencils determined by C_1, C_2, and C'_1, C'_2 respectively have a common circle $C_{1,2}$, the pencils determined by C_2, C_3 and C'_2, C'_3 respectively have a common circle $C_{2,3}$, and the pencils determined by C_3, C_1, and C'_3, C'_1 respectively have a common circle $C_{3,1}$, then, if $C_{1,2}$, $C_{2,3}$, $C_{3,1}$ belong to a common pencil, the three pencils determined by C_1, C'_1, C_2, C'_2, and C_3, C'_3 respectivley have a circle $C_{1,2,3}$ in common.

These two last examples depict two of the infinitely many possibilities of conceiving intuitive space as an instance of the species determined through formal space: in this case the projective intuitive space S'_{3p} as an instance of the species S_{3p}.

II Intuitive Space

Intuitive space is an order structure whose formal type we can certainly delimit conceptually but, like everything intuitible, not its particular nature. Here we can only point to contents of experience, namely to intuitively spatial forms and relations: points, linear segments, surface-elements, volume-elements, the lying of a point on a line or in a volume, the intersection of two lines, etc. The psychological question of how such representations arise is not at issue here, but rather that concerning the logical foundation of our cognition of intuitive space: more precisely, the axioms, since the further propositions are derived from them in a formal-conceptual manner. Experience does not furnish the justification for these axioms; they are independent of experience. More precisely (Driesch), they are independent of the "quantum of experience": i.e., their cognition is not, as with experiential propositions, made ever more secure by often repeated experience. For here, as Husserl has shown, we are certainly not dealing with facts in the sense of experiential reality, but rather with the essence ("Eidos") of certain data which can already be grasped in its particular nature by being given in a single instance. Thus, just as I can establish in only a single perception--or even mere imagination--of three particular colors dark green, blue, and red, that the first is by its nature more akin to the second than to the third, so, I find by imagining spatial forms that several lines [i.e., curves] pass through two points, that on each such line still more points lie, that a simple line segment, but not a surface-element, is divided in two pieces by any point lying on it, and so on. Because we are not focussing here on the individual fact--shade of color seen here-now--but on its atemporal nature, its "essence," it is important to distinguish this mode of apprehension from intuition in the narrower sense, which is focussed on the fact itself, by calling it "essential insight" (Husserl). In general, however, the term "intuition" may also include essential insight, since it is already used in this wider sense since Kant.[14]

We shall now investigate which axioms about spatiality can be established by appeal to intuition. Only the axioms need to be obtained intuitively. To be sure, we *can* also obtain the derived propositions intuitively, at least the first few of them. However, in order to follow the principle of scientific economy and build our theory on only as many assumptions as are absolutely required, we obtain from intuition the fewest possible propositions: just enough to uniquely determine the spatial structure, i.e., so that a determinate formal order-structure can be coordinated with it. One should also avoid utilizing intuitive statements about non-simple forms, because they very quicly become more uncertain and indefinite in content as the complexity of the form increases. If, for example, one wished to obtain the Pythagorean theorem immediately from intuition instead of inferring it from simple axioms, then one would certianly only be able to assert an inequality concerning the sides and an approximate equality concerning their squares.[15]

Intuition always relates only to a limited spatial region. Therefore, it can only yield cognitions about spatial forms of limited magnitude.[16] On the other hand, we have complete freedom with respect to the total structure we construct from these basic forms. If, for example, a form is of such a kind as to permit a second form of the same kind to be added to it in a certain way, then we can postulate that this process of addition should be possible without end. In this way we can construct the *concept of* an infinite straight line from the straight segment, and, in a certain derived sense, also the *intuition*, namely, as a consciousness, based on knowledge of the rule of connection, of the possibility of apprehending each segment of the line in intuition. But then not only the infinite line corresponds to the concept so obtained, but also the finite, but unbounded, closed line of elliptical space. Neither intuition nor the above postulate decides between the two. Intuition and postulate together certainly help us to transcend the finite, but still leave certain questions about the infinite open. Let us investiage these relationships further.

We shall first assemble the axioms founded upon intuition. Next, we shall explain the postulates which have to be added to these axioms in order to obtain a spatial global structure, and we then have to investigate the types of structures that result thereby.

Because of the above-mentioned impossibility of conceptually delimiting, here in the sphere of intuition, the meaning of the basic concepts, either of the basic forms (points, lines, angles, etc.) or their relations (lying on, intersecting, equality), they could only be made comprehensible by pointing to a few intuitable characteristics--that is the sense of the Euclidean definitions. They do not seem to be required here.[17]

Hilbert's well-known construction of geometry from axioms that contain precisely the presuppositions required for the later proofs (which, as is well-known, is not true for Euclid's axioms) will now be searched, in order to discover which axioms arise from the intuition of a *limited* region.[18] Let us assemble them here in a brief form:

A. In a limited spatial region the following *axioms* hold (Hilbert I, 1-8; II, 1-4; III, 1-4):

Axioms of Connection:

1. Through any two points there is (at least) one line.
2. Through two points there is only *one* line.
3. On any line there are at least two points; in any plane there are at least three points not all lying on one line.
4. Through any three points not all lying on one line there always passes (at least) one plane.
5. Through three points not all lying on one line there passes only *one* plane.
6. If two points of a line lie in one plane, then so do all the rest.
7. If two planes have one point in common, then they have at least one other in common.
8. There are at least four points not all lying in one plane.

Axioms of Order:

9. If a point lies on a line between A and B, then it also lies between B and A.
10. If A and C are any two points on a line, then there is always at least one point B lying between A and C, and at least one point D such that C lies between A and D.
11. Among any three points on a line there is always one and only one that lies between the other two.
12. If a line and three points not on that line all lie in a plane, and if the line intersects one of the three segments determined by the points, then it also intersects one of the other two segments.

Axioms of Congruence:

13. On any line and from any point on that line in either direction, there is always one and only one segment that is congruent to any given segment.
    Every segment is congruent to itself.
14. If two segments are congruent to a third, then they are congruent to each other.
15. Two segments are congruent if they consist of pairs of congruent partial-segments.
16. For any given angle, in any plane, on either side of any ray, there is always one and only one congruent angle.
    Every angle is congruent to itself.

Remarks. On 3 and 8: The corresponding findings of intuition obviously testify to the presence of many more than two, three, or four points respectivley. Since, however, the presence of further points can clearly be inferred from the remaining axioms (10 in particular), this weakest possible form for axioms 3 and 8 must be chosen in order to satisfy the requirement of mutual independence of axioms. On 7: It follows from this axiom that the space of intuition is limited to three dimensions.

Whereas axioms 13-16 determine only the formal properties of the concept of equality (congruence), there now follow two axioms that make contentful assertions about the equality of determinate segments and angles respectively. In view of the fact that this is possible for intuition only with respect to neighboring structures (indeed, strictly speaking, intuition only asserts the agreement of structures that are brought into coincidence), the following axioms must be used for our purpose in place of Hilbert's axioms III, 5 (congruence of triangles) and IV (parallel postulate):

17. If two neighboring triangles agree in any two sides and in the angle they enclose, then they also agree in the other two angles.
18. If two neighboring lines in a plane do not intersect, then the two angles produced on the same side of any line that intersects them both are equal.

From 18, together with 16, the uniqueness of (neighboring) parallels follows.

On the basis of these findings of intuition, which relate only to a limited region, we now have to construct a complete structure whose unlimited validity is laid down by postulation. We designate it by S_{3m}'. In order to guarantee its completeness and consistency--both in itself and with the testimony of intuition--the following postulates are to be fulfilled.[19]

B. *Postulates* for the construction of an unlimited system (S_{3m}'):

1. In every limited partial region axioms A1-18 are to hold.
2. Moreover, axioms A1 and A4 are to also hold for the global region.

3. The operation of marking off a segment on a line can be repeated arbitrarily many times on both sides of any given point.
4. By this operation one can always reach a segment in which any arbitrary given point of the line lines.
5. The formal properties of the equality relations between segments and angles determined by A13-16 are to retain their validity in the extended system.
6. The equality relations expressed in A17, 18 for neighboring places are to be extended for non-neighboring places so that, in place of equality, a relation is introduced that depends on the mutual places of the structures under consideration and continuously approaches equality in the limit as the places in question approach one another (symbolically espressed:
$$\lim_{P_2=P_1} f(P_2) = f(P_1).)$$

Remarks. On 3: It does not follow from this that the extension always leads to new points. On 4: Archimedean axiom (Hilbert V,1). On 5: This is not already expressed in 1. One must distinguish precisely between the validity of a proposition in every limited partial region and its validity in the entire structure.

It follows from B1 that those axioms which only asset something about the limited partial region itself also retain general validity for the extended system. These are axioms A3, A7-10. Further, according to B2 and B5, the following continue to be valid: A1, A4, A13-16. On the other hand, the remaining axioms (A2, A5-6, A11-12) are indeed valid--according to B1--in each limited partial region, but not generally.

If we choose any very *small* region--which is to mean that only neighboring structures (in the sense of A17-18) are contained in it--then requirement B4 for the global structure will obviously also be correct for this partial region. Further, A1-18 hold without limitation for such a region. Now these together comprise Hilbert's axioms, concerning which he proves that they suffice for the construction of Euclidean geometry (the one axiom not employed here, V, 2, is also not used by Hilbert in his construction; see the conclusion of §8). Our global structure is therefore so constructed that Euclidean geometry is everywhere valid in the small. Riemann, who calls this property "planeness in the smallest parts,"[20] was the first to show how many different possibilities for the global structure are consistent with it. These different types of S_{3m}' are characterized by a certain three-valued function of places: i.e., a coordination of three numbers to every point of space (the "*measure of curvature*" for three surface-directions at this point). The significance of these numbers in the context of our postulates will now be explained.[21]

Our postulates require that the limited spatial region, whose spatial properties are given in intuition and expressed in axioms A1-18, shall be extended on all sides. The properties of the spaces so extended can best be characterized by stating the properties of the planes lying within them. Investigation shows that axioms A1, A3, A9-10, A13-16 remain valid in all such planes. For the rest, however, the spatial relations in the various possible planes can be as different from one another as those on curved surfaces. And the latter can be characterized in a well-known way by giving their Gaussian curvature at every point. So, if the spatial relations in any region of one of our extended planes are to be characterized, this can also be done by means of a coordination of numbers to the individual points. It is thereby shown that in this portion of the plane the same inner relations hold as in that region of a curved surface to whose points the same numbers are coordinated as measure of curvature. These numbers, which characterize the metrical relations within the plane (not the relations to points lying outside it) and are assigned one to each point, are now also called, as in the case of the curved surface, the (Riemannian) measure of curvature of the plane at the point in question. But this should not be misunderstood to imply that we are dealing here with a curved surface lying within the extended space S_{3m}'. Rather, the plane characterized by such curvature-numbers is throughout a *plane*, in the sense that any two of its points can always be connected by a straight line lying entirely within the plane. Then, do we mean *curved lines* by "straight lines"?--especially since we are speaking of closed lines of finite length. No, even though every point on such a line also has a number coordinated to it as "curvature." It is a *straight line*, in so far as any small segment AB on it is shorter than any other piece of any other line of our space between A and B; while for a curved line a chord is always shorter then the arc of the curve.

The metrical relations of our three dimensional space are thus completely characterized by giving this measure of surface curvature at every point for three differently directed surfaces--we will assume it, for example, for three mutually orthogonal surfaces. Now, if the same numbers can be assigned to every point of the space, then the same metrical relations that govern any region in the space also govern every other. In this case, the space is called *homogeneous*. Any plane in this space then has the same curvature at each point ("planes of constant curvature"), although not all planes need have the same curvature. On the other hand, if the three numbers assigned to any point of the space are all equal to one another, then all directions in the space are equivalent. In this case, the space is called *isotropic*. If both conditions are satisfied, then all points and all directions are equivalent; all characterizing numbers of such a *homogeneous* and *isotropic* space are equal to one another: *the* curvature of the "space of constant curvature." In this case, all planes in the space are not only planes of constant curvature, but each is equivalent to all the others; the curvature is the same for all and, indeed, the same as that of the space itself. (Cf. our overview of space-types.)[22]

The planes of constant curvature are designated as hyperbolic, parabolic (or Euclidean), and elliptic respectively, according to whether the curvature is negative, zero, or positive. The spatial relations in limited regions of these planes are the same as those on the following surfaces in ordinary (Euclidean) space: (1) the so-called pseudo-sphere or everywhere saddle-shaped surface, (2) the Euclidean plane, (3) the sphere. In all three types of plane a congruence axiom holds in the following form (instead of the limited form A17): If two arbitrary triangles agree in any two sides and in the angle they enclose, then they also agree in the other two angles. Therefore, these space-types are also called *congruence-spaces*. The three cases are distinguished by the angle-sum in a triangle, which is smaller than, equal to, or greater than two right angles respectively. They are also differentiated by the number of parallels: in a plane there are several, or (respectively) one, or no lines through a point that do not intersect a given line. In the first and second cases the lines, planes, and the entire space are infinite. In elliptic space, however, these three structures are indeed unbounded (i.e., they have no terminus anywhere) but of finite magnitude because they are closed. The same is true of a subtype of elliptic space, spherical space, in which there is not always only one line passing through two points (in the partial regions of these two spaces the same relations hold; they differ only in their global interconnection).[23]

From the facts provided by intuition concerning limited spatial regions, with the help of certain postulates, we have ascertained the various types of complete spatial structure in whose limited regions all the facts of intuition hold good. The reason we have not established stricter postulates, by which we could have arrived only at the simplest of the extended systems--namely, flat Euclidean space--will only become evident when we discuss physical space. Let it merely be noted here that this would certainly be possible: e.g., via the postulate that axioms A1-18 should hold not only in limited partial regions, but also in the global structure, and further, that A17-18 hold not only for neighboring structures, but in general.

The structure S_{3m}' under consideration, three dimensional intuitive space, is still capable of *generalization* in various ways and, from a certain point of view, is also in need of it. The mathematical treatment of these various subspecies of S_{3m}', whose immediate juxtaposition and disjunction as mutually exclusive possibilities was highly unsatisfactory from the point of view of scientific unity, has led to the realization that it is possible to construct a four-dimensional structure S_{4m}' which contains these different types of S_{3m}' as parts--not, however, in the sense in which S_{3m}' contains three dimensional volume-elements as parts, but rather in the way that S_{3m}' contains planes, spheres, and a wide variety of other surfaces. Intuition, however, was not even able to grasp S_{3m}' as a whole, let alone S_{4m}', of which, indeed, it cannot even grasp limited regions. Nevetheless, since the four-dimensional structures in such a region can be constructed from intuitively given three dimensional structures with the help of conceptual relations, a type of representation akin to intuitive apprehension--one combining intuitive and conceptual elements--is still possible here. In the previously discussed formal structure S_{4m}, and further in S_{5m}, . . , S_{nm}, we already have constructed the frames into which the intuitive terms need only to be inserted. In this way, we can ascend from S_{4m}' to S_{5m}', etc. and finally to S_{nm}': the intuitive space with arbitrarily many dimensions. Even this structure should still be called an intuitive space, in spite of the impossiblitiy of intuitively apprehending its forms--in so far as they have more than three dimensions; for, firstly all the intuitive structures we are acquainted with in S_{3m}' also occur in S_{nm}', and secondly these higher-dimensional forms are put together from intuitively given terms.

This ascent to higher-dimensional spaces is one way of generalizing S_{3m}' and thereby unifying its different subspecies. The other way consists in remaining on the level of three dimensional structures, but proceeds to more general classifications that attend only to those spatial properties that do not rest on *metrical* relations. For it is precisely these latter, as stated in axioms A13-18, which distinguish the different subspecies of S_{3m}'. A spatial structure built up from the primitive concepts point, line, plane and their relations of lying in or on one another--without using the relations of segment and angle equality--can therefore be so constructed that these differences here disappear. Such a structure is called *projective* space S_{3p}'.[24] The corresponding formal structue S_{3p} has already been mentioned and clarified through several examples--the third of which presented precisely the application to this S_{3p}'. A still more general structure is S_{3t}', *topological* intuitive space. In the construction of the latter we also forsake the primitive concepts of line and plane; besides the concept of point we use only the more general concepts of curve and surface, and we investigate their relations of lying in or upon one another and their interconnections.[25]

In the same way in which the *three dimensional* metrical space S_{3m}' has been generalized here to S_{3p}' and then S_{3t}', we can also generalize the metrical space with arbitrarily many dimensions S_{nm}' to projective S_{np}' and topological S_{nt}'.[26] Just as S_{nt} is the most general structure of formal order-relations, so S_{nt}' presents the most general structure constructed from intuitive terms: the *most comprehensive intuitive space*, which contains all other possible intuitive spaces--some as parts and some as particularizations (specializations) by means of further primitive forms and relations (see again our overview).

III Physical Space

In the processes given to us in experience--i.e., in "nature"--we ascertain, in addition to relations of another kind, also those that are customarily called spatial: relations of before, within, between, near, far, and so on. These relations will here be called physico-spatial. The theory of physical space therefore has the task of establishing which of these relations hold for the particular things that confront us in experience. The possibilities of solving this problem will now be investigated.[27]

It has been emphasized since ancient times, and has lately often been taken into account even in mathematical investigations, that the spatial forms whose names we customarily employ to designate physico-spatial relations--e.g., the straight line, the circle, the right angle--are never to be found in nature and that, moreover, if they were their existence could not be established with complete precision. Now since in what follows we shall speak of another impossibility for establishing certain physico-spatial relations, it might mistakenly be supposed that we mean this impossibility, which rests partly on the irregular shapes of natural bodies and partly on the necessarily limited precision of our technical instruments. In order to avoid such confusion, we make the fictitious assumption that errors inherent in the production of regularly defined bodies (e.g., a straight edge) or in their measurement can be reduced at will to any prescribed level. Since the following investigation will show how *little* we are able to assert about physico-spatial relations on the basis of observation, this assumption cannot lead to false consequences. We shall therefore also speak simply of "points" in physical space, without paying attention to the fact that any designated, or even recognized place in physical space, however we may designate or even merely notice it, must have an extension--no matter how small--that depends on the precision of our means of observation.

Similarly, the difficulty that the structures in physical space are treated as continuous, while physics teaches a discontinuous construction of bodies from separate parts, will not be discussed here, since it has had no essential effect on the theory of physical space according to the current state of our knowledge. It is not impossible, however, that this could take place sometime in the future.

Let us first investigate the question of whether and in what way a *straight line in physical space* can be established. For example, let the edge of a body be presented to us, or a light ray be exhibited (e.g., through a few different places at which the edge of a shadow is intercepted on a moving screen), or even just three or more points be indicated. The question is: are these lines straight or do these points lie on one straight line? One normally establishes this by either "sighting along" the line to be tested, laying a ruler along it, or something similar. It is therefore already assumed that either the light-ray or the edge of the ruler in question is straight. The problem is obviously not solved in this way, but merely pushed back; for we now have to ask further how it is known that the lines used for comparison--light-rays and straight edges (among which we should also count the thread of a resting plumb line and other such things)--are themselves straight. It is in principle impossible to establish such things if one relies only on what proceeds unequivocally from experience, without coming upon freely chosen stipulations about the objects of experience. Such stipulations, which are set up by postulation without any possibility of confirmation or refutation by experience, and which are to make it possible to test physical lines for straightness (more precisely: for whether they should count as straight), can be of two kinds. In the first type, it is directly stipulated that a class of lines presented by some definite natural objects or processes is to count as straight; let this be called a "*straightness stipulation*." The conditons that such a class of lines must satisfy for this purpose will not be discussed here, since this case is the less important one.[28]

The second way consists in "*metric stipulation*." Roughly speaking: a body is specified that is to count as rigid; more precisely: a specific body and two specific points on it are chosen, and one then stipulates which measure-number is to be coordinated with the interval between these points under various conditions (temperature, place, direction, pressure, electric charge, etc.). Example of a metric stipulation: it is stipulated that the two marks on the standard meter bar in Paris present a segment of $100 \times f(T,f,l,h; \ldots)$ cm; or of so many feet, yards, etc. In other words, a unit must also be chosen--but this is not what concerns us here; we are concerned, rather, only with the stipulation of the body itself and the function $f(T, \ldots)$.

On the basis of such a stipulated metric one can now test a physical line for straightness in a great variety of ways. For example, one can investigate via measurements made with the help of the stipulated metric-body whether the line-segment to be tested is shorter than all other lines connecting its endpoints. Or one can establish with the help of the metric-body that two other bodies are rigid: i.e., that all intervals between any two points on their surfaces remain equal (for this purpose, however, one does not need to test all of the intervals, but only a certain number of them). Then, if three or more points of one such rigid body are in contact with just as many points of the other, and the former can be so moved relative to the latter that all these points of contact are undisturbed, then all these points lie on a straight line. Furthermore, measurement in a coordinate system and yet other procedures are possible.[29]

One might now object that such a metric can by no means be freely chosen but rests, rather, on facts of experience. It is known through experience, for example, that heating an iron bar by 1° (to consider here only this most important influence) expands its length by

0.000011%. One may thence infer that two specific points on such a bar having an interval of *a* units at temperature T_0 will always have an interval of $a(1 + 0.000011(T - T_0))$ units at temperature T. If I now take other bodies made of iron or any other material and insert the corresponding coefficients of expansion, I can regard all the resulting metrics as equivalent, because in proceeding from them I always arrive at the very same measure for any physical segment. Let these equivalent metrics be therefore counted as one and designated as M_1. But now what has become of our freedom of choice? Where are the other possible metrics which lead to other results of measurement than those customarily employed in physics--yet without contradicting particular facts of experience?

