Bernays Project: Text No. 4

Reply to the Note by Mr. Aloys Müller, "On Numbers as Signs" (1923)

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(Erwiderung auf die Note von Herrn Aloys Müller: Über Zahlen als Zeichen. Mathematische Annalen 90, 1923, pp. 159-63.)

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Comments:

Mr. Aloys Müller's criticism of the conception of number theory as a theory that deals with meaningless signs ("number signs") consists basically in three objections. A discussion of these objections is useful for clarifying the standpoint of intuitive number theory, and thereby also that of Hilbert's proof theory.

1. The first objection is terminological and is directed against the use of the word "sign" for something meaningless [*bedeutungslos*]. If the objects of number theory, such as 1, $1 + 1$, have no meaning, then they are not signs, so the objection goes, but rather figures or, "as we would rather want to say," *shapes* [*Gestalten*].

The first part of this objection must be conceded: Indeed, it corresponds better to linguistic usage to say that the objects of intuitive number theory, the “number signs,” are *figures*. On the other hand, we must thoroughly avoid using the word “shape” in the same sense as “figure.” Figures *are* not shapes; rather, they *have* a shape. (Moreover, they also have individuality.) We must be able to speak about the fact that a figure a has the same shape as another figure b.

2. The second argument is the following. Since the figures under consideration have no meaning, nothing can hinge on the particular form [*Form*] of the individual constituents. For example, instead of the figures

$$1+1, \quad 1+1+1$$

one might just as well choose different ones, for example,

○●○

○●○●○

Moreover, the objection proceeds, we are not bound to the “serial form of the arrangement” [*Reihenform der Zusammensetzung*], and just as little to the number |[Mancosu: 224] of elementary shapes. Any specification relating to this would already introduce a contentual element [*inhaltliches Moment*]. (“One thinks of members of the -sequence, of the position of members in the sequence, and thereby a meaningful content [*Bedeutungsgehalt*] again unnoticeably attaches itself.”) Thus completely arbitrary kinds of arrangements of discrete constituents are admissible. However, the examination of such shapes does not lead to number theory.

This much is correct about the foregoing, namely, that the special shapes [*Formen*] “1” and “+” are inessential. If we disregarded the connection to habit, it would even be advisable, in order to emphasize the principle, to take as numerical signs figures of the type

$$\cdots\cdots$$

(which are thus constituted merely of points). And, of course, stars, vertical strokes, circles, and other shapes could just as well be chosen instead of points. One could also take a time sequence, say, of similar noises, instead of a spatial sequence.

But it is essential that *specimens of equal shape be joined in the same sort of arrangement* [*Zusammensetzung*]. In this respect we are indeed bound to the serial form of the arrangement. However, thereby we are in no way surreptitiously obtaining a meaning for the numerical signs. For, the intuitive-contentual elements that occur in the *description* of a figure need not be ascribed as meaning to the figure itself.

In order to indicate what kind of figures the "number signs" should be, we need the idea of a determinate, concretely exhibitable, form of succession. The inessential elements of the shape and of the arrangement that occur here will then be, as it were, eliminated by the sort of consideration that is applied to the number signs, for they play no role in the occurring relations

Thus intuitive number theory can indeed be viewed as a fundamental chapter of the theory of shapes. Its delimitation and problematic is also by no means arbitrary from the point of view of the shape, but rests on a natural abstraction, on a choice of certain, simplest elements. The necessary abstraction is here not at all obtained surreptitiously but rather is displayed by means of manifest objects (whereby, of course, the possibility of communication through language is a prerequisite). The principle of abstraction is however something contentual—an intellectual discovery—but this content is thereby not yet a meaning of the objects about which we think.

The claim “that mere shapes do not suffice as a basis for number theory” may be conceded. But the meaningless figures are not supposed to constitute the *basis* but only the objects of number theory.

3. To elucidate his standpoint, Mr. Aloys Müller advances the following third objection: “$3 > 2$” means, according to Hilbert’s explanation, that the number sign 3, that is, $1 + 1 + 1$, extends beyond the number sign 2, that is, $1 + 1$, or that the latter figure is a segment of the former. However, according to this spatial interpretation, the claim $3 > 2$ is not correct under all circumstances. For example, the objection goes, if one writes the two figures in the form

$$
\begin{array}{ccccc}
1 & & + & & 1 \\
1 & + & 1 & + & 1
\end{array}
$$

|[Mancosu: 225] one beneath the other, then the second does not extend beyond the first, and also the first is not a segment of the second. “Whoever contests that is again secretly attaching a sense to $1 + 1$ and $1 + 1 + 1$, e.g., that the first shape contains two units, and thus one unit less than the second.”

