Bernays Project: Text No. 11

Methods for proving consistency proofs and their limitations (1932)

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Comments:

none

The methods that were used to prove the consistency of formalized theories from the finite standpoint can be surveyed according to the following classification [*Einteilung*].

1. Method of valuation [Wertung]. It has obtained its fundamental development in the *Hilbert*ian procedure of trying [*Ausprobierens*] the valuation. Using this procedure *Ackermann* and *v. Neumann* proved the consistency of number theory — admittedly under the restriction that the application of the inference from n to $(n+1)$ is only allowed on formulas with free variables.

2. Method of integration [Ausintegrierens]. This can only be applied to such domains [*Gebiete*] that are completely controlled mathematically [*mathematisch vollkommen beherrscht*]. It allows for these not only the question of consistency to be answered in a completely positive sense, but also those of completeness and decision [*Entscheidbarkeit*]. Such domains are in particular:

a) the calculus of one place functions [*einstellige Funktionenkalkul*], which has been handled concludingly [*abschließend*] by *Löwenheim Skolem*, and *Behmann*.

b) Subformalisms [*Teilformalismen*] of number theory. *Herbrand* and *Presburger* have applied the method to such. It has been shown that *Peano*'s axioms do not suffice to develop number theory based on the function calculus [*Funktionenkalkul*] of "first order" (with axioms for equality). Only the addition of the recursive equations [*Rekursionsgleichungen*] for addition and multiplication brings full number theory about [*kommt zustande*][1].

3. Method of elimination. The idea for it can be found already in *Russell* and *Whitehead*, in particular applied to the concept "that, which" [*derjenige, welcher*]. However, the actual implementation [*Durchführung*] of the thought [*Gedankens*] is laborious [*mühsam*]. A fundamental simplification is effected by an approach by *Hilbert*, which follows the introduction of the "ϵ-symbol".

Firstly, — as has been shown by *Ackermann* — this yields, in a simpler way, again the result of the method of valuation.

Moreover, a new proof for a theorem can be reached from here, that has been discovered and proved for the first time by *Herbrand*, and which consists

[1]The situation differs if the standpoint of class logic [*Klassenlogik*] is taken as basic [*zugrunde legt*] at the outset, like *Dedekind*; however, this contains stronger assumptions than are needed for number theory.

in a reversal of *Löwenheim*'s famous theorem about the satisfiability in the countable realm [*im Abzählbaren*]. It also yields a general procedure for the treatment of questions about consistency.

The present narrowness [*vorliegende Begrenztheit*] of the results presents itself as fundamental, because of the new theorem on the limits of decision procedures [*Entscheidbarkeit*] for formal systems by *Gödel* in connection with the conjecture by *v. Neumann* that followed it, despite the manifold [*mannigfachen*] of insights that have been obtained.