One should remember, first of all, that a different metric was indeed customary when it was still usual to disregard the influence of heating on measuring rods. In those days the metric (M_0) was stipulated as follows: the interval between these two marks on this iron bar A remains always the same (and thus independent of temperature). Subsequently, the phenomenon of heat-expansion was encountered: i.e., it was found that the interval between two points on another body B--in which no other alteration whatsoever was noticed--did not prove equal on different measurements. Rather, it always came out shorter if the measuring rod A was warm. It would nevertheless have been possible to retain the metric M_0 by expressing the result as follows: the segment marked on body B alters its length with time--even if all known state-magnitudes of B itself (termperature, chemical composition, electrical charge, etc.) remain unaltered--provided only that the temperature-interval between B and A changes. We therefore have here an action-at-a-distance, which is in principle no less absurd than are the actions of electrostatic and gravitational forces, with which we have long been satisfied. Yet there was still a very important reason for not remaining with M_0 and instead establishing M_1--which incorporates dependence on temperature. Certainly, it would have been possible to master all facts of experience with M_0 as well: i.e., to present them without contradiction in the form of natural laws. But these natural laws would have taken on a very much less simple form than the usual laws of heat-expansion which present the facts on the basis of M_1. To discover the metric that leads to the simpler form for natural laws it is not necessary for the whole series of possible metrics to be set up experimentally in each such case in order to develop the laws of nature from them. Rather, the choice usually proceeds to some extent by instinct but also (and this is what we always strive for) in many cases consciously, according to principles of scientific procedure. These principles themselves, however, have still hardly been brought into a form valid for all different cases but are, even where the choice of metric or other stipulation is consciously made, usually tacitly contained in the justification. In our example the situation is this: In measuring different bodies B that are in thermal equilibrium with the heated measuring rod A it is revealed (even before we can have any idea of expressing the effect in a natural law) the remarkable circumstance that, not only is there an action-at-a-distance affecting all bodies B, but that this action is actually numerically identical in all of them, no matter what material they are made of. But here the following principle of scientific procedure comes into play: If, relative to a reference body, the other bodies exhibit numerically identical behavior no matter what their individual differences, then, in order to simplify the presentation of laws, one should attempt to treat this agreement as merely apparent by attributing the opposite behavior to the original reference body. It is this principle, a special case of Mach's principle of scientific economy, which gives preference to the conception of the earth's rotation, the earth's orbiting around the sun, and the sun's motion relative to the fixed stars over the older conception to the contrary. This same principle, in another application, has also led us from the fact that all bodies fall with the same acceleration to Einstein's principle of equivalence for gravitational force. This principle now prompts us to prefer M_1 to M_0 as well. But, and this is our point here, the facts of experience cannot force us to do this. In this sense, the choice of metric is free and independent of experience. It is not arbitrary, however, but is guided by principles similar to the above and can thereby take the facts of experience into account.

It is important for our investigation to distinguish the question, within what limits the choice of metric is possible at all, from the question, which metric is most expedient given the special facts of experience before us. Thus, the only thing that is required is that the metric should lead to a non-self-contradictory presentation. On closer consideration (which will not be carried out here) it turns out that a stipulated metric may choose any two points on the surface of any arbitrary natural body--where the latter may experience equally arbitrary changes of shape according to the usual way of looking at the matter--subject only to the condition that the two points never come in contact.

Suppose, for example, we choose a rubber body C, which may change its shape in many ways, although the two measure-points on it are never to touch each other. Is the metric stipulation (M_2)--"These two points on C have always the same interval"--contradicted by any facts of experience? Certainly not. To be sure, the measurements made on the basis of M_2 will yield very peculiar results: all other bodies will undergo violent changes of shape which cannot be brought into agreement with the customary laws of nature--and therefore require others. Can these changes of shape, then, be always brought into a law-governed system, or will there perhaps be some that contradict the principle: "the same things happen under the same circumstances"? This cannot occur. For this principle is of course satisfied if M_1 is applied as metric (i.e., in the customary physics); so here, by that principle, the interval between the two measure-points on C will therefore change only if some other circumstances change as well, which are then called the causes of such a change of interval on C. Now, if M_2

is applied instead, and the point-interval on C is conceived as unchanging, then the other bodies, in so far as they are counted as rigid according to M_1, will undergo changes of shape precisely when and only when these circumstances occur. The latter will now be conceived as causes of the change in the other bodies. Thus, causeless change does not occur even when M_2 applies. But how can we deny the obvious "factual" changes of shape of C? They are not "factual" if they cannot be established, and they can only be established by means of measurement with another body, such as an iron measuring rod D. This latter can be viewed as suitable for measurement, however, only if we have, by a freely chosen metric stipulation, declared the interval between the two measure-points on D to be unchanging. As we have seen, the facts do not force us to do this; so no contradiction arises even if D is declared to be non-rigid on the basis of M_2.

We now summarize the results of the foregoing discussion. The question whether three or more given physical points lie on a straight line cannot be answered on the basis of the facts of experience alone without a certain stipulation which we are free to choose, and hence, without reference to such a stipulation, it is meaningless. The required stipulation takes place here either through a straightness stipulation or a metric stipulation. In the latter case the interval between two arbitrary physical points (which, however, are never to come in contact) is set equal to any arbitrary function of state. The metric stipulation gives us more than the straightness stipulation: not only the means of deciding as to the straightness of physical line-segments, but also as to their relations of magnitude.

This result puts us in a position to recognize the conditions for erecting physico-spatial systems. In S_{3t}' we had a structure of intuitive space, which was constructed without using the concepts of straight line or segment congruence. We can therefore construct the corresponding structure S_{3t}''--(three dimensional) topological, physical space--without having to decide about the straightness or relations of magnitude of physical lines--hence, without straightness-stipulations or metric-stipulations. The only relations required for the ordering of physico-spatial elements given by experience in such a structure are the relations of lying-in-or-on-one-another (incidence) of points, lines, surfaces, and spatial volumes. These relations between physico-spatial elements can be gathered from experience without an agreement on any choice of stipulation.

Since no experience so far obliges us to choose a structure of more than three dimensions, we shall not consider S_{nt}'' here. However, that there is no physical spatial form of more than three dimensions is not an absolute certainty, but only an experiential probability. Still less is three-dimensionality any condition for the possibility of an object of experience in general. For it is easy to say what facts of experience (thinkable in principle, only not arising so far) would have to occur for us to regard them as four-dimensional forms.

We have seen that only *topological* space (by which we shall now always understand the three dimensional S_{3t}") univocally reproduces what lies before us in experience. Even the projective space S_{3p}" is not univocal, since we must choose and establish a *straightness stipulation* for its construction, for which there are always different possibilities. And still less are we limited in the construction of a metrical spatial structure S_{3m}"; there are infinitely many types here. Which type is realized in our completed structure depends on the metric we choose.

This connection between a freely chosen metric and the physical spatial structure erected upon it constitutes a central point in the entire question and requires a more detailed discussion. For this purpose we must introduce an important distinction. It concerns the analysis of what is present in the completed experience into two components, which arise from two different sources. The analysis I have in mind is related to that into matter and form of experience, but is not the same. For the latter analysis does not divide the completed experience on hand into two parts, but names two constituent forces through whose combined effect each particular element of completed experience is first possible. But they cannot be indicated separately: unformed matter cannot be exhibited but is a mere abstraction. Instead of that, we wish to make a division within the realm of form between necessary and optional form. Let matter which is certainly not unformed, but appears only in *necessary* form, be called "*matter of fact*" [*Tatbestand*] of experience. This can be subjected to a still further formation in terms of optional form. In order to test an experiential statement for whether it is a statement of matter of fact or not, and, in the latter case, what in it pertains to matter of fact and what depends on the form determined by choice, we have to investigate whether the experiential statement remains valid for all possible formations, which means, for our investigation, for all possible types of spatial structure. Mathematically expressed, this will be the case if the content of the experiential statement is unchangeable (invariant) under one-one, continuous spatial transformations. Now this holds for all *topological* statements and for these only: i.e., for statements concerning incidence and connection of spatial forms, and thus for all statements with respect to the topological space S_{3t}" and with respect to this only. On the other hand, all statements with respect to S_{3p}" and S_{3m}" are not unchangeable under these spatial transformations, and so do not hold for all possible formations, which result from the various metrics. They are therefore not purely statements of matter of fact, but depend on form determined by choice. And, in fact, nothing contained in the concepts of straight line and plane for S_{3p}" nor in the concepts of segment and angle congruence for S_{3m}" belongs to matters of fact. Statements of matters of fact are, for example: "this porcelain body is surrounded on all sides by this glass body" or "the contact-surface of this body (table) with this body (floor) consists of three separated parts"--for these always remain valid no matter which metric may be used for measuring the bodies. On the other hand, the experiential statement "these two points of this body have the same interval as those two points of this other body not in contact with the first" is no statement of matter of fact, but depends on form determined by choice. If I were to apply the metrics in our earlier examples (M_0, M_1, M_2), this statement would certainly not need to remain always valid; it is valid only with respect to a specific group of metrics and thereby with respect to particular sub-species of S_{3m}".[30]

The concept of matters of fact, as the entire content of such corresponding statements, allows us to recognize more precisely how, *from the choice of a metric stipulation, a specific metric space for nature*, a particular sub-species of S_{3m}" arises. Although S_{3t}" has the advantage over them of emerging uniquely from the matters of fact, the structures S_{3m}" are still incomparably more important for both natural science and everyday life, because it is here a question of measurement.

We therefore choose a metric stipulation, e.g., "these two points A and B on this piece of iron are to serve as measure-points." We thereby assert that the interval between the points is to be conceived as remaining always the same and can therefore serve as a measuring rod. The reasons for choosing this particular body and these particular points A and B are not important here--they will be discussed later. Here, all that matters is the result of our earlier discussion: namely, that the establishment of this metric can never lead to a contradiction with experience, or, as we can now express it more precisely, with the matters of fact of experience. We shall now make experiments that are to tell us which type of physico-spatial strucure S_{3m}" it is that is consistent with both the chosen metric and the matters of fact of experience. In so doing, we must take care to use no other presuppositions than the afore-mentioned metric stipulation and, with respect to experience, only statement of matter of fact. Thus, we must be particularly careful not to introduce unnoticed a knowledge of Euclidean geometry.

The experimental procedure that we wish to set up is rough and laborious. The theorems of the geometry of S_{3m}' (which hold for all of its sub-species and might therefore be applied here without thereby having already presupposed a particular sub-species like the Euclidean) could offer us simpler procedures for determining S_{3m}". But we here value the perspicuity of the procedure more than its economy, so we choose a procedure that employs one of the previously given distinguishing characteristics for the sub-species of S_{3m}": namely, the angle-sum of a plane triangle.

In order to avoid an ambiguity in the implementation of our procedure, we must also pay attention to the following important circumstance. What we designate here as physical space S" is not itself the form of spatial events, but rather only a three dimensional projection of this form: namely, a projection of the four-dimensional space-time manifold. Now, three dimensional projections can be constructed from the latter in various ways through the choice of three axial directions. Under certain circumstances (namely, when none of the three axes falls within Minkowski's "light-cone") such a projection is to be regarded as space. So numerous spatial structures are possible corresponding to different determinations of the simultaneity of two events at different points. But we can free ourselves from this ambiguity by restricting ourselves to those spatial relations that are independent of the determination of simultaneity. This can be done if in the following measurement experiments we establish the position of two spatial forms relative to one another only when they are mutually at rest. If, for example, the "simultaneous" contact of the points A, B, C, of one body with one each of the points A', B', C' of another body is to be established, then this must not be done on merely momentary contact. Rather, the following is to be assumed: an observer transports himself (with no matter what velocity) from A by way of B, C, A, B, to C; three other observers remain at A, B, C, respectively, and each of them establishes that the contact with the corresponding point A', B', or C' remains continually in effect between the two instants at which the first observer passes by him. For all possible determinations of simultaneity a simultaneous occurence of the three contacts AA', BB', CC' is thereby demonstrated. Since we need not determine anything other than coincidences in the following experiments, this cautionary rule for establishing the properties of physical space suffices to free us from the otherwise unavoidably required reference to time determinations.

We now begin to make measurements with the help of that iron body specified in our metric stipulation, which carries the two measure-points A, B. We have found or produced a (physical) surface element **s**, perhaps the upper surface of a table top, which satisfies the following conditions (1) - (5).

(1) A and B, as well as two other points C and D on the iron body, can always be brought into simultaneous contact with the four points A_1, B_1, C_1, D_1, on **s**. Through repeated experiments it appears that whenever any of the three points A, B, C or A, C, D, or B, C, D coincide with their corresponding points, the fourth pair of points also coincide. Further, A and B can always be put into simultaneous coincidence with B_1 and C_1, and also with C_1 and D_1, and also with D_1 and B_1. We shall now call an aggregate of points, a linear or surface element, or a body "rigid" relative to a certain metric stipulation if the interval between any two points in the aggregate remains always the same. According to this definition it follows that first, if we always presuppose our metric stipulation (A,B), the point-pair A, B is rigid, and further, according to the experiment, so are the pairs A_1 B_1; B_1, C_1; C_1, D_1; D_1, B_1 which can always be made to coincide with A, B. It follows also by the first experiment that the pair C, D, is rigid, and so also is the quadruple A_1, B_1, C_1, D_1 and therefore the quadruple A, B, C, D.

(2) We now bring A, B, C, D into coincidence with four other points A_2, B_2, C_2, D_2, on **s**; repeated experiments also yield here the same results as above. Thus, the quadruple A_2, B_2, C_2, D_2 is rigid as well. Moreover, in all point-aggregates we find the very same results for any

quadruple A_n, B_n, C_n, D_n on **s** that can be brought into coincidence with A, B, C, D, and we find the same behavior with those for which we make the same experiment with A_1, B_1, C_1, D_1. We find no such quadruple that is not rigid; therefore, the entire surface **s** is rigid. This situation is repeatedly tested during the following experiments and is continually found to be verified.

(3) Whereas in the first experiment the coincidence of any three of the four pairs AA_1, BB_1, CC_1, DD_1 always led to coincidence for the fourth as well, except when the fourth pair was CC_1, in this latter case that does not hold: we observe that A, B, D coincide with their corresponding points and C also at first, but then no longer, while the other three points of coincidence are preserved. The two rigid point-aggregates A, B, C, D and A_1, B_1, C_1, D_1 have therefore moved relative to one another in such a way that three point-pairs have remained in coincidence. This is the criterion for a physical straight line; hence A, B, D lie on such a line and so do A_1, B_1, D_1.

(4) We bring A into contact with A_1, and, at the same time, we successively bring B into contact with various other possible points B_1', B_1'', . . . It then never happens that D is not also in contact with a point on **s**: let these points be D_1', D_1'', . . . Then A_1, B_1', D_1' lie on a straight line, and so do A_1, B_1'', D_1'', etc.

(5) We make this same experiment as with A_1 also upon A_2, A_3, etc. and observe the same results for any arbitrary point on **s**. From this we conclude that the surface **s** carries straight lines in all directions at every point. Therefore, **s** is *plane*.

We now wish to investigate the spatial relations on this physical plane. We employ the angle-sum in a triangle as our test. As already explained, the numbers that characterize the metric relations in a plane (measures of curvature) are related to this angle-sum in a uniform way. Depending on whether the curvature at a point on the plane is equal to, less than, or greater than zero, the angle-sum of a (small) triangle there is equal to, less than, or greater than 180°--and thus the angle of an equilateral triangle is equal to, less than, or greater than 60°. Six equilateral triangles surrounding a point will therefore, in the three cases, either fit exactly, or leave an angle free, or partially overlap. Around A_1 on **s** we therefore seek seven points B_1, B_2, B_3, B_4, B_5, B_6, B_7 such that AB can be successively brought into coincidence with A_1B_1, A_1B_2, . . , and A_1B_7, and likewise also with B_1B_2, B_2B_3, . . and B_6B_7. We then check whether or not B_7 coincides with B_1. In the first case the plane **s** has zero curvature at point A_1, in the second case a positive or negative curvature (one then says briefly that the surface is curved here; however, our earlier remarks about the meaning of these analogical expressions again apply).

In order to be able not only to establish the fact of curvature but also its numerical value, we must derive a procedure for *measuring segments* from our metric stipulation. For this purpose we produce a body on whose surface a simply connected linear element **l** is shown to be a rigid straight-edge via the following experiments. We test the points of **l** with two points A, B of our metric body in the same way that we have tested the surface **s**. Corresponding to experiment (1) we show, first, that the endpoints P_0 and P_{10} of **l** constitute a rigid point-pair and, indeed, that they have the same interval as A and B. Then, by comparison with the rigid plane **s**, it is shown, corresponding to experiments (1) and (3), that for a given point Q on **l** the three points P_0QP_{10} are rigid and lie on a straight line. If, in further experiments this is always found to be confirmed for any arbitrary point on **l**, the latter is thereby shown to be a rigid straight-edge. Now in order to be able to measure in something like the usual way with tenths of the unit segment AB = P_0P_{10}, we mark nine points $P_1, \ldots, P_9$ on **l** such that the ten point-pairs P_1P_2, P_2P_3, . . . , P_9P_{10} prove to be equidistant--by being successively brought into coincidence with a point-pair MN on **s**. Similarly, we could also divide each of these segments into ten equal parts, and so forth, depending on the precision required for the measurements to be made with this measuring rod or observationally possible.

We can now measure on another body an interval between any two points K, L that is smaller than the interval AB (otherwise we would have to set up a longer measuring rod using obvious procedures): we apply the measuring rod so that K coincides with P_0 and L with a second point on **l**; if, for exampe, the latter lies in the partial segment P_6P_7, then the interval KL is equal to 0.6 with the precision determined by this division, where AB is taken as the unit of measurement. To measure more precisely one must refer to a sub-division of the segment P_6P_7. These considerations exhibit a fundamentally important point: once a measuring rod is set up on the basis of the metric stipulation, only topological properties are established in measuring segments, and thus only "statements of matters of fact": namely, the mutual contact of points and the lying of a point in a segment.

With the measuring rod thereby set up we now return to the surface **s**. We had delineated around point A_1 the angles of equilateral traingles, without also being able to establish the sides with the means hitherto employed. This can now be accomplished simply with the help of the straight-edge **l**. We bring P_0P_{10} successively into contact with A_1B_1 and B_6B_7 and mark all those points on **s** that come in contact with these points of **l**. We then establish whether the segments A_1B_1 and B_6B_7 have a point in common. In this case the curvature is either zero, namely, if B_1 and B_7 coincide, or positive, if other points coincide. In the two segments have no common point the curvature is negative. (The cases of extraordinarily strong curvature--curvature around ±1--may be left out of consideration here, since they contain nothing fundamentally different; it will merely be the case that other segments will intersect and the later calculations will be changed.) We now measure the interval *a* of the points B_1 and B_7 with the measuring rod **l**, thus in fractions of the unit segment AB; we then give the number *a* positive or negative sign depending on whether the curvature has been established as positive or negative by our criterion. We could now have interepreted the number $\pm a$ directly as the measure of curvature. But, in order to conform to the usual notation, we can also calculate the Gauss-Riemann curvature k, which is proportional to *a* when the curvature is not excessively strong: $k = 2/\sqrt{3}a$.

We have thereby determined the curvature of the planar element **s** at the place A_1. In order to determine the metrical relations in the entire region **s** we must carry out the same experiments at various other points. How many points are chosen for this purpose is not a matter of principle, but depends on the degree of precision aimed at in presenting the global metrical relations of **s**, and will also take into account whether large differences of curvature are found for different places. Once the findings for **s** are completed, all the rest of space must be measured in the same way: we must everywhere set up surfaces that are to be proved rigid and plane just like **s**, and the curvature is to be determined from them. And, in fact, this must be done for three different directions at every point in space. When the entire universe has been so measured, then we have unambiguously found the particular metrical space, a wholly determinate subspecies of S_{3m}", that agrees with the matters of fact of experience and the chosen metric stipulation. If we choose any other metric stipulation then we also find unambiguously, by the same procedure, a determinate, but in general different, physico-spatial structure.[31]

The presentation of this experimental procedure is simply designed to show first, that the establishment of metrical relations in physical space only has a sense when a freely chosen metric stipulation has been set up and, second, that this establishment then employs only the matters of fact from experience: that is, only topological properties of the physico-spatial structure are observed and evaluated, and no assumptions are made about the straightness of any physical lines (such as light-rays), about parallelism, about the homogeneity of space (in the sense explained earler), and the like. We thereby overcome the frequently raised objection that the experimental establishment of metrical relations is circular, since it makes assumptions that it then attempts to prove.[32]

This is the significance in principle of our procedure. In actual practice, by contrast, the experiments would be carried out quite differently and would be simpler in some respects, more difficult in others. We can significantly simplify our extremely laborious procedure, which in larger regions is quite impossible to carry out. Just as we have tested the surface **s** with the measure-points A, B, so we can also test the properties of light-rays and then use these rays in a more convenient fashion for constructing figures. Nor do we need for this purpose to take the six equilateral triangles carrying thirteen segments--one of which is to be measured--but can use simpler figures. It proves to be possible and especially suitable to choose a very large figure on which only very small segments need be measured. This is the case, for example, if we choose a large triangle constructed from light-rays whose angles are determined by measuring certain segments with surveying instruments set up at each anlge. This procedure was applied by Gauss with the help of the triangle determined by Inselsberg-Brocken -Hoher Hagen. We should point out, however, that, in using light-rays in this way, the neglect of temporal determination is not permissible, since the above mentioned cautionary rules that entitled us to do this are not applicable in this case.