To this it should first be remarked that, according to Hilbert’s explanation, the claim $3 > 2$ indeed has a spatial sense, but not thereby a *metric* sense. In characterizing the attitude required in the case of intuitive number theory, Hilbert does stress (in a passage quoted by Mr. Aloys Müller himself) that “insignificant differences in the construction [*Ausführung*] of the figures should be disregarded.” The separation of the constituents of the figure $1+1$ from one another by larger or smaller distances is such an insignificant difference. That this difference is to be viewed as “insignificant’ already follows from the fact that the same shape must always be denoted by "2." This shape is completely described by the fact that I stands first, after that "+," and after that again "1." The figure $1 + 1 + 1$ is to be described correspondingly. And that the figure $1 + 1$ coincides with a constituent of the figure $1+1+1$, in such a way that the latter results by affixing something, namely, " +1," to the former, is now a fact that can be grasped intuitively.

But by this observation it is surely not the case that a sense is "secretly" conferred upon the figures. Indeed, it is only a question of a purely external relationship between the figures. Moreover, the circumstance addressed by Mr. Aloys Müller in the quoted sentence, viz. that the shape $1 + 1$ contains two units (by unit the component I is probably meant), and thus one less than $1+1+1$, does not constitute the sense of these signs, just as the observation that the word "chair" contains five letters contributes nothing to the sense of the word "chair."

However, when he speaks of a meaning of the number signs, Mr. Aloys Müller is apparently not at all thinking of the kind of meaning as it occurs in the words of a language. Rather he is thinking of the meaning that befits the number signs *within the formalism of number theory.* But senseless figures are equally capable of such meaning, because of the external properties that are found in them and of the external relationships that can be observed between them.

It should also be noted that the contentual character of the Number [*Anzahl*] concept is indeed compatible with the purely figural character of the number signs. The figures are used as tools for *counting*, and by counting one arrives at the determination of number. Incidentally, this way of introducing the concept of Number [*Anzahlbestimmung*], together with the required intuitive considerations, can already be found in the earlier literature (e.g., in Helmholtz, "Zählen und Messen").

One here has to recognize that the Numbers are only defined in connection with the entire *Number statement.* For example, it will not be explained what "the number five" is, but only what it means for the Number five to apply to a given totality of things.

Of course, it cannot be claimed that the method of intuitive number theory and the Number definition associated with it represents the only grounding of number theory compatible with Hilbert's basic methodological direction. (Indeed, in the construction of his more comprehensive theory, Hilbert himself replaces this grounding with a different and more formalized one.)

It is by no means compatible, however, with Hilbert's basic thoughts to introduce the numbers as ideal objects "with quite different determinations from those |[Mancosu: 226] of sensory objects," "which exist entirely independently of us." By this we would go beyond the domain of the immediately certain. In particular, this would be evident in the fact that we would consequently have to assume the numbers *as all existing simultaneously.* But this would mean assuming at the outset exactly that which Hilbert considers to be problematic.

However, the objects of intuitive number theory, the number signs, are, according to Hilbert, also not "created by thought." But this does not mean that they exist independently of their *intuitive construction*, to use the Kantian term that is quite appropriate here. But the construction always only yields either a single de- terminate figure or a procedure for obtaining a further figure from a given one (e.g., by affixing "+1"). But it does not lead to the idea of a simultaneous existence of "all" the number signs. That the idea of the number series as a closed totality [*Inbegriff*] can be applied in mathematical inferences without danger of a contradiction is precisely what is shown by Hilbert's proof theory.

Hilbert's theory does not exclude the possibility of a philosophical attitude that conceives of the numbers as existing, nonsensory objects (and thus the same kind of ideal existence would then have to be attributed to transfinite numbers as well, and in particular to the numbers of the so-called second number class). Nevertheless the aim of Hilbert's theory is to make such an attitude dispensable for the foundation of the exact sciences.

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