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Comments:

Has to be re-read!

Introduction

The aim of this book is to provide an exhaustive [*eingehende*] orientation over the current material [*gegenwärtigen Stoff*] of HILBERT's proof theory. Despite the fact that very little has been achieved here so far compared to the goals of the theory, there still are plenty of succinct [*prägnant*] results, viewpoints, and proof ideas that seem worthwhile to make aware of.

The purpose of the book led to two main themes for the design of the content of this second volume.—On the one hand, to present in detail the principal proof-theoretic approaches of HILBERT that follow from the ϵ-symbol together with their implementations.

A substantial part of the investigations presented here has not been published yet at all, except for some bare hints. Thus, not only there is interest in the subject matter, but also the scientific obligation of the HILBERT-school to justify the various previous announcements of proofs by actually providing these proofs—this demand is all the more pressing in this case, since one was initially (until the year 1930) mistaken about the consequences of the proofs by ACKERMANN and v. NEUMANN which resulted from one of the approaches of HILBERT mentioned above.

These hitherto unpublished proofs are now presented in detail in §§ 1 and 2. In particular, the restriction which is here still imposed on the consistency proof for the number-theoretic formalism is clearly made apparent.

With the help of one of the methods presented here a simple approach to a series of theorems arises at the same time by which the proof-theoretic investigation of the predicate calculus is satisfactorily rounded off and which also allow for remarkable applications to axiomatics. A theorem of theoretical logic that has been first formulated and proved by J. HERBRAND, for which we obtain a natural and simple proof by the mentioned route, lies at the center of these considerations.

The discussion of the applications of this theorem also offers the opportunity for some considerations about the decision problem. Following these a proof-theoretic sharpening [*Verschärfung*] of GÖDEL's completeness theorem is proved in § 4.

The second main theme is provided by the considerations [*Auseinandersetzung des Sachverhalts*] which have led to the necessity of extending the limits of the contentual forms of inference [*inhaltlichen Schlußweisen*] that allowed in proof-theory beyond the previous demarcation of the "finite standpoint". GÖDEL's discovery that every sharply delineated [*scharf abgegrenzt*] and sufficiently expressive formalism is not deductively complete [*deduktiven Unabgeschlossenheit*] stands at the center of these considerations. Both of GÖDEL's theorems which express this matter of fact are discussed in detail with regard to their relation to the semantic paradoxes, to the conditions for their validity and the implementation of their proofs—GÖDEL only hints at the proof for the second theorem—, and to their applicability to the full number-theoretic formalism.

The discussion regarding the extension of the finite standpoint is followed by the consideration of GENTZEN's recent consistency proof for the number-theoretic formalism. Sure enough, only what is methodologically novel in this proof is presented in detail and discussed, namely the application of a particular kind of CANTOR's "transfinite induction".

The mainly external reason for not presenting the entire proof was that the newer, first really clear version of GENTZEN's proof had not been published at the time of the printing [*Drucklegung*] of this volume. By the way, GENTZEN's proof does not relate directly to the number-theoretic formalism discussed in the book. L. KALMÁR recently succeeded in modifying this proof so that it became directly applicable to the number-theoretic formalism developed in our book (in § 8 of the first volume), whereby also certain simplifications arise.

W. ACKERMANN is currently developing [*ausgestalten*] his earlier consistency proof (presented in § 2 of the present volume) by applying the kind of transfinite induction that is used by GENTZEN in order for it to be valid for the full number-theoretic formalism.

If this succeeds—for which there is a great chance—, HILBERT's original approach would be rehabilitated with respect to its effectiveness. In any case, already GENTZEN's proof justifies the view that the temporary fiasco of proof theory was merely due to the fact that too strong methodological requirements had been imposed on the theory. For sure, the final decision about the fate of proof theory will be based on the task of proving the consistency of analysis.

A few separate considerations to the train of thought developed in §§ 1–5 of the present volume are added as "supplements". Two of these complement the considerations in § 5: Supplement II which is about making precise the notion of computable function (that has been successfully implemented recently with various methods) and presents the facts related to this problem space which can be easily developed following the remaining considerations of the book. A. CHURCH's theorem about the impossibility of a general solution to the decision problem for the predicate calculus is applied in this connection; furthermore, Supplement III in which some questions pertaining to the deductive propositional logic are discussed, and which also contains additional remarks to the considerations about the "positive logic" formulated in § 3 of the first volume.

Various deductive formalisms for analysis are set up in the Supplement IV and it is shown how the theory of the real numbers and also that of the numbers of the second number class are obtained from them.

Supplement I contains an overview of the rules of the predicate calculus and its application to formalized axiom systems, as well as remarks about possible modifications of the predicate calculus, and a compilation of various concept formations and results from the first volume.

In view of the already huge amount of material various proof-theoretic themes could not be addressed in this book: In particular, the theme of the multi-sorted predicate calculus, which was dealt with first in HERBRAND's thesis and recently in more detail by ARNOLD SCHMIDT (in *Mathematische Annalen*, vol.115).

Certain considerations that could be found in HILBERT's lectures and in discussions with HILBERT, but that only remained isolated remarks or that had not been sufficiently clarified, are not presented: In particular, the approaches regarding the definitions of numbers of the second number class by usual (i.e., not transfinite) recursion, and those concerning the use of symbols for species [*Gattungssymbolen*], in particular those that are introduced by explicit or recursive definitions.

The present volume has been formulated following closely the first volume; the connection with it is also strengthened by frequent references to page numbers. On the other hand, the compilation of terms and theorems from the first volume given in Supplement I and the recapitulation in part b), section 1 is intended to render the reading of the second volume largely independent of the first volume. The reader who is already somewhat familiar with logical formalization and with the questions addressed by proof theory will be able to follow the considerations of the second volume without knowledge of the first one.

At any rate, it is recommended for the reader of the present volume to start with § 1 of Supplement I. Furthermore, he should make use of the page references only when he feels the need to do so in the particular passages.

In addition to the hint given in § 2 about a possible omission of the reading, also the rather tedious section 2 of § 4 can be left out.

Regarding the statements of paragraphs, the numbers from 1 to 5 refer to the present second volume if nothing else is indicated, while the numbers from 6 to 8 occur only in the first volume.

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