On the other hand, the difficulties that emerge in the actual execution of such experiments rest on the fact that (if we presuppose the customary metric M_1) the curvature proves to be everywhere either zero or extremely small, so that the procedure of moving about the iron rod would be far too imprecise, and only such procedures as permit the measurement of large regions--that is, only involving light rays--offer any prospect for success.[33]

The reciprocal relations of dependence between matter of fact, metric stipulation, and type of physical space have so far been considered by choosing a metric stipulation and then finding the properties of physical space. We now wish to proceed in the opposite direction and show how it is also possible *to determine in free choice the metric relations of physical space* and how then *a metric stipulation is found* that yields the chosen space-type on the basis of the matters of fact--and according to which the form of particular natural laws is adjusted. Just as, in presenting the experiments in the above example, we assumed much stronger curvatures than are apparent in actually performing the experiments--so as to make the matter of principle stand out more clearly--so we shall now also choose a spatial structure which diverges very strongly from the usual one. The matter of principle--the optional character of the space-type--is thereby made clearer than if we were to choose, say, the flat (Euclidean) space S_{ih}". In reality, when starting from a choice of space, one would most likely always choose the latter since it is the simplest.

In contrast to the above example, however, we shall now employ, not the results of imaginary experiments, but those of the existing physics. This should be borne in mind especially at those points where, owing to a different *interpretation* of physical observation, completely different metrical relations are asserted of things, such as the earth, from those in the usual physics; topological relations--the only ones that physical observation can actually establish--are always to remain in agreement with physics.

In order to take an intuitive *example*, let it be determined that we shall consider *the earth's surface E as a plane*.[34] Since E turns out to be a sphere on the basis of the customary metric M_1 (for simplicity we leave oblation out of account), a different metric will prove to be required here. But our freedom to choose with respect to the spatial structure is not yet exhausted by the establishment of the above determination. For we are still free to determine what curvature the plane E is to have at the various points and how the rest of space is to be constituted. We could thus determine, for example, that E is to have zero curvature everywhere. We could then regard the earth's surface as infinitely large with the Euclidean geometry of the plane holding everywhere upon it. For brevity, we shall not proceed to work out this seemingly strange conception nor to demonstrate that it actually contradicts no physical experience--and therefore no results of geodetic measurements made by optical or mechanical procedures, provided only that they are interpreted differently than is customary on the basis of M_1 and are instead interpreted according to M_e. For this purpose M_e whould have to have something like the following form: "these two points A, B on this iron body present an interval that is not to count as always the same but (aside from temperature, magnetization, etc. in such and such a degree) is to depend above all on the place on E at which the body is located in such and such a manner." This dependency would have to be so stated that some one particular point on E, e.g., the place at which the iron body is found just now, or the North pole, say, appears priviledged in a certain way. If we do not wish such preference for a point on E to appear in our metric, then we have to determine that E is to be considered as a plane with positive curvature everywhere whose magnitude is equal to that which E has as a sphere on the basis of M_1. Our metric will then no longer contain any dependence upon the place on E.

We can now go on to prescribe values of the curvature for the entire remaining universe. We wish to choose the simplest structure that can be brought into agreement with the determination already hit upon: space is to be homogeneous and isotropic and is thus to have the same curvature at all points and on every directed-surface. Since this has already been determined as positive on E, our choice has thereby fallen upon a species of $S_{ih>}$".

We now have to investigate which metric is appropriate to this chosen spatial structure. Rather than a derivation, let us state the result at once, and then test it to see whether it can indeed be harmonized with both the spatial structure and the matters of fact of physical experience--and thus, in particular, with all astronomical and terrestrial space measurements. The metric (M_s) runs as follows: "these two points A, B on this iron body present an interval that is to be certainly counted as independent of the place on E, but (aside from temperature, magnetization, etc. in the same degree as for M_1) as dependent on the height h_s above E: $1 = l_0(1 - \sin(h_s))$."

We shall discuss units and numerical values in a moment. First, we note that M_s agrees precisely with M_1 so far as the relations on E itself are concerned. It follows that all segments on E that are counted as equal in physics are also to be viewed as equal here. Now according to M_1 the great-circles on the terrestrial sphere are the shortest lines on E; hence they are also such according to M_s. And, since in this case, as we intend to confirm by measurements outside E, the latter is a plane, these same lines are straight lines here. On account of the agreement between M_1 and M_S on E, physical meter sticks, so far as they are applied to E, are also valid for M_S-measurement. However, in order to obtain simpler formulas for relations in the whole of space, we shall choose units of length of 6,370 km rather than 1 cm or 1 m. In these units the total length of every straight line on E--e.g., the equator or the meridians--equals 2π. One can easily see that through any two points on E there passes, in general, only one straight line; but if these points are opposing poles (e.g., North and South pole or observation-point and antipode) then there are infinitely many straight lines passing through them. The plane E is thereby characterized as a special sub-species of the elliptical plane ("spherical plane"). Since we have determined the entire space to be homogeneous, it is thereby itself also a "spherical space"; so let the generating metric and the segments thereby measured be designated by M_S, l_S, h_S.

All straight lines in this space have length 2π. The greatest distance that can separate two points is π; the greatest interval between a point and a plane is $\pi/2$; this is therefore also the greatest possible value for h_S. The space itself is certainly unlimited, i.e., any straight segment anywhere can always be extended on both sides, but space is not infinite: its total volume is $2\pi^2$. It is divided into two halves by any plane and, therefore, by E: both the body of the earth and also the total space outside the earth therefore have the volume π^2 (or 400 million km^3).

We now return to our metric M_S. We wish to test what follows from it about the metric relations outside E and, in particular, whether E is confirmed as a plane by measurements interpereted according to M_S. The units of measurement for segments appearing in M_S are already determined. But M_S contains an apparent circularity: the determination of length outside E is made to depend on the height h_S, and this height can itself only be measured if we already have a length. However, the circularity vanishes on closer examination. What is asserted is completely unequivocal, for it says: a physical segment that behaves just like the measure-segment referred to in M_S and which has the length a as measured on E, will have the length x_S when perpendicular, where a is related to x_S via the equation:

$$\int_0^{x_S} \frac{dx}{1-\sin(x)} = a.$$

For, according to our stipulation, every element dx of such a perpendicular rod that is found at a height x, will have the length $dx/(1-\sin(x))$ if brought to earth. Integration yields $a = \tan(x_S) + 1/\cos(x_S) - 1$; from which it follows that $\tan(x_S) = a/2 \times (2+a)/(1+a)$ (where one chooses the first or fourth quadrant depending on whether *a* is positivie or negative, i.e., on whether the segment is directed outward from E or towards the earth's center). The length of a perpendicular segment, whose length is measured when lying on E, is thus uniquely determined.

The distance of a star at the zenith--which, according to M_1, may be arbitrarily large--remains always less than $\pi/2$ according to M_s. The statement of matter of fact to be inferred from astronomical measurements, that one must place many billion iron rods each measuring 1 m on E (thus in our units 1/6,370,000) perpendicularly one upon another in order to reach such a star, is interpreted as follows according to M_s: the rods placed one upon another all undergo a contraction which is very small in the neighborhood of E but becomes ever greater with increasing distance, and thereby simulates a very great "apparent length" of the segments. Thus, for example, the moon has an "apparent" distance of 59.3 from E; but the distance according to M_s is $\pi/2$ - 0.0335. As the distance approaches the maximum value $\pi/2$ (= 10,000 km) according to M_s, the "apparent" distance increases to infinity. An arbitrarily great distance established through astronomical measurements therefore creates no contradiction with the properties of our finite space.

It also follows from M_s that the interval between two measure-points--and consequently every other physical segment behaving precisely like this measure-segment as well--undergoes an *expansion* when brought into the earth's interior instead of a contraction, while the "apparent" length of the segment (namely, the length measured according to M_1) remains unchanged. Thus, for example, the segment between two antipodes, and therefore the diameter of the earth, has the length π instead of the "apparent" length 2.

Hence we are now able to set up a test for whether E is confirmed as a plane via measurements on M_s. According to M_1 it could be shown, e.g., by the following experiments, that E is not a plane, but a curved surface whose concave side is directed towards the earth's interior: any two points on E are, on the one hand, to be connected by the shortest possible line on E and, on the other hand, by a straight tunnel; the tunnel is then always found to be shorter than the connecting line on E. This result, which does not in fact arise from a direct measurement but can be unequivocally inferred from geodetic measurements on the basis of M_1, must be interpereted as follows according to M_s: the tunnel is only apparently shorter--namely, as a consequence of the expansion of rods that are used as standards according to M_1. Calculation shows that, if the metric M_s is used, the tunnel is always longer than a shortest possible line connecting the two points on E. Closer examination also shows that, not only such a tunnel (which is therefore certainly not a straight line according to M_s), but also any other arbitrary line connecting the two points--whether it runs outside or inside the earth--is always longer according to M_s than the shortest connection on E. This latter connecting line is therefore shown to be a straight segment, and, since the same holds for any point on E, E is shown to be a plane; and it is thereby shown that in this respect M_S corresponds precisely to the postulates we have set up for our chosen space.

The objection that the behavior of light-rays (objects emerging on the horizon, the circular shadow of the earth during lunar eclipses, etc.) allows us to unequivocally recognize the curved shape of the earth can be easily answered on our account. For these so-called proofs of course depend on presupposing the straightness of light-rays. We know, however, that the straightness of any physical lines whatever holds only for certain metrics. Now the straightness of light-rays relative to M_1 has certainly been demonstrated by a multidude of facts of experience, even in everyday life. On the basis of these same facts, however, light-rays are not to be conceived as straight relative to M_S, but as curved lines; and, indeed, as closer examination shows, as circles that all go through the "zenith point" Z. The straight lines perpendicular to E according to M_S, which all are also straight and perpendicular to E according to M_1, and which thus point towards the zenith at every point on E, therefore intersect one another according to M_S--not only at the distance $\pi/2$ at the center of the earth, but also at the same distance from E at the "zenith point" Z outside the earth. Z is therefore the antipode of the earth's center. Here too we again see that, relative to M_S, E behaves exactly the same on both sides--towards the interior and exterior of the earth--which is the case for no other surface but a plane. According to M_1 there is only one possible path for a light-ray between any two points, namely, the straight path. On M_S there is a similar uniqueness: for any two points there is only one circle that also goes through Z.

On the metric M_S we must obviously arrive at different natural laws from the customary ones, which are based on M_1.[35] The required alterations are not of equal magnitude in the different areas. Thus, for example, the circular form of light-rays according to M_S allows us to retain the wave theory of light and, in particular, the electromagnetic theory and therewith all laws of optics. We are merely obliged to attribute to so-called empty space, not the refractive index of 1 at all points, but a value dependent on the distance from E: $n = 1/(1 - \sin(h_S))$.

We shall show how, according to the chosen metric M_S, natural laws have to take on a different form from the customary one in the example of the energy principle for mechanics (the "principle of the conservation of living force") as well, since a natural consideration first appears to lead to the result that this fundamental principle cannot be sustained according to the measurements made on M_S. Imagine an apparatus used to fling a small sphere by means of the compressed energy in a spiral spring. The energy of this spring does not change when the apparatus is taken from one place to another. Therefore, wherever the test is made--so physics tells us--it will impart to the sphere the same kinetic energy each time--which is measured, on the customary definition, by the fact that the sphere flies off with the same initial velocity, say

10 m/sec, wherever the appratus is set up. However, if we measure according to M_s we do not find the same velocity everywhere, for a segment that is 10 m long according to M_1 will of course be measured as $10(1 - \sin(x_s))$ according to M_s if it is found at a distance x_s from E: for example, as 9.986 m on the highest mountain on earth, and as only 0.55 cm on the moon. Thus, if the kinetic energy L_s of a mass m_s having velocity v_s is determined as $1/2 m_s v_s^2$, then the kinetic energy of our sphere will be smaller at great heights than on E, even though it absorbs the same compressed energy from the spring. Therefore, the energy principle is not satisfied.

Reflection suggested by this example leads on closer consideration to the result that either the principles of the customary mechanics do not remain valid on M_s or the basic concepts of mechanics must, to some extent, be given other definitions. If we follow the second way then, in place of the magnitudes measured according to M_1: length l, time t, mass m, velocity v, acceleration a, force F, work W, kinetic energy ("living force") L, and potential V, we can define the magnitudes measured according to M_s: l_s, t_s, m_s, etc. in such a way that the defining equations for v_s, a_s, W_s, L_s correspond precisely to the usual ones:

$$v_s = \frac{dl_s}{dt}, \quad a_s = \frac{d^2 l_s}{dt_s^2}, \quad W_s = F_s \times l_s, \quad -\frac{\partial V_s}{\partial l_s} = F_s$$

(where for simplicity we have given only one component of the relevant vectors). However, instead of the definition F = ma we must put

$$F_s = \frac{m_s \times a_s}{(1 - \sin(x_s))^2},$$

and, instead of $L = 1/2 mv^2$:

$$L_s = 1/2 m_s \times \frac{v_s^2}{(1 - \sin(x_s))^2}.$$

By this we insure that if a certain process is observed and measured via both M_1 and M_s then, not only time and mass, but also work, kinetic energy, and potential energy receive the same values in both measurements: i.e., $t_S = t$, $m_S = m$, $W_S = W$, $L_S = L$, $V_S = V$. For length, velocity, acceleration, and force, on the other hand, different values result from the two measurements: $l_S = l(1-\sin(xs))$, $a_S = a(1-\sin(x_S))$, $F_S = F/(1-\sin(x_S))$. The equations $L_S = L$ and $V_S = V$ are especially important; from these it follows that the energy principle of mechanics does remain valid: $L_S + V_S = \text{const.}$, and so does another fundamental principle of mechanics, Hamilton's principle:

$$\int_{t_{S,0}}^{t_{S,1}} \delta(L_S - V_S)dt_S = 0.$$

From these laws it follows that, if the gravitational potential of a mass m_S relative to the earth is found to be

$$V_S = V = -mg/r = -\frac{m_S\, g \times \cos(x_S)}{1 + \sin(x_S)},$$

then all processes influenced by the earth's gravity: projectile motion, pendulum motion, (monthly) motion of the moon, and so on are calculated with the same results as they are observed to have, even though the magnitudes measured in these processes in part deviate considerably from the usual ones measured according to M_1: thus, for example, the moon orbits around the zenith point Z at a distance of 213 km from it and appears as a sphere of diameter 1.92 km.

We now conclude this example. It is clear that the chosen spatial structure is not at all convenient for presenting the facts of experience and will therefore never be chosen seriously. But here we are not dealing with the question of convenience, which will be explored later. Rather, the significance of our example is that it shows the *possibility* in principle of choosing a structure for physical space that is quite different from the customary one, but is equally capable of presenting all facts of experience without contradiction.

We have now considered the *relation of dependence* subsisting between the matters of fact of experience F, the metric stipulation M, and the metrical spatial structure S (namely, the particular sub-species of S_{3m}") from two different points of view. Our earlier example (measurement of the surface **s)** showed how a particular S results from F by means of a choice of M. In the last example (E as plane) we proceeded in the opposite direction: S was chosen, and it then turned out that there was a certain M that brought the matters of fact F into the form of the chosen structure. Putting both procedures together, we can therefore say: F, S, and M stand in a *functional relationship* to one another such that, if two of them are given, the third determination is thereby uniquely given: $S = f_1(M,F)$; $M = f_2(S,F)$. The third case $F = f_3(M,S)$ then also holds: F is uniquely determined by M and S. This, in fact, is the basis on which the spatial matters of fact are presented in scientific theory: it is asserted that for a specific M, physico-spatial forms are ordered in a particular metric structure S according to a certain M; and by this statement the matters of fact of experience F are completely described with respect to spatial relations. Still, this third case is quite essentially different from the others in that, while either S or M may indeed be freely chose, F may not: the matters of fact are uniquely given.[36]

An important question now arises for the scientific procedure of presenting F through S and M. We can indeed *freely choose either S or M*, whereby the other determination in question uniquely results with regard to F. But which of these two paths are we now to follow? And, if we take one of them, from what points of view should the choice of S or M then be made? The second question is easier to answer than the first, even though it has to decide between many possibilities. For it turns out that among the possibilities for both S and M there is always one case that clearly proves to be the simplest. Thus, if we really wished to choose freely one of the two determinations without reference to the other resulting thereby or to the presentation of F that proceeds from this, then there would be no doubt about the decision: for the choice of S the Euclidean spatial structure $S_{ih=}$" would be preferred; for the choice of M the above mentioned metric stipulation M_0 (metal rod with no dependence on temperature or other influences) would be simplest.

Now, do systems of physics appearing historically proceed sometimes from $S_{ih=}$" and sometimes from M_0? No. To be sure, the first choice is customarily made, although this is usually tacit. The latter, on the other hand, is never chosen, now that the matter of fact designated as temperature-expansion relative to M_1 has become known. And, even in those cases where a different spatial structure from $S_{ih=}$" has been either established (Einstein) or conceived as possible (Gauss, Riemann, Helmholtz, Schwarzschild) M_0 has never been used as metric stipulation.

This fact must cause us to wonder whether the correct scientific procedure should consist in pursuing either one of these two paths: free choice of the simplest S or of the simplest M. Bearing in mind that the postulate of simplicity governing the procedure of scientific presentation relates to the total presentation of matters of fact, we recognize that the greatest possible simplicity for those optional determinations that are independent of the matters of fact is to be postulated only to the extent to which we thereby achieve greater simplicity for the resulting structure. The latter is always the final measure: simplicity of the building takes precedence over simplicity of the building process and its tools.

Thus, neither S nor M is to be freely chosen without reference to F, even though the conceptual possibility of such a choice must always be retained. Rather, a middle way, as it were, is to be followed, that proceeds neither from the simplest S nor from the simplest M, but receives its justification only by virtue of its goal, in that it leads to the simplest structure constituted from our current knowledge.

The way to express the dependence of the spatial structure on experience--more precisely, the metrical structure S_{3m}" on the matters of fact F--is therefore as follows: S is not determined by F itself, and so in this sense is not dependent on experience; yet it certainly is so determined if we add to F a goal-directed viewpoint concerning procedure (a teleological and methodological principle), namely that of simplicity. Nor is S determined by this principle alone, but only by it and F together--and, in fact, uniquely, if it is possible to determine the satisfaction of this postulate of simplicity in the individual case according to generally valid rules. If we reckon the postulate of simplicity as belonging to the definition of science, then the relation between S and F might be expressed as follows: in theory S is freely chosen and independent of experience, but determined by F in scientific practice; or better: is to be determined, in order to thereby make clear that a particular S does not already lie implicit in the matters of fact F, but has first *to be set up* on the basis of that postulate.[37]

Since only this middle way leads to the goal, the simpler metric M_0 was rightly abandoned in favor of the less simple M_1 as soon as certain cognitions of matter of fact (temperature-expansion) came to light. Yet, in precisely the same way, a different spatial structure from the Euclidean would have to be introduced, if the total system were to be simplified thereby. Even those of the opinion--still represented by Poincaré, for example--that such a condition will never be realized, will agree to this postulate. According to the present state of knowledge, moreover, the unrealizability of this condition should certainly not be asserted without closer investigation.[38]

The question that such an investigation would have to resolve has been forced upon us by *the general theory of relativity*. We shall not answer it here, but merely present the alternatives clearly. For the concepts clarified in the foregoing discussion put us in a position more precisely to grasp the question of the justification of non-Euclidean geometry in physics that is connected with this theory. The theory itself will not thereby be put to the test; we shall assume it to be correct, in the sense that no facts of experience contradict it. This assumption appears, on the one hand, to be supported by numerous observational confirmations and, on the other, to be useful, since in the treatment of relativity theory by philosophy and theory of science our attention is rightly directed in a higher degree to the wider problem of the form in which the theory is to be presented.[39] For us, the assertions contained therein will only be considered in so far as they touch on spatial relations. In doing so, however, we should again remember that to consider them separately implies a conceptually permissible abstraction from the unitary space-time-structure, but one that is not univocal with respect to the observations themselves (the "matters of fact"), since it only makes sense for a particular conceptual stipulation concerning simultaneity. Yet it is possible in special cases to arrive univocally--i.e., independently of the definition of simultaneity--at assertions about spatial relations: if, namely, we apply the cautionary rules mentioned above in the measurement of surface **s** and limit ourselves to such assertions about long-lasting point-coincidences as are established in the described fashion by several observers. For brevity, we shall in fact simply use the following mode of expression: according to the metric M_n we find such-and-such an interval for the two points A, B, or we establish that the three points A, B, C lie on a straight line. But in so doing it should always be noted that these assertions can always be traced back to point-coincidences (as in measurement of surface **s**) and that the simultaneity of different point-coincidences that occurs here has the sense described earlier.[40]

The two forms (among infinitely many others that are less simple) in which, if the general theory of relativity is assumed, the spatial relations in a gravitational field, e.g., in the neighborhood of the sun, can be presented are now as follows:[41]

(1) We again choose M_1 as our metric, as has always been customary in physics. Observations based on M_1 yield the result that the length of any measuring rod certainly depends on temperature, magnetization, elastic stresses, etc. but not on place or direction in the gravitational field. The measurement of space to be made in the gravitational field corresponding to the measurement we described for **s** (where we again abstract from the non-fundamental and merely technical difficulties due to the small curvature) would yield the result that here the curvature is not everywhere null, but, for example, becomes more and more strongly negative on a plane through the center of the sun as we approach the surface of the sun from outside, and indeed does so with circular symmetry.

