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Preface to the *Abhandlungen zur Philosophie der Mathematik* (1974)

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(Vorwort zu *Abhandlungen zur Philosophie der Mathematik*)

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Comments:

none

The Wissenschaftliche Buchgesellschaft has kindly offered to publish a collection of my essays on the philosophy of mathematics, which have appeared in various journals. I accept this offer gladly, in particular since several of these essays are not easily accessible.

The present volume can also serve as a temporary substitute for a comprehensive treatment of the philosophy of mathematics.[1] This is possible because, during the period in which these articles were published, my views on the relevant questions have changed almost exclusively in response to new insights gained from research in the foundations of mathematics.

This collection of various essays should provide the reader with an adequate characterization of my views on mathematics.

[1]A monograph on this topic, to be published by Dunker & Humbolt, has long been planned but has not yet been written.

Especially with regard to what has been called the “foundational crisis” of mathematics it will become clear, I hope, that according to my view we cannot justifiably speak of a crisis, at least not in the sense that classical mathematics has been shown to be questionable. Problematic aspects have, of course, presented themselves in various respects.

First of all, we have become conscious of the fact that the idea of the triviality (Selbstverstaendlichkeit) of mathematics is not justified unless we consider as trivial (selbstverstaendlich) simply what has become familiar to us through use and practice. Even ideas that are not really trivial can become familiar to us in this sense, and in their use we can acquire practical certainty. The very idea of an absolute certainty for human reason is presumably illusory in any case.

Going beyond the trivial is involved especially in all those idealizations which are characteristic for the formation of mathematical concepts.

It has become clear that even the general concept of natural number and the related notion of the number series are based on an idealization.

Already here, we also meet with an opposition that calls for a restriction of methods of proof. The restricted methods of Brouwer’s “intuitionism” as well as those of the “finitist” standpoint—as Hilbert called it—avoid the inference according to which either a numerical predicate applies to all numbers or else there exists at least one number to which it does not apply. This kind of application of the “tertium non datur” is avoided here, all the more, for predicates of sets and of functions. However, even when only such restricted methods are accepted, in generating functions recursively one goes beyond what is concretely, computationally feasible.

Avoidance of the above-mentioned application of the "tertium non datur" has, incidentally, no essential impact on elementary number theory. For analysis, however, it amounts to a considerable restriction.

It was initially believed that the methods of Dedekind and Cantor provided an entirely arithmetical foundation for classical analysis. However, viewed from the standpoint of the requirement of a strict arithmetization, classical analysis was soon criticized. And this critique grew under the influence of the demands of finitist and intuitionistic methods. Various programs for a more strictly arithmetical treatment of analysis have since been developed within research in the foundations of mathematics.

It would be unjust not to recognize that these various kinds of a more strongly arithmetized analysis are of definite mathematical interest. Yet, it should also be admitted that it is a prejudice to believe that it is absolutely necessary to arithmetize analysis completely. In analysis, after all, geometrical ideas are made conceptually precise. The methods of Dedekind and Cantor, referred to above, succeed in basing analysis on number theory, but not without the addition of set-theoretical concepts. If one understands clearly that these concepts are not completely arithmetical, then the procedure involves nothing objectionable.

To be sure, the methods of classical analysis contain strong idealizations. But these do not detract from practical certainty. On the contrary, a kind of intuitability is gained here that confers great certainty on our reflections.

Problems have also arisen in connection with the discovery of the formalizability of mathematical proofs by means of symbolic logic. This formalization can be viewed as a sharpening of the axiomatic method. Indeed, formal systems have been successfully set up for number theory, analysis, and set theory. These formalizations consist of a symbolism (a formal language, ed.) and rules of deduction, and are organized in such a way that they permit the formal representation of the known proofs of the respective theories.

Formal-deductive systems can also be set up independently of already existing theories, and then we have the reverse situation, namely that we can try to find meaningful interpretations (models) for them. That is a topic for "semantics" which is generally concerned with the relations between theories and formal systems.

A different kind of research tied to the formalization of mathematical theories takes formalized proofs as the object of mathematical investigations. This is the topic of Hilbert's proof theory. It deals above all with the investigation of the internal consistency of formalized theories. For the theories of the infinite, there arises the possibility of a reduction of methods (used in consistency proofs, ed.) because formalized proofs are, after all, finite objects and because consistency can be formally characterized. Consistency proofs of this kind have actually been given successfully for formalized number theory and formalized analysis, but they use by no means methods as elementary as Hilbert had sought. He wanted to restrict the methods of such proofs to combinatorial ones in accord with the finitist standpoint. Stronger methods of providing constructive proofs had to be used, however.

This necessity of going beyond the elementary "finitist" methods in consistency proofs is related to another difficulty. Both came to light as consequences of results obtained by Kurt Gödel and Thoralf Skolem. It was shown that a formal system, if it is to satisfy the conditions of strict controllability, couldn't completely express the particular intended theory. This is particularly clear from the fact that the formal system, apart from its normal interpretation by the intended theory, also permits deviant interpretations, the so-called "nonstandard" models.

Within foundational research one has dealt with this fact in different ways, either by studying nonstandard models more closely or by considering possibilities of excluding nonstandard models by an extended kind of formalization. For number theory two ways of extending formalizations have been considered: "infinite induction" and the admission of infinite conjunctions and disjunctions. In either case, the finite character of proof figures is lost.

As regards fundamental reflections, it emerges that the role of formalization is not so simple as it was originally intended and, at the same time, that we do not have to lay down so unconditionally the requirement of formalization. In any case, semantics, in keeping with its purpose, uses set-theoretical reasoning that is not bound to a formal system.

As we see, there is no dearth of problems for the philosophy of mathematics. Nevertheless, what I said in one of my essays[2] remains true: "If we... start from the position which holds that mathematics is the science of idealized structures, then we have an attitude for research in the foundations of mathematics that will save us from exaggerated aporiae and forced constructions and that cannot be contested, even if foundational research brings to light many astonishing facts."

Zürich, December 1974 Paul Bernays

[2] "The Schematic Correspondence and the Idealized Structures"