(2) Our earlier discussions have shown that one can always find a metric stipulation on the basis of which the matters of fact can be brought into the form of a Euclidean spatial structure $S_{ih=}$". So there must also be a metric stipulation M_E that leads to a Euclidean structure for the behavior of bodies in the gravitational field. However, for this purpose it is necessary to include in the specification of the metric not only, as in M_1, the temperature (T) and other physical state-magnitudes, but also place and direction, more precisely: the distance (r) from the center of the mass (m) generating the gravitational field and the angle φ between the measuring-segment and r. That this dependence on a determination of length (r) in the metric does not lead to a circular argument has been shown above for a similar example (M_S);

precisely the same holds for the determination of the angle φ, as one can easily convince oneself by reducing angular measurement to the measurement of length. Whereas M_1 runs as follows: "these two points A, B of this iron rod shall be measure-points; at temperature T their interval presents the segment $l = l_0(1 + b(T - T_0))$" (where for simplicity we introduce only temperature among the state-magnitudes and restrict ourselves to the simplest form of dependence), M_E takes the form: "these two points A, B of this iron rod shall be measure-points; at temperature T and distance r from mass m at angle φ from r, their interval presents the segment:

$$l = l_0(1 + b(T - T_0))[1 - C((m/r)\cos(\varphi))],$$

where C is a constant (in cm-gr-sec units $C = 3.72 \times 10^{-29}$)." If we measure according to this metric stipulation there is, just as in M_1, an expansion of all rigid bodies on heating and, indeed, one that differs according to their composition; unlike in M_1, however, there is also a contraction of all bodies in the direction of the line connecting the body in question to the center of m (but not at right angles to this direction). Furthermore, this contraction is the same for all bodies at the same distance from m, independent of their composition. If a (very long) rod is established as straight according to M_E, then it does not in general remain straight when it changes place or direction, but acquires a curvature.

We deliberately refrain from discussing the curvature of light-rays here, because it is not possible to measure them subject to the cautionary rules that allow us here to ignore time.

The question as to the spatial form in which the matters of fact should be presented in a gravitational field therefore comes down to a choice between the two metric-stipulations M_1 and M_E. We limit ourselves here to characterizing the situation by way of these alternatives, without wishing to make a decision on this question, which is one not of truth but of scientific convenience. We shall merely point out that the decision brings into play the above mentioned rule of scientific procedure: if possible, numerically identical behavior of the most diverse bodies is to be presented as merely apparent, namely, as a consequence of a corresponding property of that to which this behavior is related--in this case, the metric or spatial structure. Let it also be recalled once more that we have here abstracted space from the total space-time structure; if the decision is to be valid not only for this extract, the spatial relations, but also for the total construction of the structure of natural processes, if can only be so through investigation of the question whether one or the other of the two metrics yields the simpler form for the four-dimensional space-time-structure.

We now briefly *summarize* the results of our investigation of physical space. The three dimensional topological space S_{3t}" is given to us among the matters of fact of experience--not, however, a metrical space. Such a space results only from a metric stipulation, so that either the latter itself or the metrical spatial structure can be freely chosen. The best way to proceed, however, is to choose neither one nor the other, but rather to so determine the metric stipulation and the associated spatial structure that the matters of fact can thereby be presented as simply as possible.

IV The Mutual Relations Among Formal, Intuitive, and Physical Space

In discussing the different types of structure that likewise result for each of the three meanings of space, the relations that exist among these meanings has already become clear at various places, so that we need only to survey the connections briefly once again.

We consider the three propositions:

(1) Multiplying one number by another yields the same result as multiplying the second by the first.

(2) Three groups of any four things comprise precisely as many things as four groups of any three things.

(3) Here are 3 boxes, the number of balls in each is 4; there are 4 boxes with 3 balls in each; so there are just as many balls here as there.

The relation of (1) to (2) and of (2) to (3) is that of a general rule to its *application*, but in a different sense. The former is a limitation of a general conceptual rule to a special case, although one to which generality still belongs with respect to reality; the latter is an application of this limited generality to a particular case in reality, in which no more generality resides. This distinction will be designated by the terms specification (substitution) and subordination (subsumption); since in the first case determinate relational terms are substituted for indeterminate ones, while in the second case experienced reality is subordinated to the determinate rule.[42]

The relations among geometries may now be grasped with the help of these definitions. The relation of specification holds between the theory of formal and that of intuitive space; the relation of subordination holds between the latter and the theory of physical space. The same three-termed relation, which is of fundamental importance for the theory of science, also holds generally between logic (in the sense of the theory of order), the theory of magnitude (not only spatial), and physics. It corresponds (in Husserl's terminology) to the step-wise progression: formal ontology (Leibniz's "*mathesis universalis*"), regional ontology, factual science; and also to the first steps of the scientific pyramid in Ostwald's theory of science. Examples of particular realms of science that stand to one another in such relations are: general theory of relations, general theory of kinship, historical genealogy; general mathetics, mathetic theory of color, physical theory of color (as named and developed by Ostwald); and so too with the geometries.[43]

Corresponding to the relations among the three realms of science are the relations among their objects--and so here among spaces in the three meanings S, S', S". Both the relation of S to S' and that of S' to S" is that of species to individual, but in a different sense. Here also the relations can be designated as specification and subordination, for these two expressions (or else substitution and subsumption) are not only used in the theory of judgement but also (in another, but closely corresponding meaning) in the theory of classes. The relation of S to S' is that of the species of structures with determinate order-properties but undetermined objects to a structure with these same order-properties but determinate objects--viz., intuitively spatial forms. The relation of S' to S" is that of a form of intuition to a structure with this form made up of real objects of experience.[44]

From this it can now also be seen why the different types of S'--especially the different sub-types of S_{3m}'--and the corresponding types of S are constructed. The point and purpose of these constructions lies in S". The spatial relations of experience are to be brought into a consistent structure S", for this the general form S' is constructed first, and for this in turn the still more general conceptual form S. Now, since the different types of S_{3m}" prove to be possible for S", depending on the choice of metric, the corresponding types of S' must also be constructed. As previously explained, these are then generalized to, and at the same time brought together in, the comprehensive structures S_{nm}' or S_{3t}" and finally S_{nt}'. And for these latter structures we construct the formal framework of the corresponding S up to the most general one, S_{nt}.[45]

V The Relations Between Spatial Cognition and Experience

(a) *The Sources of Spatial Cognition*

From the view thus obtained of the three different meanings of space and the corresponding types of spatial structure encountered under these meanings--especially the topological and metrical structures--we can answer the question about the dependence of spatial cognition on experience and, more generally, about the sources of this cognition.[46]

The theory of formal space is an extension of a special domain of the theory of relations; its propositions, just like those of number theory, are derived from the basic laws of deductive logic and are wholly independent of experience.[47]

The case is not so simple for intuitive space. Here the theorems are derived purely conceptually from certain axioms; so the remaining question is simply what the cognition of these axioms is based on. We have distinguished here between axioms in the narrower sense and postulates. The former are the findings of a certain type of "essential insight" (in Husserl's sense) and therefore, like all cognitions from this source, they are not demonstrated by the accumulation of facts of experience and so are not to be called experiential cognitions; yet they are also not independent of all experience, in so far as they are obtained from some or another representative of the type of object in question.[48] The postulates, by contrast, are not cognitions at all, but stipulations advanced in order to obtain a complete global structure, "space," from cognitions which by nature appear limited to an incomplete region. Various possibilities emerged for these extensions to the complete structure. Topological space presents what is common to all of them and is therefore to be viewed as the form of what can be grasped in the essential insight of the spatial. The metrical intuitive spaces, on the other hand, are also still dependent on the choice of those postulates; they therefore lack the property of unconditioned validity that belongs to topological intuitive space, as to all cognitions arising from this source.

Cognition of the structure of physical space is experiential cognition: it is based on the "matters of fact" of experience and is obtained through induction--i.e., through the assembling and processing of facts of experience--and can therefore never itself arrive at unconditioned certainty, but can merely approach ever more closely to this as a limiting value. Cognition of topological space thus arises on the basis of the matters of fact, while its transformation into one of the metrical structures is possible only by the addition of a freely choosable metric.[49]

So far we have intentionally avoided the *Kantian* terminology of a priori and empirical cognitions, and of analytic and synthetic judgements--partly because these terms are not interpreted and applied in the same way by all sides, partly also because the situation in our problem appears capable of sharper expression with the aid of other given definitions. However, in order to clarify our relation to views that employ these concepts--especially to the question of synthetic a priori judgements--let us briefly indicate how the results of our present investigations relate to these concepts. Once again, however, the only question here concerns the axioms of the theory of space, since the theorems are derived from them without help from either intuition or experience.

The axioms governing *formal* space are obviously a priori. They are not synthetic but analytic, since they are derived solely from logical axioms and therefore assert of every concept of a "spatial structure" (in the formal sense) that occurs therein only what is already posited in its definition. The axioms of *intuitive* space are likewise a priori. Following Kant's well-known distinction between "arises from experience" and "begins with experience" this does not indeed mean: comprehensible without experience, but rather: "independent of the aggregate of experience" (Driesch)--and therefore does not contradict the fact that the givenness of experience is required for essential insight, either immediately in perception or mediately in imagination. In these axioms of intuitive space we have before us the synthetic a priori propositions whose existence is asserted by Kant. However, the same does not hold generally for the theorems derived from them, but only in so far as they pertain to topological space; for those that relate to one of the metrical spaces depend not only on the axioms, but also on the postulates on the basis of which the complete structure of intuitive space results. Thus, such theorems depend on determinations that are not a priori cognitions, because they are not cognitions at all but rather stipulations. Hence Kant's assertion is indeed correct, but does not hold for the entire range of those propositions for which he himself asserted it. Finally, the propositions governing *physical* space are likewise synthetic, but certainly not a priori; rather, they are a posteriori in that they rest on induction.

Therefore, apart from the determinations added by means of freely chosen stipulations, the propositions governing formal, physical, and intuitive space are analytic a priori, synthetic a posteriori, and synthetic a priori, respectively. The old controversies between mathematicians, who disputed Kant's assertion, and philosophers, who defended it, were thus obviously unable to reach any result, because the two sides were not talking about the same object. The former had partly formal space in mind (e.g., Couturat) and partly physical space (Riemann, Helmholtz, Poincaré), the latter intuitive space. So both parties were correct and could have been easily reconciled if clarity had prevailed concerning the three different meanings of space.[50]

To return to our own definitions, instead of the a priori/a posteriori distinction, we can express the intellectual (i.e., ground providing) sources of a cognition by a formula, in which I_1, A_1, or F_1 are to mean that the cognition in question rests on essential insight, arbitrary stipulation, or matters of fact of experience and I_0, A_0, or F_0 that the cognition is free from these determinations. Propositions concerning the various types of space then have the following *source-formulas*:

S_{nt}, S_{3t}, (S_{3m}):	$I_1A_0F_0$
S_{3m} (see below):	$I_1A_1F_0$
S_{3t}':	$I_1A_0F_0$
S_{nt}':	$I_1A_0F_0$
S_{3m}':	$I_1A_1F_0$
S_{3t}'':	$I_1A_0F_1$
S_{3m}'':	$I_1A_1F_1$.

I occurs throughout, but is properly "spatial" only in the latter cases; in the first two, on the other hand, it is formal in nature (Husserl: "formal ontology"). S_{3m} is then free from A if this structure is derived from S_{3t} and thereby has uninterrupted connection with the logical axioms, but is not so if its axioms are postulated indendently as formal conditions on a relational structure resting on freely chosen stipulation. We noted above that this latter procedure is customarily applied. That S_{3t}' is free from A, even though constructed with the help of postulates stipulated by free choice, rests on the fact that this structure contains only those spatial determinations that result from each of the various possible stipulations, and is therefore not dependent on the choice of stipulation.

(b) *Space as a Condition of Experience*

According to Kant, space is the condition for the possibility of every (outer) experience as such. Is this true for the spatial determinations of all the structures we have distinguished? To decide this we must consider which spatial determinations are necessarily to be met with in *every* (outer) experience, and thus also when that experience has not yet been brought, on the basis of freely chosen determinations, into a *special* form that goes beyond the necessary form. Now, we have called experience, in so far as it is presented only in the univocal necessary form that contains no arbitrary stipulation whatsoever, "matter of fact." Therefore, only the spatial determinations contained in matters of fact can be conditions for the possibility of experience. And these, as we have seen, are only the topological, but not the projective and above all not the metrical relations.

The transformation of a statement of matter of fact from one metrical space-form into another--e.g., from the Euclidean into one of the non-Euclidean--has been aptly compared to the translation of a proposition from one language into another. Now, just as the genuine sense of the proposition is not its presentation in one of these linguistic forms--for then its presentation in the other languages would have to appear as derivative and less original--but is merely that in the proposition which remains unaltered in translation; so too the sense of the statement of matter of fact is not one of its metrical presentations, but that which is common to all of them (the "invariants of topological transformations")--and that is precisely its presentation in merely topological form.

In treating this question it has often been correctly pointed out that this "transcendental function" of space--the grounding of experience--can be attributed only to a unique space-form, and that therefore the non-Euclidean space-forms could not be considered for this purpose.

From this correct assertion it should not, however, be concluded that therefore only Euclidean space can assume this role. For this space is on a par with the others and possesses as little or as much uniqueness as any of the non-Euclidean spaces--the one, say, with a constant curvature of -20. Rather, the correct inference from our premise can issue only in topological space, for only the latter is both superordinate to these and also completely unique: the matters of fact of experience cannot appear in several different topological forms.

The topological spatial relations that form the condition of the possibility of every object of experience cannot be those of physical space, since the latter is not independent of the matters of fact of experience but rather merely presents the non-necessary, merely actual findings thereof: e.g., this particular physico-spatial structure stands to that one in a particular topological relation (of contact, connection, inclusion, etc.). The determinations of topological intuitive space, in their independence of experience and in the general validity accruing to them in virtue of their cognitive source (and, consequently, also those of formal topological space--that general relational structure of undetermined things of which topological intuitive space forms a particular special case), can alone have this experience-constituting validity.

The much disputed question, whether the three-dimensionality of space belongs among those determinations that are the condition of every object of experience is to be answered in the negative. As we have seen in our construction of intuitive space, it emerges as a finding of intuition that the spatial forms of the realm of intuition have up to three dimensions. However, in the extension of this realm to the global space it turns out that, if a form of k dimensions is present, one can certainly conclude that the global structure to which it belongs has at least k dimensions, but the upper limit to the number of dimensions of the global structure cannot be inferred. From this finding of intuition it thus follows only that the global intuitive space has at least three dimensions. Still less can it be certainly concluded from the cognition of physical space, which possesses no necessity but only experiential probability, or from that of formal space, for which the number of dimensions is obviously not limited, that it is a condition of the possibility of any object of experience to have at most three dimensions. The view that this conclusion can be drawn by arguing that only by the three-dimensionality of spatial forms will the uniqueness of experiential determination be guaranteed is likewise off the mark. Rather, the situation is precisely the reverse: spatial determination becomes equivocal if we allow an upper limit to the number of dimensions, in accordance with the plurality of possibilities for such limits. And, in order to avoid this equivocality, the unlimited number of dimensions has to be postulated as possible, so that arbitrarily many dimensions for objects of experience are consistent with its possibility as such a structure.

It has already been explained more than once, from both mathematical and philosophical points of view, that Kant's contention concerning the significance of space for experience is not shaken by the theory of non-Euclidean spaces, but must be transferred from the three dimensional Euclidean structure, which was alone known to him, to a more general structure. However, to the question what this latter is now to be, the answers are partly indeterminate, in that only isolated characteristics of the three dimensional Euclidean structure are proposed as requiring generalization, and partly contradictory, chiefly because of a failure to distinguish the different meanings of space and insufficient clarity about the conceptual relationship of the space-types themselves--especially the relation of the metrical to the superordinate topological ones. According to the foregoing reflections, the Kantian conception must be accepted. And, indeed, the spatial structure possessing experience-constituting significance (in place of that supposed by Kant) can be precisely specified as topological intuitive space with indefinitely many dimensions (S_{nt}'). We thereby declare, not only the determinations of this structure, but at the same time those of its form of order, S_{nt}, to be conditions of the possibility of any object of experience whatsoever.[51]

NOTES

[1]The treatment of formal space goes back originally to Leibniz ([147], [149], [152]); see also [146] and compare [6], Vol. 2, Chap. I, §11, [22], Part I, Chap. 1. Leibniz considered formal geometry as a special case of his planned "*mathesis universalis*"--a universal theory that presents the formal law of any contentful particular theory ([151]): compare Couturat [30], Chap. VII, Chap. IX; Husserl [118], §60, §69; Cassirer [24], Book 5, Chap. 7, §II, [25], Chap. III, §III. But essential steps towards this goal have only been taken in the last decades, through construction of the formal theory of relations or order by means of a procedure imitating the mathematical; see: Royce [216], §23; Gätschenberger [75], Chap. 11. Here compare the following works and those mentioned in connection with particular points below.

Couturat [31] above all provides a good overview, which reports on the works of Russell [218], Whitehead [268], and others; Cassirer [23] provides a much briefer discussion of the main questions from [31] and [218]. Whitehead and Russell [270] is the most fundamental work on the construction of formal logic and the theories of relations, series, numbers, and magnitudes; Volume 4 on geometry has not yet appeared. This work is apparently much less well known in Germany than the older [218] and [268] on whose preliminary work it is constructed; nevertheless [268] still remains important for its detailed applications to geometry and mechanics, and [218] above all remains important for its discussion of logical principle. [31], [218], and [270] are also relied upon in the first instance in all following paragraphs on formal space.

Hilbert [110] can be considered as a treatment of formal space; it is so conceived in [25], Chap. III, §III, [40], Chap. 1, §1, [260], §13; however, see the discussion also in §II below on intuitive space. On the distinction of the two points of view compare Frege's criticism [73].

See also: Graßmann [84], [85], [86] and the comments in [169], [222], [30], Appendix V (§§1, 2); Riemann [215]; Vahlen [252]; Peano; [193], [194], [194]; Whitehead [269]; Veronese [254]; Wellstein [260], §§8-13; Husserl [118], §69; Schlick [223], §7. See further: [232], [96], Chap. 7, §1, [88], pp. 110-368, [89], [175], §4, [176], [177], [143], [174], [263], [262], pp. 64ff., [219], [55], pp. 8ff., 18ff., [182], pp. 13-40, 127-147, [198], §85, [106], p. 23.

On the reduction of arithmetic and geometry to logic and the theory of relations compare also, however, the following *critical remarks*: Jakowenko [123]; Poincaré [206], Book II, Chap. III-V (and Lindemann's notes to the German translation), [205], Part 1, Chap. 1; Klein [136], Vol. 2, Part 3, Chap. II, §2, #3 ("Modern geometric theory of axioms"); Geißler [80]; Aster [2], Chap. IV, §10.

(For *all* following paragraphs of §I see *also* [31], [218], and [270]!)

[2]On *judgements* see: Couturat [34], [33], §§I-II; Schröder [231], [232], Vol. 2, [233]; Peano [193], §1, [197], §1; Cohn [29], Part I, Chap. II, §3; Mally [163].

[3]For *concepts* our derivation depends on Frege [71], Vol. 1, §3, [69], [70], and also [73]; see also Bauch [4]. Our theory of concepts therefore corresponds to the "*theory of classes*": Couturat [34], [33], §III; Schröder [233]; Peano [197], §1; Mally [163], pp. 3ff.; Royce [216], §19; König [141], Chap. 2, §§11-12; Russell [219], §VII.

Here only one side of the concept, its *extension*, is used. This theory of classes must therefore find its necessary completion either in the already constructed theory of judgement (see above) or also in a special theory of the content of concepts: i.e., a theory of the *determination* of objects. Here compare, e.g.: Husserl [117]; Mally [163], p. 77; Dingler [43]; Gätschenberger [75], Chap. 11; but also Couturat [30], Chap. VIII, §29.

[4]On *relations* see: Schröder [232], Vol. 2; Frege [71], Vol. 1, §4; Couturat [33], §IV; Royce [216], §18; Russell [221], Chap. II, [219], §§VII, VIII; Cassirer [23], §§I, II.1, [25], Chap. II, §II; Cohn [29], Part I, Chap. III, §6; Ostwald [185], §107, [184], pp. 70-95; Dingler [41], pp. 7ff.; Gätschenberger [75], Chap. 7, Chap. 11.

[5]*Set theory* originates with Cantor [20], [21]. Schoenflies [229], [230] presents a comprehensive account; the best textbook is Hausdorff [96]. For particular consideration of logical questions see: Hessenberg [108]; Fraenkel [67]; Couturat [31], pp. 231-240. Compare also: Schoenflies [228]; Klein [136], Vol. 1, Supplement, Chap. II; Voß [256], Chap. 5; Cassirer [23], §III, [25], Chap. II, §4; König [141].

For *critical remarks* see: Natorp [179], Chap. 4, §4; Weyl [263], [265]; Ziehen [275]; Geißler [80], pp. 96ff.; Cohn [29], Part II, Chap. 4, §11; Bergmann [9]. Here, however, compare also Bernstein [11]. On the so-called paradoxes and their solution see Russell [219] and Zermelo [274].

[6]On *number* see: Dedekind [36]; Frege [71], Vol. 1, especially §§38-42, [72], [68]; Schröder [232], Vol. 1; Couturat [33], §IV; Russell [219], §IX; Hausdorff [96], Chap. 4, §§1, 2; Klein [136], Vol. 1, Part One, Chap. I, §3; Weber [257], §§1, 2; Kerry [127], Chap. 3; Voß [256], Chap. 5. See also: Natorp [179], Chap. 3, §2, [178], §22;

Cassirer [25], Chap. III; Cohn [29], Part II, Chap. IV, §9; Driesch [50], Chap. II, §4a.

For *criticism* compare: Husserl [116], Chap. 6; Weyl [263].

For derivation from *proper axioms* rather than from the theory of relations see: Stolz [247], Chap. II, §3; Peano [196], [197], §2; Hilbert [111], [112].

[7]On *series* see: Frege [67a], Part III, [71], Vol. 1, §§43-46; Veronese [254], Introduction, Chap. 2, §2; Kerry [127], Chap. 2; Russell [219], §X. See also: Natorp [179], Chap. 3, §1, [175], §2; Cassirer [25], Chap. II, §II; König [141], Chap. 3, §§1, 2; Royce [216], §20; Ostwald [185], §§111, 112.

[8]On *ordinal numbers* see: Frege [71]; Hausforff [96], Chap. 4, §§1, 2; Kerry [127], Chap. 3. See also: Natorp [179], Chap. 3, §2; Cassirer [25], Chap. II, §§I, II; König [141], Chap. 9, §§1, 2.

For *criticism* of the purely logical derivation of the theory of numbers see: Rickert [211]; Nelson [180], §21; Hartmann [94].

[9]On *continuity* and the *continuum* (*irrational number*) see: Dedekind [35]; Cantor [20], §§9, 10, [21], §11; Frege [71], Vol. 2; Bolzano [14], §38; Du Bois-Raymond [12], §47; Stolz [248], Chap. II, §5, Chap. III, §§6, 7; Kerry [127], Chap. 6; Klein [136], Vol. 1, Part One, Chap. II, §3, and Supplement, Chap. II, §2, [134], pp. 234ff.; Veronese [254], Introduction, Chap. 6, §10; Peano [197], §§3, 94; Voß [256], Chap. 5. See also: Hessenberg [107], §VIII, [108], Part 2, Chap. IX, §§26-29; Hausdorff [96], Chap. 4, §5; Fraenkel [67], §5; Weber [257], §24; Russell [221], Chap. V. See further: Cassirer [23], §II.2, [25], Chap. II, §IV; Herbertz [105], pp. 16ff.; Henry [104], Part II, §4.

For *criticism* of the formal derivation of continuity see: Weyl [265], [266]; Natorp [179], Chap. 4, §§3-6, [178], §26; Driesch [50], Chap. II, §4c; Cohn [29], Part 2, Chap. VI, §15; Isenkrahe [120], pp. 99ff., [121]; Schmied-Kowarzik [225], Part I, Chap. 3; Sigwart [240], §§66, 15; Hankel [92], §§12, 16; Bergmann [9], §§10, 11; Wernicke [262], pp. 66ff.; Henry [104], Part II, §4; Schmitz-Dumont [227], pp. 116ff.

[10]For multi-leveled number-structures see Weierstraß [258]. For *space as a structure of complex numbers* see: Riemann [214]; Hankel [92], §28; Stolz [248], Chap. X, §§1, 2; Burkhardt [19]; Wellstein [260], §12; Couturat [31], Chap. VI.A; Natorp [175], §4, [177]; Hilbert [110], Chap. II. But compare also: Cohn [29], Part II, Chap. V, §13; Stallo [243], Chap. XIV; Wundt [273], Part III, Chap. 3, §4; Geißler [80], pp. 55ff.

[11]For S_{nt} (compare also note 25 below on the concept of topology) see: Riemann [215]; Tietze [250], R. Graßmann [90] (but not H. Graßmann [84]). Here see the *theory of point-sets*: Schoenflies [229], [230]; Hausdorff [96], Chap. 7, §1.

For S_{np} (compare also note 24 below on the concept of projective geometry) see Pieri [201]. For S_{nm} see Hausdorff [96], Chap. 8, §5, Example IV.

[12]For S_{3t} see Enriques [57]. For S_{3p} see: Veblen [253]; Wellstein [260], §15; Whitehead [269]. For S_{3m} see H. Graßmann, jr. [87].

[13]With respect to the *multiple applicability of formal space* to geometrical structures (of intuitive space), Wellstein [260] gives a great number of examples (our example 4 below is taken from him); see also Müller [174],

pp. 11ff. The application of formal space to non-spatial structures does not appear in the literature; it should here (examples 1 and 2) only more forcefully illustrate the complete indeterminateness of the relational terms. Compare also Frege [73] and Korselt [143].

[14]On *intuition, essential insight* see: Husserl [119], especially §§3-5; Aster [2], Chap. 4, §2. For the rest see note 50 below, (1), concerning *synthetic a priori judgements*. See the Introduction to Hilbert [110] on "logical analysis of spatial intuition," but compare also note 1 above *vis à vis* formal space.

On the *contrast with formal space* see: Graßmann [84], Introduction, §A.3, Part 1, Chap. 1, §A.13 ("geometry -- the theory of extension, formal science"); Wernicke [261], pp. 9ff. ("mathematics of extension -- general mathematics as theory of forms"); Riehl [212], Chap. 2, §8 ("spatial manifold -- manifolds in general"); Enriques [57], p. 239 ("intuitive space (visual) -- analytic space"); Stallo [243], Chap. XIX; König [140], Chap. 5, §2; Korselt [143]; Geyser [82], Book 2, Part 2, Chap. 13, §2 ("spatially determinate elements of sensible intuition -- mathematical order system of places in a number-aggregate"); Schlick [223], §29 ("system of intuitive spatial structures -- system of pure judgements and concepts"); Henry [104], Part II, §7; Sellien [238], Part II, Chap. I. On the relations of mathematics to both domains compare Klein [133a].

On the *distintion from empirical intuition* see Nelson [180], §§3, 4, [181].

The "physiological space of sight" is to be distinguished from "intuitive space"--compare, e.g., Sterneck [246]. The "physical space" contrasted with this "space of sight" by Russell ([220], Chap. III) corresponds to the order structure S with particular reference to its application to S' and S''. The same holds for Schlick's ([223], §29) "visual space" -- "physical-objective space." Moreover, our "intuitive space" is not equivalent to the space so-designated by Veronese ([254], Part I, Book 1, Chap. 1, §1), which corresponds rather to our physical space S''.

[15]On the *imprecision of intuition* see: Klein [132], Vol. 37, §4, [133a], p. 85, [134], pp. 7f., 18f., 39ff.; Hölder [113], endnote 64; Wellstein [260], §14; Enriques [58], Chap. 4A, §7; Study [249], Chap. 4; Christiansen [27], p. 31. *On the other hand*, see: Nelson [180], §19, [181], Part II, §§8, 9; Geißler [80], p. 39.

[16]On the *limitedness of the domain of intuition* see: Klein [136], Vol. 2, Part Three, Chap. II, §2.5 ("Significance of non-Euclidean geometry from standpoint of philosophy"), [135]; Pasch [190], §1 (here under "empirical observation" is meant intuition--for, although experience is incorrectly viewed as the cognitive source, physical space is not yet in question); Schmied-Kowarzik [225], Part II, Chap. 1; Voß [256], Chap. 6.

[17]On the *impossibility of definition* see: Wellstein [260], §6; Pasch [190], §1; Couturat [31], Chap. VI; Driesch [50], Chap. II, §6a; Veronese [254], Preface; Mollerup [170], before §1; Wernicke [262], p. 73; Schlick [223], §29, [224], §X. *On the other hand*, see Geißler [80], pp. 18f, 30ff.

[18]On *the axioms* see, in the first place, Hilbert [110]; compare here Frege [73] and Korselt [143]. See also: Euclid [65], [56], pp. 6-14; Pasch [190]; Killing [128a], §1, [128b], pp. 128ff.; Lie [153], especially Part V, §95; Whitehead [269], §§4-6; Schur [235], Introduction; Mollerup [170], §1; Klein [134], p. 15; Veronese [254], Introduction, Chap. 1, §9; Enriques [59], §1; Couturat [31], Chap. VI.C;

Veblen [253], §9; Geißler [79]. See further: Poincaré [204], Chap. III; Cohn [29], Part 2, Chap. V, §§13, 14; Husserl [119], §72; Nelson [180], §§5-7; Hessenberg [106]; Ostwald [185], §153; Wernicke [261], pp. 25ff., [262], pp. 73ff.; Gerstel [81]; Henry [104], Part II, §1.

[19]On the question of the *expansion of the spatial domain* see: Killing [128a], §11, [129], Vol. 1--see especially Chap. 1, §24, Chap. 4, §10; Pasch [190], §1; Kerry [127], Chap. 4; Driesch [50], Chap. II, §§6a, 6b.

[20]On the *validity of Euclidean geometry in the small* see: Riemann [215], §II.2; Killing [128a], §10 . With reference to the physical space of general relativity (see note [39] below) see: Weyl [267]; Cassirer [26], Chap. VI; Reichenbach [209], Chap. III.

[21]On the Riemannian "*measure of curvature*" of space see: Riemann [215], §II; Helmholtz [99]; Killing [128a], §10.

The measure of curvature is very frequently *misunderstood as change of direction*--see, for example: Lotze [160], §136; Pietzker [202]; Kirschmann [130], e.g., Part I, §IV; Riehl [212], Chap. 2, §8; Schmitz-Dumont [227], p. 150; Geißler [80], p. 58; Driesch [49], Chap. B.2; Medicus [164], p. 23n, [165], pp. 12f.; Natorp [179], Chap. 6, §5; Weinstein [259], p. 36; Isenkrahe [120], pp. 32ff., [122], Chap. VII, Chap. XII; Cornelius [29a], Part 2, Chap. III.D. *On this point compare*: Helmholtz [102]; Christiansen [27], p. 139; Hartmann [93], Part II, Book I; Born [17], Chap. VII.7.

[22]On *the metrical space-types* (*non-Euclidean geometry*) see especially: Wellstein [260], §11; Klein [132], [133]; Killing [128a], [129]; Liebmann [154]. See also: Vahlen [252], pp. 237-298; Mollerup [171]; Veronese [254], Part I,

Book 2, Chap. 3, §1; Simon [241]; and compare the schema in Heymans [109], §46. The *most fundamental* treatments are: Helmholtz [99]; Poincaré [204], Chap. III; Russell [217]; Geiringer [78]. For *space-types not treated by us* see: Hausdorff [95]; Killing [129].

On the historical develoment see: Bonola [15], [16]; Engel [56].

See: Euclid [65]; Simon [242]; Lobatchevsky [157], [158]; Bolyai [13]; Gauß [77], [76], pp. 157-268.

The error of supposing that non-Euclidean geometry can only be defined with reference to Euclidean geometry is very widespread; see, for example: Delboeuf [18], Chap. I.2; Kirschmann [130], Part I, Chap. IV; Riehl [212], Chap. 2, §8; Sigwart [240], §67; Geißler [80], pp. 52f., 84; König [140], Chap. 5, §2; Medicus [165], pp. 12f.; Aster [2], Chap. IV, §9; Cornelius [29a], Part 2, Chap. III.D; Driesch [50], Chap. II, §6b. See *on the other hand*: Russell [217], §97; Wellstein [260], §15; Voß [256], Chap. 6; Christiansen [27], pp. 138f.

[23]On *homogeneity and isotropy*, congruence-spaces (*constant curvature*) see: Riemann [215], §II.4; Helmholtz [97], §4; Poincaré [204], Chap. III. IV (and Lindemann's note 34 in the German translation). (Some narrower, non-standard meanings of homogeneity are found in Cohn [29], Part 2, Chap. V, §14; Delboeuf [18], Chap. I.2--here distinguished from isogeny = homogeneity in our sense. A wider meaning, holding also for a space of unequal measures of curvature, is found in Reichenbach [209], Chap. III.)

The curious *error of supposing that only Euclidean space is homogeneous* is found very frequently among philosophers: Lotze [160], §137 (here see Russell [217], §95); Pietzker [202], p. 29; König [140], Chap. 2, §2; Natorp [179], Chap. VI, §6; Cassirer [25], Part I, Chap. III, §IV ("The conceptual principles of pure space"), [26], Chap. VI; Driesch [50], Chap. II, §§6b, 3c.

[24]On S_{3p}' and the concept of *projective geometry*: Projective geometry as pure geometry of position goes back to Leibniz's plan for an *Analysis Situs* ([150], [144], pp. 69ff., [148], [149]; compare Couturat [30], Chap. IX and Cassirer [24], Book 5, Chap. 2, §II ("The geometrical characteristic")); here no concept of measure is used, but rather the concepts of line and plane. Therefore projective geometry is to be regarded as the realization of the Leibnizean plan, not topology--which is also frequently called *Analysis Situs* (see below); here compare Couturat [31], Chap. VI.A, [30], Chap. IX, §20. See also: Steiner [245]; Staudt [244]; Reye [210]; Pasch [190], §§1-12; Vahlen [252], pp. 55-169; Killing [129], Vol. 1, Chap. 2, §1, Vol. 2, Chap. 6, §1; Russell [218], Chap. XLV, [217], §102-140; Klein [131], §3; Enriques [59], §17; Veblen [253]; Wellstein [260], §15.

For projective geometry *as generalization of metrical geometry* see: Killing [128a], §65; Wellstein [260], §15; Vahlen [252], p. VI f.; Lindemann, note 24 to the German translation of [204]; Cassirer [25], Chap. III, §II (especially "Geometry and the group theory"). (In these important expositions the generalization with the help of transformation-invariants is carried out only up to projective space; the same procedure constitutes a very appropriate route to topological space.)

[25]On S_{3t}' and the concept of *topology* (also called "*Analysis Situs*," but compare note 24 above) see: Couturat [31], Chap. VI.A; Poincaré [205], Part I, Chap. 3, §2; [206], Chap. I, §II ("Geometry of Position"), [207], Chap. 3, §1. See also: Riemann [214]; Enriques [59], §13; Dehn

[37], [38]; Klein [131], §8, [136], Vol. 2, Part Two, Chap. III.3.

On the *succession of levels--metrical, projective, topological space*--see: Klein [136], Vol. 2, Part Three, Chap. I.1; Enriques [61], §5, [58], Chap. 4A, §27, [59], §12.

[26]On *higher dimensions* see in general (also for history and literature) Segre [237]. On S_{nm}' see: Helmholtz [97], §1; Veronese [254], Part 2, Book 2, Chap. 1, 2; Killing [128a], §60. On S_{np}' see Killing [128a], §§48, 49. On S_{nt}' see: Poincaré [207], Chap. 3, §6; Tietze [250]. (Compare also above under S_{nm}, S_{np}, S_{nt}.)

For attempts to prove the *logical impossibility of higher spatial dimensions* (compare also note 51, (1), below) see: Lotze [160], §§132-135 (here compare Russell [217], §94); Pietzker [202], pp. 64ff., 87f., [203]; Schmitz-Dumont [226], pp. 45f. These derivations mostly contain formal fallacies which are easy to detect if one guards against presuppositions holding only for three dimensional space. A complete misunderstanding of the concept of dimension is found in Kirschmann [130]. Compare Müller [174].

[27]In the first instance see Dingler [40], especially Chap. I, Part II, §4, Chap. III, Part I, §1, Chap. III, Part II, §5, Appendix I, and [47a]. See also: Helmholtz [103]; Clifford [28], Chap. 2; Dittrich [48].

On the *logical construction* of S'' from the elements of sense perception see Russell [221], Chap. III, IV; compare Bergmann [10].

On *physico-spatial relations* see: Helmholtz [99]; Mach [161], Chap. VI, VII, [162], Chap. 22; Enriques [58], Chap. 4A, §6; Einstein [52], §I.

On the *contrast* between *physical* and *pure* (intuitive or formal) space (see also note 45 below: S'' as goal of S and

S') see: Russell [218], §352 ("geometry as the study of actual space -- geometry as a pure *à priori* science"); Couturat [31], Chap. VI.C; Einstein [54] ("practical geometry, a natural science -- pure axiomatic geometry, free creation of the human mind"), [52], §I ("propositions about the relative position of practically rigid bodies -- pure geometry"); Natorp [179], Chap. 6, §8 ("spatial order of the empirical -- pure geometrical space"); Cassirer [25], Part I, Chap. IV, §VI [under "Hertz's system of mechanics"] ("physical space of bodies -- geometrical space of lines and distances," with a reference to Leibniz), [26], Chap. V ("empirical -- pure space"), Chap. VI ("relations of measurement of the empirical -- space of pure intuition"); Medicus [164], pp. 19ff. ("empirical space -- pure form of intuition"); Dingler [40] and other works ("empirical -- logical geometry"); Meinong [166], §17 ("our space of reality and physics -- space of geometry"); Liebmann [155], Part 1, Chap. 2; Enriques [61], §2 ("physical -- intuitive space"); Kleinpeter [138], Chap. IV, §2 ("geometry as theory of the spatial proerties of bodies, part of physics -- geometry as formal science"); Study [249], Chap. V ("natural (concrete) -- abstract geometry"), Chap. IV ("empirical space -- space of our world of representation"); Ostwald [185], §148 ("natural -- mathematical space").

[28]On the *establishing of physical straight lines* see: Dingler [40], Chap. I, Part II, §4; Poincaré [205], Part I, Chap. 3, §1; Einstein [52], §I. See also: Helmholtz [103]; Clifford [28], Chap. 2, §5; Study [249], Chap. VII; Wellstein [260], §13; Born [17], Chap. VII.6; Geiringer [78], Part III.

On the distinction between straightness stipulation and metric stipulation compare also Klein [132], Vol. 37, §IV.

[29]Concerning *metric stipulation*: The possibility of free choice is not recognized by: Russell [217], §74, [220], Chap. III; Hölder [113], pp. 5, 30; L. Poincaré [208], Chap. 2, §2; Aster [2], Chap. IV, §6.

Metric stipulation may appeal only to a *point-pair* (compare Einstein [52], §I; Schlick [223], §31), and *not* to a *rigid body,* as is customary--see, for example: Helmholtz [97]; Poincaré [204], Chap. IV, [207], Chap. 2; Natorp [179], Chap. VI, §8; Dingler [42], Part II, Chap. 2, §§5, 6, Part III, Chap. 1, §3, [40], Chap. I, Part II, §6, Chap. III, Part I, §1, [45], [46], [47a]; Wellstein [260], §14; Ostwald [185], §148; Wien [271], Lecture 2; Schlick [224], §V.

On metric stipulation with dependence on place and time ("reference mollusc") see: Einstein [52], §XXVIII; Born [17], Chap. VII.7.

[30]Concerning the concept of "*matter of fact*": On the circumstance that *only spatio-temporal encounter* ("coincidence") can be physically *established* see: Einstein [52], §XXVII, [51], §3; Born [17], Chap. VII.6-7; Petzoldt [200], §35; Schlick [223], §31, [224], §VII; Cassirer [26], Chap. V. On the idea that therefore *only topological determinations are unique* see: Poincaré [205], Part I, Chap. 3, §2, [207], Chap. 3, §1; Schlick [224], §§IV, VII.

[31]Concerning *establishing the physical spatial structure through experiments*, measurement of the *surface* ***s***, and the establishing of curvature see: Helmholtz [103]; Einstein [52], §XXIV; Born [17], Chap. VII.4-5. Dingler [40], Chap. III, Part I, §3 is only apparently in contradiction to this--compare the "manual construction of geometry," Chap. III, Part II, §5 and [46], §3, and see below: from F *and* M, S results uniquely. Precisely the same holds for Poincaré [204], Chap. V: Poincaré's objection, that space could

equally well be found to be Euclidean or non-Euclidean, is correct; this depends precisely on the choice of metric stipulation. By the way, however, without such a stipulation the given point-coincidences still determine a topological physical space--even if they do not determine a metric space.

[32]Through our presentation of experiments the conception *that such experiments always already presuppose the result*, or that they must necessarily turn out as Euclidean, is *contradicted*; see: Müller [172], §83; Weinstein [259], Chap. V, §4; Hönigswald [115a], Part II, §9; Natorp [179], Chap. VI, §5; Cornelius [29a], Part 2, Chap III.D. This conception is correctly disputed by Study [249], Chap. IX and Medicus [164], p. 35f.

[33]On *actual execution of* (astronomical) *experiments* see especially Schwarzschild [236]. See also: Engel [56], p. 216; Enriques [58], Chap. 4A, §10; Study [249], Chap. VII; Poincaré [204], Chap. V.3 (and Lindemann's note to the German translation).

[34]Concerning *the choice of space-type, determination of the metric stipulation belonging thereto*: This procedure has been universally customary in physics so far, and, in fact, the choice of Euclidean space. However, it has only been articulated in detail by Dingler ([40], Chap. III, Part I, §1, [46], [47a], Part I, Chap. 2).

Our example, the earth as plance, is not to be confused with the example of a non-Euclidean world of Poincaré [204], Chap. IV, which is based on an imagined physical experience rather than on our actual physical experience--precisely as in the case of Helmholtz's example in [99] (and similarly following him repeatedly in the literature).

The presented possibility, of conceiving the earth as place according to alternative metric stipulation, has naturally noting to do with Barthel's theory of *The Earth as Total-plane* (Leipzig, 1914), which rests entirely on unscientific speculation.

[35]Concerning alteration of the laws of nature on the basis of an alternative space-form see: Helmholtz [99]; Killing [128]; Schlick [224], §§III-V.

[36]Concerning *the functional relationship between space-type, metric stipulation, and matter of fact*:

(1) *S and F are not uniquely coordinated to one another*: Helmholtz [99]; Wellstein [260], §14; Bauch [3], Chap. III ("Experience and geometry . . ."), §II; Natorp [179], Chap. VII, §§5, 7; Cassirer [25], Part I, Chap. III, §IV, [26], Chap. VI; Poincaré [204], Chap. V; Dingler [42], Part II, Chap. 2, §5, Part II, Chap. 1, §3, Part III, Chap. 1, §3, [46], pp. 119ff., 128, [47a], Part I; Nelson [180], §§14, 15; Hölder [113], endnote 64; Wien [271], Lecture 2; Kleinpeter [138], Chap. IV, §5; Becher [7], Chap. XI, §5; Petzoldt [199], [200], §§24, 35; Aster [2], Chap. IV, §9; Schlick [223], §38, [224] §§III-IV; Geiringer [78], Part III. But it is frequently overlooked (although not by Helmholtz, Wellstein, Petzoldt, Dingler, Schlick, Geiringer) that *if a metric stipulation is set up*--and this is always the case tacitly in physics--this *coordination **is*** unique: see (2) below.

The *ambiguity does not follow only from the* necessary *imprecision* of measurement, as it sometimes appears to be conceived, for example, by: Killing [128a], §10, [129], Vol. 1, Chap. I, §8; Russell [217], §141; [218], §353, [220], Chap. XIV; Hausdorff [95], Part II, Book 1; Wellstein [260], §11; Natorp [179], Chap. VI, §7; Henry [104], Part II, §5.

On the idea that, in general, it is not possible to recognize any bridge between non-Euclidean geometry ("chimera") and experience see: Pietzker [202]; Kirschmann [130]; Sigwart [240], §67; Geißler [80]; Driesch [49], Chap. B.2, Chap. B.3, Chap. C.2i, Appendix 17; König [140], Chap. 5, §2; Herbertz [105], pp. 29ff.; Cornelius [29a], Part 2, Chap. III.D; Wundt [273], Part 3, Chap. 3, §2b (but here compare: Killing [129], Vol. 2, Chap. 7, §6; Voß [256], Chap. 6).

(2) But *each of the three determinations follows uniquely from the other two*:

(a) *M from S and F*: Dingler [42], Part II, Chap. 2, §§5, 6; [40], Chap. III, Part I, §1, [45], [46], §III, [47a], especially Part I, Chap. 2.

(b) *S from F and M*: Dingler [40], Chap. III, Part I, §3; [46], §III (however, since M is determined from (Euclidean) S and F, no other S but Euclidean space results--but the other possibilities also subsist: [46], §III, [47a], Part IV); Helmholtz [99]; Einstein [52], §I. [54]; Schlick [224], §§III-V, VII.

[37]On the *postulate of simplicity of the total presentation* see: Volkmann [255], p. 407; Schlick [223], §38, [224], §§IV, X; Cassirer [26], Chap. VI--*not the simplicity of the first stipulations* (S or M), as Dingler proceeds (choice of simplest S): [42], Part II, Chap. 2, §5, Part III, Chap. 1, §1, [40], Chap. I, Part II, §4, Chap. III, Part II, §4, [44], §7b, [45], [46], §III, [47a].

[38]That the *Euclidean space-type must be given up if necessary* on the basis of the postulate of simplicity is *conditionally* admitted by: Poincaré [204], Chap. IV, [207], Chap. 2; Cassirer [25], Part I, Chap. III, §IV; Becher [7], Chap. XI, §5; Wien [271], Lecture 2; Wellstein [260], §14; Aster [2], Chap. IV, §9; Hönigswald [115a], Part II, §9;

Bauch [3], Chap. 3 ("Experience and geometry . . ."), §IV (here not as a case of greater simplicity but as sole possibility). Heymans [109], on the other hand, incomprehensibly proceeds--after explicit discussion of the works of Riemann and Helmholtz(!)--always from the presupposition that "our" space is Euclidean: one never has had need of measurements (§47), the probability for the non-Euclidean case is infinitely small (§59).

For the idea that the abandonment of Euclidean space does not lead to choice of a determinate non-Euclidean space but rather to ascent to a *more general structure* see: Einstein [52], §XXVII; Freundlich [74], §5b; Born [17], Chap. VII.6-7; Geiringer [78], Part III.

[39]For the *example: general relativity (space in relativity theory)* see in the first instance Weyl [264]. See further: Einstein [52], [53]; Freundlich [74], §5a; Born [17], Chap. VII; Sellien [238]; Geiringer [78], Part III. For epistemological discussions see: Cassirer [26], Chap. IV; Petzoldt [200]; Reichenbach [209]; Einstein [54]; Schlick [224]; Haas [91]. Compare also: Riemann [215], §III, and Weyl's remarks thereto [267], §6.

[40]On the *separate consideration of space* (without time) subject to particular limitations see: Minkowski [168]; Einstein [51], §3, [52], §§VIII, IX, XII; Cassirer [26], Chap. V. This was already noted earlier by Czolbe [34a], Chap. 7 ("Time as the Fourth Dimension of Space") and Palágyi [188], §§1-3. On the possibility of a separate consideration see Weyl [264], §§22, 29.

[41]On the *spatial relations in a gravitational field* see: Weyl [267], §29; Born [17], Chap. VII.9-10; Freundlich [74], §5a; Reichenbach [209], Chap. III. Compare the remarkable presentiment by Clifford [28], Chap. 4, §19. These are *non-*

Euclidean on the basis of M_1: Einstein [51], §22, [54]; Weyl [264], §§31, 32; Born [17], Chap. VII.6,9; Flamm [66]. On the in principle *possibility of retaining Euclidean space* see: Born [17], Chap. VII.6; Schlick [223], §38, [224], §§V, VII; Einstein [54]. Reichenbach emphasis on the impossibility of this ([209], Chap. I, III, VIII) is thoroughly in agreement, since here, as is customary in physics, M_1 is always tacitly presupposed as metric stipulation. For considerations in favor of the retention of Euclidean space see: Dingler [42], Part III, Chap. 1, §1, [46], §III, [47], [47a]--but compare note 37 above.

[42]On the relation of "specification" see Ostwald [183], p. 19 ("coordination"); for the contrary relation see Dingler [47a], Part II, Chap. 1 ("logical delineation"). On the contrast between "specification" and "subordination" see Husserl [119], §13 ("deformalization or filling out -- subsumption"). On "subordination" compare Bauch [3], Chap. I ("Relation between philosophy and natural science"), §IV on the necessary presupposition of a "subsumption-universal" for induction, and Bauch [4], §V on the concept as condition of possibility of the concrete.

[43]On *the theories of S, S', and S''* as cases of the general scientific relationships: "*formal ontology, regional ontology, factual science*" see: Husserl [119], §§15, 16, 72, [118], §60 (reference to Lebniz's *mathesis universalis*, compare [151] and also [146]), §70 (but here the erroneous conception--which, however, is not essential to the process of thought of the indicated exposition--is to be contested that "our space of the world of appearance," and thus S'', is to be viewed unconditionally as Euclidean); Driesch [50] ("general theory of order -- theory of natures (including theory of space as a determinate order-particularity) -- theory of the order of natural

actualities"); Ostwald [186], [183] (the three first steps of the scientific pyramid), [187], Chap. 2. (application to theory of color); Kleinpeter [138], Chap. IV, §2 ("operative sciences (combinatorics, arithmetic, logic) -- geometry -- empirical sciences (physics, etc.)"); Cohn [29], Part 3, Chap. VIII, §22 ("pure constructive -- general reconstructive -- particular reconstructive sciences").

[44]Our meanings of space correspond approximately to those given by: Russell [217], §140 ("complex of relations -- intuitive space -- actually given space"); Müller [172], §88 ("space of mathematics -- psychological space (=S'?) -- empirical space"); Pasch [191], p. 185 ("hypothetical geometry -- (geometry of) mathematical points -- (geometry of) physical points"); Enriques [59], §1 ("abstract space -- customary, intuitive space -- physical space"). On the other hand, they do not correspond to those given by König [140] ("intuitive -- geometrical -- physical space"): the first and third are not treated by us, the second is S'; nor to those given by Hausdorff [95] ("mathematical, empirical, absolute space"): the second is not treated by us, the third can as such be no object of experience, and the first comprises all our three meanings--compare p. 6: "in three relations therefore--in thought, in experience, in intuition--we have full scope and freedom of choice among numberless forms of mathematical spaces."

[45]On the idea that *the purpose of constucting S and S' lies in S''(order of experience in spatial respect)* see: Poincaré [205], Part I, Chap. 4, §6, [207], Chap. 3, §4; Cassirer [23], §VI, [25], Part I, Chap. IV, §VI (under "Hertz's system of mechanics") and other places, [26], Chap. V, VI; Kneser [139], p. 13; Hausdorff [95], p. 4; Wellstein [260], §15; Schlick [223], §7. The purest expression of this relationship appears to be the Kantian conception of S'

(perhaps also of S, where a few declarations can be indicated: cf. Bauch [5], pp. 178, 182n) as synthetic lawfulness of the order of experience and thus of S''. Compare Bauch [5], Chap. II, Part III; Natorp [178], §5, §29; Christiansen [27], pp. 140f. On the significance of this lawfulness as *function* see Cassirer [25] and Bauch [4].

[46]For *the distinction between sources of cognition* in the sense of logical grounds of justification and (psychological) *generation* see: Kant [125], B1 ("arising from -- beginning with"); Bauch [3], Chap. 2 ("Problem of general experience"), §V ("grounding -- descending from"); Meinong [166], §14 ("legitimation -- origin").

[47]On the derivation of *S from principles of logic, independently of experience* see the references cited in §I above, especially Couturat [31].

[48]On the derivation of *S' from essential insight, independently of experience* see the references cited in §II above, especially Husserl [119], §§3-5.

[49]On the derivation of *S'' from induction, as empirical cognition* see: Lobatchevsky [158]; Riemann [215]; Kleinpeter [137], p. 44, [138], Chap. IV, §5; Study [249], Chap. VII; Enriques [58], Chap. 4A, §6; Medicus [164], pp. 34ff. Here, however, the required metric stipulation is either left out of account or, as is customary in physics, only tacitly presupposed. On the other hand, since without establishing a metric stipulation S'' (as metrical structure) cannot be determined through experience, the most often represented conception of the independence from experience of S'' is correct in certain respects: here see the references cited in note 36 under (1) ("S and F are not uniquely . . .") and, further, König [140], Chap. 5, §2.

The decision between the two conflicting conceptions therefore depends on the circumstance--which is usually not discussed--of whether a metric stipulation is or is not presupposed.

[50]On the *contradictory conceptions concering the sources of spatial cognition as a consequence of the diversity of meanings of space:*

(1) *With reference to S' and Kant's "synthetic a priori judgements"* see Kant [125] (Transcendental Aesthetic, Axioms of Intuition), [126], §2c.2, §§6-13; compare Bauch [5], Chap. II, Part III and Cassirer [24], Book 8, Chap. 2, §III. See also: Bauch [3], Chap. 2 ("Experience and geometry . . ."), §II (including an explicit distinction between S' and S''); Heymans [109], §40 (on S: p. 164); Husserl [119], §16; König [140], Chap. 5, §2; Nelson [180], §11; Natorp [179], Chap. 6, §§5, 7, 8; Gerstel [81], pp. 108ff.; Kirschmann [130]; Tobias [251], pp. 38-77; Sigwart [240], §67; Hönigwald [114]; Aster [2], Chap. IV, §§4, 7--most explicit opposition to Gauß, Riemann, Helmholtz.

(2) *In opposition to Kant:* (a) *In reference to S* see: Russell [217], §§58, 140, [218], Chap. LII; Couturat [32], ("The Geometrical Judgement"), [31], Chap. VI.C; Poincaré [204], Chap. III ("On the Nature of Axioms"); Wellstein [260], §13; Driesch [50], Chap. II, §6d; Petzoldt [198], §85; Bergmann [8], Müller [173], p. 343; Schlick [223], §38.

(b) *In reference to S''* see: Gauß [76], p. 177, [56], p. 227 (Paragraph 3 of a Letter from Gauss to Bessel 9 April 1830); Helmholtz [99], [101], [103] (but it is an error to suppose that S' is otherwise not possible); Wellstein [260], §§13, 14; Kleinpeter [137], p. 42; Mach [162], Chap. 22, §§1-3; Erdmann [64]; Bonola [15], §43; Study [249], Chap. IX; Born [17], Chap. VII.5-6; Geiringer [78], Part III.

---

[51]*Kant's theory* of the transcendental-logical significance of *space as condition of the possibility of experience* is in fact *not overthrown* by the development of geometry (Natorp [179], Chap. 6, §7; Nelson [180], §11; Sellien [238], p. 56; Helmholtz [100], [103]; Müller [172], §88), but *is to be transferred* from three dimensional Euclidean space *to a more general structure.*

The following *do not belong* to the characteristics of that spatial structure which presents the *spatial lawfulness constituting the objects of experience:*

(1) *Three-dimensionality*--see: Russell [217], §159; Poincaré [204], Chap. IV ("Visual space"), [205], Part I, Chap. 3, §3, Part I, Chap. 4, §5, [206], Book II, Chap. I, §§IV, V, [207], Chap. 3, §§4, 6; Medicus [164], pp. 14f., 25; Simon [241], pp. 26ff.; Aster [2], Chap. IV, §9; Isenkrahe [122], Chap. VII; Dingler [47a], Part I, Chap. 2.

On the other hand, the following require three-dimensionality: Kant [124] (in §9 he rejects an attempted proof of Leibniz, but he attempts his own derivation in §10); Helmholtz [99]; Kirschmann [130], Part II, §6; Schmitz-Dumont [227], p. 149; Killing [129], Vol. 1, Chap. 3, §15; Liebmann [155], Part I, Chap. 3; Riehl [212], Chap. 2, §4; Wundt [273], Part 3, Chap. 3, §2; Natorp [179], Chap. 6, §6, [175], pp. 383f., [178], §32, [177], pp. 7f. (this detailed derivation contains a formal mistake); Couturat [32], "The Axioms of Geometry"; Schultz [234], p. 29; Herbertz [105], pp. 35f; Driesch [49], Chap. B.2; Geißler [80], p. 135.

(2) *Euclidean constitution* ("planeness")--see: Helmholtz [99]; Russell [217], §58; Poincaré [204], Chap. III, IV; Wellstein [260], §14; Medicus [164], p. 15; Christiansen [27], p. 138.

On the other hand, the following retain Euclidan space: Kirschmann [130]; Schmitz-Dumont [227], pp. 148ff.; Sigwart [240], §67; Liebmann [155], Part 1, Chap. 3; Geißler [80];

Hönigswald [114], p. 891 (but compare [115a], Part II, §9); Wundt [273], Part 3, Chap. 3, §2; Driesch [49], Chap. B.3; Bauch [3], Chap. 3 ("Experience and geometry . . "), §IV; Natorp [179], Chap. 6, §§6, 7 (against Natorp: Müller [172], §86); Schultz [234], pp. 26f.; Meinong [166], §16; Sellien [238], Part III, Chap. 1. A reason often adduced for this is that only the laws of Euclidean space are independent of an absolute length--see: Kirschmann [130], Part I, §IV; König [140], Chap. 5, §2, and note 19 pertaining thereto; Geißler [80], p. 54; Cohn [29], Part 2, Chap. V, §14; Gerstel [81], p. 110; Cornelius [29a], Part 2, Chap. III.D; and compare Aster [2], Chap. IV, §§7, 8.--Against this see: Russell [217], §§98-101; Müller [172], §86; Dittrich [48].

Euclidean constitution cannot be inferred from the requirement of *uniqueness*, as the following maintain: Pietzker [202], p. 6; Natorp [179], Chap. 6, §§6, 7, 8; Bauch [3], *loc. cit.*--Against this see Cassirer [26], Chap. VI. Still less can such be inferred from the requirement of *homogeneity* (compare note 23 above); moreover, this requirement is itself doubtful: see (3) below.

(3) *Constant curvature* (homogeneity and isotropy)--see: Medicus [164], pp. 17ff.; Hausdorff [95], p. 10; Delboeuf [18], §I.2; Hartmann [93], Part II, Book 1. (Compare also note 39 above: relativity theory.)

On the other hand, the following require homogeneity: Riehl [212], Chap. 2, §§4, 8; Russell [217], §130, §§143-145; Aster [2], Chap. IV, §8; Henry [104], Part II, §6.

(4) *Metrical properties in general*--see: Poincaré [205], Part I, §2; Cassirer [25], Part I, Chap. III, §II ("The concept of space and the concept of order")--but from this exposition it follows with even greater justice that topology, rather than projective geometry, is to be declared the "universal *a priori* science of space" (see note 24 above on the concept of projective geometry).

From the negative determinations (1) - (4) it follows: *that the experience-constituting spatial structure* is S_{nt}',

by which the transcendental significance of its formal lawfulness is also given, namely to S_{nt}.

REFERENCES

1. *Abhandlungen der Fries'schen Schule*, N. F., Bd. I. 1904-1906. See [107], [108], [180].
2. Aster, E. v. *Principien der Erkenntnislehre: Versuch einer Neubegründung des Nominalismus.* 1913.
3. Bauch, B. *Studien zur Philosophie der exakten Wissenschaften.* 1911.
4. Bauch, B. "Über den Begriff des Naturgesetzes," *Kantstudien* 29 (1914): 303-337.
5. Bauch, B. *Immanuel Kant.* 1917.
6. Baumann, J. *Die Lehren von Raum, Zeit und Mathematik in der neueren Philosophie nach ihrem ganzen Einfluß dargestellt und beurteilt.* 2 Bde. 1868, 1869.
7. Becher, E. *Naturphilosophie. = Kultur der Gegenwart* III, VII, 1. 1914.
8. Bergmann, H. "Über den analytischen Charakter des Existenztheorems in der reinen Mathematik," *Ann. d. Naturphil.* 8 (1909): 495-502.
9. Bergmann, H. *Das Unendliche und die Zahl*. 1913.
10. Bergmann, H. "Bertrand Russells 'Erkenntnis der Außenwelt'," *Kantstudien* 25 (1920): 50-56.
11. Bernstein, F. "Die Mengenlehre Cantors und der Finitismus," *Jahresb. d. Math.-Ver.* 28 (1919): 63-78.
12. Du Bois-Raymond, P. *Allgemeine Funktionentheorie*. 1882.
13. Bolyai de Bolya, J. *Appendix. Scientiam spatii absolute veram exhibens* (in the *Tentamen* of F. Bolyai). 1832. Translated by G. Halsted, *The Science Absolute of Space*, as a supplement to the English version of [15].
14. Bolzano, B. *Paradoxien des Unendlichen*. Aus der Nachlaß hrsg. von F. Prihonsky. (1850) 1889. Translated by D. Steele, *Paradoxes of the Infinite*. 1950.
15. Bonola, R. *La Geometria non-Euclidea: Esposizione Storico-critica del suo Svilluppo.* (1906). Translated by H. Corslaw, *Non-Euclidean Geometry.*1955. German by H. Liebmann. 1908.
16. Bonola, R. "Sulla teoria delle parallele e sulla geometria non-euclidee," in [60].
17. Born, M. *Die Relativitätstheorie Einsteins und ihre physikalischen Grundlagen. Gemeinverständlich dargestellt*. Naturw. Mongr. u. Lehrb. Bd. III, 1920. Revised

edition in collaboration with G. Liebfried and W. Biem, *Einstein's Theory of Relativity*. 1962.

18. Brulez, L. "Delboeufs Bedeutung für die Logik," *Kantstudien* 24 (1920): 52-106.
19. Burkhardt, H. "Beiträge zu den Untersuchungen über die Grundlagen der Geometrie," *Göttinger Nachr.,* Math.-phys. Kl. (1895): 114-118.
20. Cantor, G. "Über unendliche, lineare Punktmannigfaltigkeiten. Teil 5," *Math. Ann.* 21 (1883): 545-591.
21. Cantor, G. ""Beiträge zur Begründung der transfiniten Mengenlehre," *Math. Ann.* 46 (1895): 481-512; 49 (1897): 207-246. Translated by P. Jourdain, *Contributions to the Founding of the Theory of Transfinite Numbers*. 1952.
22. Cassirer, E. *Leibniz' System in seinen wissenschaftlichen Grundlagen.* 1902.
23. Cassirer, E. "Kant und die moderne Mathematik. Mit Bezug auf Russells und Couturats Werke über die Prinzipien der Mathematik," *Kantstudien* 12 (1907): 1-49.
24. Cassirer, E. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit.* Bd. I (1906) 1910, Bd. II 1907.
25. Cassirer, E. *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der Erkenntniskritik.* 1910. Translated by W. Swabey and M. Swabey, *Substance and Function*. 1923.
26. Cassirer, E. *Zur Einsteinschen Relativitätstheorie. Erkenntnistheoretische Betrachtungen.* 1921. Translated in [25], *Einstein's Theory of Relativity Considered from the Epistemological Standpoint*.
27. Christiansen, B. *Kritik der Kantischen Erkenntnislehre.* 1911.
28. Clifford, W. *Common Sense of the Exact Sciences*. 1885. German by H. Kleinpeter. 1913.
29. Cohn, J. *Voraussetzungen und Ziele des Erkennens. Untersuchungen über die Grundfragen der Logik.* 1908.

29a. Cornelius, H. *Transzendentale Systematik. Untersuchungen zur Begründung der Erkenntnistheorie.* 1916.

30. Couturat, L. *La logique de Leibniz d'après des documents inédits*. 1901.
31. Couturat, L. *Les Principes des mathématiques*. 1905. German by C. Siegel. 1908. Compare here [23].
32. Couturat, L. "La Philosophie des mathématiques de Kant," *Rev. d. Métaph. et Moral,* 1904. Separately appearing as an Appendix to [31].

33. Couturat, L. "Les Principes de la logique," German by V. Plinke in [63]. Translated by B. Meyer, "The Principles of Logic," in the English version of [63].

34. Couturat, L. *L'Algèbre de la Logique.* (1905) 1914. Translated by L. Robinsion, *The Algebra of Logic*. 1914.

34a. Czolbe, H. *Grundzüge einer extensionalen Erkenntnistheorie. Ein räumliches Abbild von der Entstehung der sinnlichen Wahrnehmun.* 1875.

35. Dedekind, R. *Stetigkeit und irrationale Zahlen.* (1872) 1912. Translated by W. Beman, *Continuity and Irrational Numbers*, in *Essays on the Theory of Numbers.* 1901.

36. Dedekind, R. *Was sind und was sollen die Zahlen?* (1887) 1918. Translated by W. Beman, *The Nature and Meaning of Numbers*, in *op. cit.*

37. Dehn, M. and Heegaard, P. "Analysis situs," in [62]. 1907.

38. Dehn, M. and Heegaard, P. "Tologogie," in [189], 1910.

39. Delboeuf, J. *Prolégomènes philosophiques de la géomterie et solution des postulats.* 1860.

40. Dingler, H. *Die Grundlagen der angewandten Geometrie. Eine Untersuchung über den Zusammenhang zwischen Theorie und Erfahrung in den exakten Wisenschaften.* 1911.

41. Dingler, H. *Das Prinzip der logischen Unabhängigkeit in der Mathematik, zugleich als Einführung in die Axiomatik.* 1915.

42. Dingler, H. *Die Grandlagen der Physik. Synthetische Prinzipien der mathematischen Naturphilosophie.* 1919.

43. Dingler, H. "Über die axiomatische Grundlegung der Lehre vom Ding," *Jahresber. d. Math.-Ver.* 28 (1919): 138-158.

44. Dingler, H. "Über den Begriff der 'Einfachstheit' in der Methodik der Physik und der exakten Wissenschaften," *ZS. f. Physik* 3 (1920): 425-436.

45. Dingler, H. "Der starre Körper," *Phys. ZS.* 21 (1920): 487-492.

46. Dingler, H. "Ein Grundproblem der modernen Physik," *Ann. d. Naturphil.* 14 (1920): 112-134.

47. Dingler, H. "Kritische Bermerkungen zu den Grundlagen der Relativitätstheorie," *Phys. ZS.* 21 (1920): 668-675.

47a. Dingler, H. *Physik und Hypothese. Versuch einer induktiven Wissenschaftslehre nebst einer kritischer Analyse der Fundamente der Relativitätstheorie*. 1921.

48. Dittrich, E. "Zur Frage nach der Geometrie der Lichtstrahlen und starren Körper," *Ann. d. Naturphil.* 10 (1910): 92-101 (but compare also the criticism by Dingler, *op. cit.*: 437-440).

49. Driesch, H. *Naturbegriffe und Natururteile. Analytische Untersuchungen zur reinen und empirischen Naturwissenschaft.* 1904.

50. Driesch, H. *Ordnungslehre. Ein System des nicht-metaphysischen Teiles der Philosophie. Mit besondere Berücksichtigung der Lehre vom Werden.* 1912.

51. (4) Einstein, A. "Die Grundlage der allgemeinen Relativitätstheorie," *Ann. d. Phys.* 49 (1916): 769-822. Also in [159]. Translated by W. Perrett and G. Jeffery, "The Foundation of the General Theory of Relativity," in the English version of [159].

52. Einstein, A. *Über die spezielle und die allgemeine Relativitätstheorie. Gemeinverständlich.* 1917. Translated by R. Lawson, *Relativity: The Special and General Theory. A Popular Exposition.* 1920.

53. Einstein, A. "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," *Preuss. Ak. Wiss. Sitzungsber.* (1917) pt. 1: 142-152. Also in [159]. Translated by W. Perrett and G. Jeffery, "Cosmological considerations on the General Theory of Relativity," in the English version of [159].

54. Einstein, A. "Geometrie und Erfahrung," *Preuss. Ak. Wiss. Sitzungsber.* (1921), pt. 1: 123-130. Translated by G. Jeffery and W. Perret, "Geometry and Experience," in *Sidelights on Relativity.* 1922.

55. Engel, F. *Der Geschmack in der neuren Mathematik.* 1890.

56. Engel, F. and Stäckel, P. *Die theorie der Parallellinien von Euklid bis Gauß. Eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie.* 1895.

57. Enriques, F. "Sulle ipotesi, che permettono l'introduzione delle coordinate in una varietà a più dimensioni," *Rend. d. Circ. Matem. Palermo* 12 (1896): 222-239.

58. Enriques, F. *Problemi della Scienza*. 1906. Translated by K. Royce, *Problems of Science*. 1914. German by K. Grelling. 1910.

59. Enriques, F. "Prinzipien der Geometrie," in [62].

60. Enriques, F. *Questioni reguardanti le geometria elementari*. 1900. German by H. Thieme 1911.

61. Enriques, F. "Über die philosophische Bedeutung der Fragen, die sich auf die Grundlagen der Geometrie beziehen," in [60].

62. *Enzyklopädie der mathematischen Wissenschaften, mit Einschluß ihrer Anwendungen*. Hrsg. im Auftr. d. Akad, d. Wiss. zu Berlin, Göttingen, Heidelberg, Leibzig, München und Wien. III. Bd.: *Geometrie*. 1907.

63. *Enzyklopädie der philosophischen Wissenchaften.* Hrsg. v. W. Windelband und A Ruge. I. Bd.: *Logik*. 1912. Translated by B. Meyer, *Encylopedia of the Philosophical Sciences*, Vol. I: *Logic.* 1913.

64. Erdmann, B. *Die Axiome der Geometrie. Eine philosophische Untersuchung der Riemann-Helmholtzschen Raumtheorie.* 1877.

65. *Euclidis elementorum libri XV*. Translated by T. Heath, *The Thirteen Books of Euclid's Elements.* 1926. German by J. Lorentz. 1781.

66. Flamm, L. "Beiträge zur Einsteinschen Gravitationstheorie," *Physik. ZS.* 17 (1916): 448-454.

67. Fraenkel, A. *Einleitung in die Mengenlehre. Eine gemeinverständliche Einführung in das Reich der unendlichen Größen.* 1919.

67a. Frege, G. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* 1879. Translated by S. Bauer-Mengelberg, *Begriffsschrift: A Formula Language, modeled upon that of Arithmetic, for Pure Thought,* in J. van Heijenoort, ed. *From Frege to Gödel: A Source Book in Mathematical Logic.* 1976.

68. Frege, G. *Die Grundlagen der Arithemtik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl.* 1884. Translated by J. Austin, *The Foundations of Arithmentic*. 1953.

69. Frege, G. *Funktion und Begriff.* 1891. Translated by P. Geach, "Function and Concept," in B. McGuinness, ed. *Collected Papers on Mathematics, Logic, and Philosophy.* 1984.

70. Frege, G. "Über Begriff und Gegenstand," *Viert. f. wiss. Philos.* 16 (1892): 192-205. Translated by P. Geach, "On Concept and Object," in B. McGuinness, *op. cit.*

71. Frege, G. *Grundgesetze der Arithmetik.* I, II. 1893, 1903. Partially translated by M. Furth, *The Basic Laws of Arithmetic*. 1967.

72. Frege, G. *Über die Zahlen des Herrn H. Schubert.* 1899. Translated by H. Kaal, "On Mr. Schubert's Numbers," in B. McGuinness, *op. cit.*

73. Frege, G. "Über die Grundlagen der Geometrie," *Jahresber. Mathem.-Ver.* 12 (1903): 319-324, 368-375. Translated by E.-H. Kluge, "On the Foundations of Geometry: First Series," in B. McGuinness, *op. cit.*

74. Freundlich, E. *Die Grundlagen der Einsteinschen Gravitationstheorie.* 1916. Translated by H. Brose, *The Foundations of Einstein's Theory of Gravitation*. 1924 (second edition, from the fourth German edition 1924).

75. Gätschenberger, R. *Symbola*. *Anfangsgründe einer Erkenntnistheorie.* 1920.

76. Gauß, C. *Werke*. Hrsg. v. d. Kgl. Ges. D. Wiss. Göttingen. Bd. VIII. 1900.

77. Gauß, C. "Besprechungen und Briefe über die Parallelentheorie. 1799-1856," in [56].

78. Geiringer, H. "Nichteuklidische Geometrie und Raumproblem," *Die Naturwiss.* 6 (1918): 635-641, 653-658.

79. Geißler, K. "Die geometrischen Grundvorstellungen und Grundsätze und ihr Zusammenhang," *Jahresber. d. Math.-Ver.* 12 (1903): 265-288.

80. Geißler, K. *Moderne Verirrungen auf philosophisch-mathematischen Gebieten. Kritische und selbstgebende Untersuchungen.* 1909.

81. Gerstel. A. "Über die Axiome der Geometrie," *Beil. z. 16 Jahresber. d. Philos. Ges. Wien* (1903): 97-111.

82. Geyser, J. *Allgemeine Philosophie des Seins und der Natur.* 1915.

83. Gmeiner, J. See [247], [248].

84. Graßmann, H. *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik.* (1844) 1878. Compare [222].

85. Graßmann, H. *Geometrische Analyse, geknüpft an die von Leibniz erfundene geometrische Charakeristik.* Gekr. Preisschr. Mit einer erläuternden Abhandl. v. A. Möbius. 1847.

86. Graßmann, H. *Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet.* 1862. Partially translated by. M. Kormes, "On the Ausdehnungslehre," in D. Smith, ed. *A Source Book in Mathematics.* 1929.

87. Graßmann, H. jr. "Punktrechnung und projektive Geometrie," *Festschr. u. Progr. d. Latina, Halle.* 1894, 1896, 1898.

88. Graßmann, R. *Die Wissenslehre oder die Philosophie.* 1. Teil, 2. Hälfte: *Die Denklehre.* 1890.

89. Graßmann, R. *Formelbuch der Denklehre.* 1890.

90. Graßmann, R. *Die Ausdehnungslehre oder die Wissenschaft von den extensiven Größen in strenger Formelentwicklung.* (1891) 1904.

91. Haas, A. "Die Physik als geometrische Notwendigkeit," *Die Naturwiss.* 8 (1920): 121-127.

92. Hankel, H. *Vorlesungen über die komplexen Zahlen und ihre Funktionen.* 1. Teil: *Theorie der komplexen Zahlensysteme, insbesondere der gemeinen komplexen Zahlen und der Hamiltonschen Quaternionen, nebst ihrer geometrischer Darstellung.* 1867.

93. Hartmann, E. "A. Einsteins allgemeine Relativitätstheorie," *Philos. Jahrb.* (Gutberlet) 30 (1917): 363-387.

94. Hartmann, N. "Über die Erkennbarkeit des Apriorischen," *Logos* 5 (1915): 290-329.

95. Hausdorff, F. "Das Raumproblem," *Ann. d. Naturphil.* 3 (1904): 1-23.

96. Hausdorff, F. *Grunzüge der Mengenlehre.* 1914. Translated (from the third edition of 1937) by J. Aumann, *et. al.*, *Set Theory*. 1957.

97. Helmholtz, H. v. "Über die Tatsachen, welche der Geometrie zugrunde liegen," *Gött. gel. Nachr.* (1868): 193-221. Translated by M. Lowe, "On the Facts underlying Geometry," in R. Cohen and Y. Elkana, eds. *Epistemological Writings.* 1977.

98. Helmholtz, H. v. *Vorträge und Reden.* Bd. II. 1903.

99. Helmholtz, H. v. "Über den Ursprung und die Bedeutung der geometrischen Axiome" (1870), in [98]. Translated by M. Lowe, "On the Origin and Significance of the Axioms of Geometry," in R. Cohen, *op. cit.*

100. Helmholtz, H. v. "On the Origin and Sense of Geometrical Propositions," *Mind* 3 (1878): 212-224.

101. Helmholtz, H. v. "Die Tatsachen in der Wahrnehmung" (1878), in [98]. Translated by M. Lowe, "The Facts in Perception," in R. Cohen, *op. cit.*

102. Helmholtz, H. v. "Der Raum kann transzendental sein, ohne daß es die Axiome sind," Beilage II zu [101]. Translated by M. Lowe, "Space can be Transcendental without the Axioms being so," Appendix 2 to [101].

103. Helmholtz, H. v. "Die Anwendbarkeit der Axiome auf die physische Welt," Beilage III zu [101]. Translated by M. Lowe, "The Applicability of the Axioms to the Physical World," Appendix 3 to [101].

104. Henry, V. *Das erkenntnistheoretische Raumproblem in seinem gegenwärtigen Stande.* Erg.-H. 34 der *Kantstudien.* 1915.

105. Herbertz, R. *Die Philosophie des Raumes.* 1912.

106. Hessenberg, G. "Über die kritische Mathematik," *Sitz.-Ber. d. Berliner math. Ges.* 3 (1904): 21-28.

107. Hessenberg, G. "Das Unendliche in der Mathematik," in [1] (1904): 135-190.

108. Hessenberg, G. "Grundbegriffe der Mengenlehre," in [1] (1906): 487-706.

109. Heymans, G. *Die Gesetze und Elemente des wissenschaftlichen Denkens. Ein Lehrbuch der Erkenntnistheorie in Grundzügen.* (1890) 1915.

110. Hilbert, D. *Grundlagen der Geometrie.* (1899) 1913. Translated from the tenth edition (1968) by L. Unger, *Foundations of Geometry.* 1971.

111. Hilbert, D. "Über den Zahlbegriff," *JB. d. Math.-Ver.* 8 (1900): 180-194.

112. Hilbert, D. "Über die Grundlagen der Logik und der Arithmetik," *Verh. III. Int. Math. Kongr. Heidelbg 1904.* (1905): 174-185. Translated by B. Woodward, "On the Foundations of Logic and Arithmetic," in J. van Heijenoort, *op. cit.*

113. Hölder, O. *Anschauung und Denken in der Geometrie.* 1900.

114. Hönigswald, R. "Über den Unterschied und die Beziehungen der logischen und der erkenntnistheoretischen Elemente in dem kritischen Problem der Geometrie," in [142].

115. Hönigswald, R. *Zum Streit über die Grundlagen der Mathematik. Eine erkenntnistheoretische Studie.* 1912.

115a. Hönigswald, R. "Naturphilosophie," *Jahrb. d. Philos.* 1 (1913): 60-98.

116. Husserl, E. *Philosophie der Arithemetik. Psychologische und logische Untersuchungen.* 1891.

117. Husserl, E. "Der Folgerungscalcul und die Inhaltslogic," *Virt. f. wiss. Philos.* 25 (1891): 168-189, 351-356.

118. Husserl, E. *Logische Untersuchungen.* I Bd.: *Prolegomena zur reinen Logik.* (1900) 1913. Translated by J. Findlay, *Logical Investigations.* 1970.

119. Husserl, E. *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie.* 1913. Translated by W. Boyce Gibson, *Ideas: General Introduction to Pure Phenomenology*. 1931.

120. Isenkrahe, C. *Das Endliche und das Unendliche. Schärfung beider Begriffe, Erörterung vielfacher Streitfragen und Beweisführungen, in denen die Anwendung finden.* 1915.

121. Isenkrahe, C. *Untersuchungen über das Endliche und das Unendliche mit Ausblick auf die philosophische Apologetik.* 1 Heft. 1920.

122. Isenkrahe, C. *Zur Elementaranalyse der Relativitätstheorie. Einleitung und Vorstufen.* 1921.

123. Jakowenko, B. "Die Logistik und die transzendentale Begründung der Mathematik," in [142].

124. Kant, I. *Gedanken von der wahren Schätzung der lebendigen Kräfte.* 1747.

125. Kant, I. *Kritik der reinen Vernunft.* 1781, 1787. Translated by N. Kemp Smith, *Critique of Pure Reason.* 1929.

126. Kant, I. *Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können.* 1783. Translated by P. Lucas, *Prolegomena to any Future Metaphysics.* 1953.

127. Kerry, B. *System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnistheorie.* 1890.

128. Killing, W. "Die Mechanik in den nichteuklidischen Raumformen," *Jahresb. Gymn. Brilon.* 1883.

128a. Killing, W. *Die nichteuklidischen Raumformen in analytischer Behandlung.* 1885.

128b. Killing, W. "Über die Grundlagen der Geometrie," *Journ. f. r. u. angew. Math.* 109 (1892): 121-186.

129. Killing, W. *Einführung in die Grundlagen der Geometrie.* I, II. 1893, 1898.

130. Kirschmann, A. "Die Dimensionen des Raumes," *Phil. Stud.* (Wundt) 29 = *Festchr. f. W. Wundt,* 1. Teil (1902): 310-417.

131. Klein, F. *Vergleichende Betrachtungen über neure geometrische Forschungen.* 1872.

132. Klein, F. "Über die sogennante nichteuklidische Geometrie," *Math. Ann.* 4 (1871): 573-625, 6 (1873): 112-145, 7 (1874): 531-537, 37 (1890): 544-572.

133. Klein, F. *Nichteuklidische Geometrie.* I, II. 1893.

133a. Klein, F. "Über Arithmetisierung der Mathematik," *Göttinger Nachr.,* gesch. Mitt. (1895): 82-91.

134. Klein, F. *Anwendung der Differential- und Integralrechnung auf Geometrie. Eine Revision der Prinzipien.* 1902.

135. Klein, F. "Zwei Besprechungen über Grenzfragen der Mathematik und Philosophie," in *Philos. Ges. Wien,* 1906.

136. Klein, F. *Elementarmathematik vom höheren Standpunkt aus.* I: *Arithmetik, Algebra, Analysis.* (1908) 1911. II: *Geometrie.* (1908) 1914. Translated by E. Hedrick and C. Noble, *Elementary Mathematics from an Advanced Standpoint.* I: *Arithmetic, Algebra, Analysis.* (1932). II: *Geometry* (1939).

137. Kleinpeter, H. "Die Entwicklung des Raum- und Zeitbegriffes in der neueren Mathematik und Mechanik und seine Bedeutung für die Erkenntnistheorie," *Arch. f. syst. Philos.* 4 (1898): 32-43.

138. Kleinpeter, H. *Die Erkenntnistheorie der Naturforschung der Gegenwart. Unter Zugrundelegung der Anschauungen von Mach, Stallo, Clifford, Kirchhoff, Hertz, Pearson und Ostwald.* 1905.

139. Kneser, A. *Mathematik und Natur.* (1911) 1913.

140. König, E. *Kant und die Naturwissenschaft.* 1907.

141. König, J. *Neue Grundlagen der Logik, Arithmetik und Mengenlehre.* 1914.

142. (Kongreß), *Bericht üb d. III. internat. K. f. Philosophie zu Heidelberg 1908.* Hrsg. v. T. Elsenhaus. 1909.

143. Korselt, A. "Über die Grundlagen der Geometrie," *Jahresber. Math.-Ver.* 12 (1903): 402-407. Translated by E.-H. Kluge, "On the Foundations of Geometry,"

in *Gottlob Frege: On the Foundations of Geometry and Formal Theories of Arithmetic*. 1971.

144. Leibniz, G. *Hauptschriften zur Grundlegung der Philosophie*. Hrsg. v. E. Cassirer. Bd. I (zur Logik und Mathematik). 1903.
145. Leibniz, G. *Mathematische Schriften*. Hrsg. C. Gerhardt. 1849-1863.
146. Leibniz, G. *Opuscles et fragments inédits*. Par L. Couturat. 1903.
147. Leibniz, G. *Dissertatio de arte combinatoria.* 1666. In [145], Vol. 5. Partially translated by L. Loemker, *Dissertation on the Art of Combinations*, in *Philosophical Papers and Correspondence*. 1969.
148. Leibniz, G. *Characteristica geometrica.* 1679. In [145], Vol. 5.
149. Leibniz, G. *Analysis geometrica propria.* 1698. In [145], Vol. 5.
150. Leibniz, G. *De analysi situs.* 1679 In [145], Vol. 5. Translated by L. Loemker, *On Analysis Situs,* in *op. cit.* German in [144].
151. Leibniz, G. *Mathesis universalis.* In [145], Vol. 7.
152. Leibniz, G. *Initia rerum mathematicarum metaphysica.* In [145], Vol. 7. Translated by L. Loemker, *The Metaphysical Foundations of Mathematics*, in *op. cit.*
153. Lie, S. (and Engel, F.) *Theorie der Transformationsgruppen.* Bd. III, Abt. V: *Untersuchungen über die Grundlagen der Geometrie.* 1893.
154. Liebmann, H. *Nichteuklidische Geometrie.* (1905) 1912.
155. Liebmann, O. *Zur Analysis der Wirklichkeit. Eine Erörterung der Grundproblem der Philosophie.* 1911.
156. Lipps, G. "Untersuchungen über die Grundlagen der Mathematik," *Philos. Stud.* (Wundt) 9 (1893): 151, 354, 10 (1894): 169-202, 11 (1895): 254-306, 14 (1898): 157-241.
157. Lobatchevsky, N. *Zwei geometrische Abhandlungen.* Hrsg. v. F. Engel. 1899--containing "Über die Anfangsgründe der Geometrie" (1830), "Neue Anfangsgründe der Geometrie" (1835-1838). Partially translated by G. Halsted, *New Principles of Geometry, with Complete Theory of Parallels.*
158. Lobatchevsky, N. *Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des paralleles.* 1856. Partially translated by. H. Manning, *Pangeometry*, in D. Smith, *op. cit.* German by H. Liebmann (1902) 1912.
159. Lorentz-Einstein-Minkowski. *Das Relativitätsprinzip. Eine Sammlung von Abhandlungen.* 1920. Translated by W. Perrett and G. Jeffrey, *The Principle of Relativity*. 1923.

160. Lotze, H. *Metaphysik. Drei Bücher der Ontologie, Kosmologie und Psychologie.* (1879) 1884. Translated by B. Bosanquet, *Metaphysics*. (1884) 1887.
161. Mach, E. *Die Analyse der Empfindungen und das Verhältniss des Physischen zum Psychischen.* (1886) 1919. Translated by C. Williams, *Contributions to the Analysis of the Sensations.* 1897. Revised and supplemented from the fifth German edition by S. Waterloo, *The Analysis of Sensations.* 1906.
162. Mach, E. *Erkenntnis und Irrtum. Skizzen zur Psychologie der Forschung.* (1905) 1920. Translated by E. Hiebart and J. McCormack, *Knowledge and Error.* 1976.
163. Mally, E. *Gegenstandstheoretische Grundlagen der Logik und Logistik.* 1912.
164. Medicus, F. *Kants transzendentale Aesthetik und die nichteuklidische Geometrie.* Diss. 1898.
165. Medicus, F. "Bemerkungen zum Problem der Existenz mathematischer Gegestände," *Kantstudien* 29 (1914): 1-18.
166. Meinong, A. *Über die Stellung der Gegenstandstheorie im System der Wissenschaften.* 1907.
167. Mill, J. S. *A System of Logic Ratiocinative and Inductive.* 1843. German by J. Schiel. 1862-1863. Abridged edition by E. Nagel, *John Stuart Mill's Philosophy of Scientific Method.* 1950.
168. Minkowski, "Raum und Zeit" (1908), *JB. d. Math.-Ver.* 18 (1909). Also in [159]. Translated by W. Perret and G. Jeffrey, "Space and Time," in the English version of [159].
169. Möbius, A. "Die Graßmannsche Lehre von Punktgrößen und den davon abhängenden Größenformen," in [85].
170. Mollerup, J. "Grundlagen der Elementargeometrie," in [189].
171. Mollerup, J. "Die nichteuklidische Geometrie," in [189].
172. Müller, A. *Das Problem des absoluten Raumes und seine Beziehung zum allgemeinen Ramproblem.* 1911.
173. Müller, A. "Die Fiktion in der Mathematik und der Physik," *Naturwiss.* 5 (1917): 341-347, 362-366.
174. Müller, E. "Über mehrdimensionale Räume," *Beil. z. 17 Jahresber. d. Philos. Ges. Wien* (1904): 1-14.
175. Natorp, P. "Nombre, temps et espace dans leur rapports avec les fonctions primitives de la pensée," *Bibl. d. Congr. Int. d. Philos.*, I: *Philos. générale et Métaphysique* (1900): 343-389.
176. Natorp, P. "Zu den logischen Grundlagen der neueren Mathematik," *Arch. f. syst. Philos.* 7 (1901): 177-209, 372-438.

177. Natorp, P. "Die erkenntnistheoretischen Grundlagen der Mathematik," *Unterr.-Bl. f. Math. u. Nat.* 8 (1902): 2-8.

178. Natorp, P. *Logik (Grundlegung und logischer Aufbau der Mathematik und mathemat. Naturwissenschaft) in Leitsätzen zu akad. Vorles.* (1904) 1910.

179. Natorp, P. *Die logischen Grundlagen der exakten Wissenschaften.* 1910.

180. Nelson, L. "Bemerkungen über die nichteuklidische Geometrie und den Ursprung der mathematischen Gewißheit," in [1], 1905.

181. Nelson, L. "Des fondements de la géometrie" (1914), in *Die Reformation der Philosophie durch die Kritik der Vernunft.* 1918.

181a. Newton, I. *Philosophiae naturalis principia mathematica.* 1686. Translated A. Motte (1729), revised F. Cajori (1934), *Mathematical Principles of Natural Philosophy*. German by J. Wolfers, 1872.

182. Orthner, R. *Entwurf einer Theorie der physikalischen Abhängigkeit.* 1919.

183. Ostwald, W. "Zur Theorie der Wissenschaft," *Ann. d. Naturphil.* 4 (1905): 1-27.

184. Ostwald, W. *Grundriß der Naturphilosophie.* 1908.

185. Ostwald, W. *Moderne Naturphilosophie.* I Bd.: *Die Ordnungswissenschaften.* 1914.

186. Ostwald, W. "Das System der Wissenschaften," Einführung zu: *Die Naturwissenschaften und ihre Anwendungen.* Hrsg. v. C. Thesing, in Bd. I: L. Graetz, *Die Physik.* 1917.

187. Ostwald, W. *Die Farbenlehre.* In fünf Büchern. I: *Mathetische Farbenlehre.* 1918.

188. Palágyi, M. *Neue Theorie des Raumes und der Zeit. Die Grundbegriffe einer Metageometrie.* 1901.

189. Pascal, E. *Repertorio di matematiche superiori.* II: *Geometria.* 1898-1900. German by P. Epstein and H. Timmerding. 1910.

190. Pasch, M. *Vorlesungen über neure Geometrie.* (1882) 1912.

191. Pasch, M. "Grundfragen der Geometrie," *Journ. f. r. u. ang. Math.* 147 (1917):184-190.

192. Pasch, M. *Mathematik und Logik.* 1919. IV: *Der Aufbau der Geometrie.*

193. Peano, G. *Calcolo geometrico secondo l'Ausdehnungslehre di H. Graßmann, preceduto dalle operazioni della logica deduttiva.* 1888. Partially translated by H. Kennedy, "The Geometrical Calculus according to the *Ausdehnungslehre* of H. Graßmann, preceded by the Operations of Deductive Logic," in *Selected Works of Giuseppe Peano.* 1973.

194. Peano, G. *Gli elementi di calcolo geometrico.* 1891. German by A. Schepp, 1891.

195. Peano, G. *I principii di geometria logicamente esposti.* 1889.

196. Peano, G. *Arithmetices principia nova methodo exposita.* 1889. Translated by J. van Heijenoort, "The Principles of Arithmetic, presented by a New Method," in J. van Heijenoort, *op. cit.*

197. Peano, G. *Formulaire de Mathématiques.* II. 1897-1899.

198. Petzoldt, J. *Das Weltproblem vom Standpunkte des relativistischen Positivismus aus.* 1912.

199. Petzoldt, J. "Die Relativitätstheorie im erkenntnistheoretischen Zusammenhang des relativistischen Positivismus," *Verh. Phys. Ges.* 14 (1912): 1055-1064.

200. Petzoldt, J. *Die Stellung der Relativitätstheorie in der geistigen Entwicklung der Menschheit.* 1920.

201. Pieri, M. "I principii della geometria di posizione composti in sistema logico deduttivo," *Mem. Accad. di Torino* 48 (1898).

202. Pietzker, F. *Die Gestaltung des Raumes. Kritische Untersuchungen über die Grundlagen der Geometrie.* 1891.

203. Pietzker, F. "Die dreifache Ausdehnung des Raumes," *Unterr.-Bl. f. Math. u. Nat.* 8 (1902): 39-41.

204. Poincaré, H. *La Science et L'Hypothèse.* 1902. Translated by J. Larmor, *Science and Hypothesis.* 1952. German by F. Lindemann and L. Lindemann. 1906.

205. Poincaré, H. *La Valeur de la Science.* 1906. Translated by G. Halsted, *The Value of Science.* 1958. German by E. Weber (with notes by H. Weber), 1910.

206. Poincaré, H. *Science et Méthode.* 1908. Translated by F. Maitland, *Science and Method.* 1952. German by F. Lindemann and L. Lindemann (with explanatory notes). 1914.

207. Poincaré, H. *Dernières Pensées.* 1913. Translated by J. Bolduc, *Mathematics and Science: Last Essays.* 1963. German by K. Lichtenecker. 1913.

208. Poincaré, L. *La Physique moderne, son Evolution*. 1907. Translated by F. Legge, *The New Physics and Its Evolution.* 1907. German by M. Brahn and B. Brahn, 1908.

209. Reichenbach, H. *Relativitätstheorie und Erkenntnis apriori.* 1920. Translated by M. Reichenbach, *The Theory of Relativity and A Priori Knowledge.* 1960.

210. Reye, T. *Die Geometrie der Lage* (1866, 1867), 3 Tle. 1886-1892.

211. Rickert, H. "Das Eine, die Einheit und die Eins. Bermerkungen zur Logik des Zahlbegriffs," *Logos* 2 (1912): 26-78.

212. Riehl, A. *Der philosophische Kritizismus und seine Bedeutung für die positive Wissenschaft.* Bd. II, 1. *Die sinnlichen und logischen Grundlagen der Erkenntnis.* 1879.

213. Riemann, B. *Gesammelte mathematische Werke und wissenschaftlicher Nachlaß.* Hrsg. v. H. Weber. (1876) 1892.

214. Riemann, B. *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen komplexen Größe.* Diss. Gött. 1851. In [213]. Partially translated in G. Birkgoff, ed. *A Source Book in Classical Analysis*. 1973, pp. 48-50.

215. Riemann, B. "Über die Hypotheses, welche der Geometrie zugrunde liegen," *Abh. d. Ges. d. Wiss. Gött.* 13 (1854); also in [213]. Neu hrsg. u. erläut. v. H. Weyl. 1919. Translated by H. White, "On the Hypotheses which Lie at the Foundations of Geometry," in D. Smith, *op. cit.*

216. Royce, J. "The Principles of Logic," in the English version of [63]. German by E. Schweitzer, in [63].

217. Russell, B. *An Essay on the Foundations of Geometry*. 1897.

218. Russell, B. *The Principles of Mathematics.* 1903. Compare [23].

219. Russell, B. "Mathematical Logic as Based on the Theory of Types," *Amer. Journ. of Math.* 30 (1908): 222-262. Reprinted in R. Marsh, ed. *Logic and Knowledge.* 1956.

220. Russell, B. *The Problems of Philosophy*. 1912.

221. Russell, B. *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*. 1914. Compare [10].

222. Schlegel, V. "Die Graßmannsche Ausdehnungslehre. Ein Beitrag zur Geschichte der Mathematik in den letzten fünfzig Jahren," *ZS. f. Math. u. Phys.* 40 (1896).

223. Schlick, M. *Allgemeine Erkenntnislehre.* Naturw. Mongr. u. Lehrb. Bd. I. 1918. Translated from the second edition (1925) by A. Blumberg, *General Theory of Knowledge.* 1974.

224. Schlick, M. *Raum und Zeit in der gegenwärtigen Physik. Zur Einführung in das Verständnis der Relativitäts- und Gravitationstheorie.* (1917) 1920. Translated by H. Brose, *Space and Time in Contemporary Physics.* 1920. Reprinted in H. Mulder and B. van de Velde-Schlick, *Moritz Schlick: Philosophical Papers.* 1979.

225. Schmied-Kowarzik, W. "Raumanschauung und Zeitanschauung," *Arch. f. d. ges. Psych.* 28 (1910): p. 94.

226. Schmitz-Dumont, O. *Zeit und Raum in ihren denknotwendigen Bestimmungen abgeleitet aus dem Satze des Widerspruches.* 1875.

227. Schmitz-Dumont, O. *Naturphilosophie als exakte Wissenschaft. Mit besonderer Berücksichtigung der mathematischen Physik.* 1895.

228. Schoenflies, A. *Mengenlehre,* in [62], Bd. I A. 1898.

229. Schoenflies, A, *Entwickelung der Mengenlehre und ihrer Anwendungen.* 1. Hälfte: *Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen.* 1913.

230. Schoenflies, A. *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten.* 2. Teil. *Jahresber. Math.-Ver.*, Erg.-Bd. II. 1908.

231. Schröder, E. *Der Operationskreis des Logikkalkuls.* 1877.

232. Schröder, E. *Vorlesungen über die Algebra der Logik (exakte Logik).* 3 Bde. 1890- 1895.

233. Schröder, E. *Abriß der Algebra der Logik.* Bearb. v. E. Müller. 2 Tle. 1909, 1910.

234. Schultz, J. "Bemerkungen zur Psychologie der Axiome," *Beil. z. Jahr.-Ber. d. Soph.-Realg. Berlin*. 1897.

235. Schur, F. *Grundlagen der Geometrie.* 1909.

236. Schwarzschild, K. "Über das zulässige Krümmungsmaß des Raumes," *Viert.-Schr. d. astr. Ges.* 35 (1900): 337-347.

237. Segre, G. *Mehrdimensionale Räume,* in [62], 2. Teil.

238. Sellien, E. *Die erkenntnistheoretische Bedeutung der Relativitätstheorie.* Erg.-H. 48 der *Kantstudien*. 1919.

239. Siegel, C. *Geschichte der deutschen Naturphilosophie.* 1913.

240. Sigwart, C. *Logik.* Bd. II: *Die Methodenlehre.* (1878) 1893. Translated by H. Dendy, *Logic*. Vol. 2: *Logical Methods*. (1895) 1980.

241. Simon, M. "Zu den Grundlagen der nichteuklidischen Geometrie," *Progr. Lyc. Straßb.* (1891).

242. Simon, M. *Euklid und die sechs planimetrischen Bücher.* 1901.

243. Stallo, J. *The Concepts and Theories of Modern Physics.* (1888) 1960. German by H. Kleinpeter (Forward by E. Mach). (1901) 1911.

244. Staudt, G. v. *Geometrie der Lage*. 1847.

245. Steiner, J. *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander.* 1832.

246. Sterneck, R. v. *Der Sehraum auf Grund der Erfahrung. Psychologische Untersuchungen.* 1907.

247. Stolz, O. and Gmeiner, J. *Theoretische Arithmetik.* (1902). I. Bd.*: Allgemeines. Die Lehre von den Rationalzahlen.* 1911.

248. Stolz, O. and Gmeiner, J. *Op. cit.,* II. Bd.: *Die Lehre von den reellen und von den complexen Zahlen.* 1915.

249. Study, E. *Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. = D. Wissensch.*, Bd. 54 (1914).

250. Tietze, H. "Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten," *Mon.-H. f. Math. u. Phys.* 29 (1908).

251. Tobias, W. *Grenzen der Philosophie, constatiert gegen Riemann und Helmholtz, verteidigt gegen Hartmann und Lasker*. 1875.

252. Vahlen, K. *Abstrakte Geometrie. Untersuchungen über die Grundlagen der euklidischen und nichteuklidischen Geometrie.* 1905.

253. Veblen, O. and Young, J. "A Set of Assumptions for Projective Geometry," *Amer. Journ. of Math.* 30 (1908): 347-380.

254. Veronese, G. *Fondamenti di geometria a più dimensioni e a più specie di unita rettilinee, esposti in forma elementare*. 1891. German by A. Schepp. 1894.

255. Volkmann, P. *Erkenntnistheoretische Grundzüge der Naturwissenschaften und ihre Beziehungen zum Geistesleben der Gegenwart.* (1896) 1910.

256. Voß, A. *Über das Wesen der Mathematik.* (1908) 1913.

257. Weber, H. and Wellstein, J. *Enzyklopädie der Elementar-Mathematik.* Bd. I: Weber, *Enzyklopädie der elementaren Algebra und Analysis.* (1903) 1906.

258. Weierstraß, K. "Zur Theorie der aus *n* Haupteinheiten gebildeten komplexen Größen," *Nachr. d. Ges. d. Wiss. Göttingen* (1884): 395-419.

259. Weinstein, M. *Die Grundgesetze der Natur und die modernen Naturlehren.* 1911.

260. Wellstein, J. *Grundlagen der Geometrie*, in *Elemente der Geometrie* = [257] II. (1905) 1907.

261. Wernicke, A. "Die Grundlage der euklidischen Geometrie des Maßes," *Beil. z. Progr. Braunschw.* (1887).

262. Wernicke, A. *Mathematik und philosophische Propädeutik. = Abh. üb. d. math. Unterr. i. Dtschl.* Hrsg. v. F. Klein, Bd. III, H. 7. 1912.

263. Weyl, H. "Über die Definitionen der mathematischen Grundbegriffe," *Math.-Naturw. Bl.* 7 (1910): 93-95, 109-113.

264. Weyl, H. *Raum-Zeit-Materie. Vorlesungen über allgemeine Relativitätstheorie.* (1918) 1919. Translated by H. Brose, *Space-Time-Matter.* 1950.

265. Weyl, H. *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis.* 1918.

266. Weyl, H. "Der circulus vitiosus in der heutigen Begründung der Analysis," *JB. d. Math.-Ver.* 28 (1919): 85-92.

267. Weyl, H. Erläuterungen zu Riemann [215]. 1919.

268. Whitehead, A. *A Treatise on Universal Algebra with Applications.* I. 1898.

269. Whitehead, A. *The Axioms of Projective Geometry.* Cambridge Tracts in Mathematics and Mathematical Physics, IV. 1906.

270. Whitehead, A. and Russell, B. *Principia Mathematica.* I-III. 1910-1913.

271. Wien, W. "Physik und Erkenntnistheorie," *Vorträge über die neuere Entwicklung der Physik und ihrer Anwendungen.* 1919.

272. Wundt, W. "Über die mathematische Induktion," *Philosoph. Studien* 1 (1883): 90-147.

273. Wundt, W. *Logik. Eine Untersuchung der Prinzipien der Erkenntnis und der Methoden wissenschaftlicher Forschung.* Bd. I: *Allgemeine Logik und Erkenntnistheorie.* (1879) 1919.

274. Zermelo, E. "Untersuchungen über die Grundlagen der Mengenlehre. I," *Math. Ann.* 65 (1908): 261-281. Translated by S. Bauer-Mengelberg, "Investigations in the Foundations of Set Theory I," in J. van Heijenoort, *op. cit.*

275. Ziehen, T. *Das Verhältnis der Logik zur Mengenlehre. = Philos. Vortr. d. Kantges.,* H. 16. 1917.