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Hilbert's investigations on the foundations of arithmetic (1935)

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(Hilberts Untersuchungen über die Grundlagen der Arithmetik, 1935.)

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Comments:

none

The first investigations of Hilbert on the foundations of arithmetic follow temporally as well as theoretically [*gedanklich*] his investigations on the foundations of geometry. Hilbert begins the paper "On the concept of number" [*Über den Zahlbegriff"*][1] by bringing to bear [*zur Geltung bringt*] the axiomatic method for arithmetic, accordingly as [*entsprechend wie*] for geometry, which he contrasts [*gegenüberstellt*] to the otherwise usually applied [*sonst gewöhnlich angewandten*] "genetic" method.

"At first let us bring to mind [*Vergegenwärtigen wir uns zunächst*] the manner of introducing [*die Art und Weise der Einführung*] the concept of number. Starting from the concept of the number 1, usually one thinks at first the further

[1] Jber. dtsch. Math.-Ver. Bd. 8 (1900); reprinted in Hilbert's "Foundations of Geometry", 3rd-7th ed., as appendix VI.

rational positive numbers 2, 3, 4,. . . as arisen [*entstanden*] from the process of counting and developed their laws of calculation [*Rechnungsgesetze entwickelt*]; then one arrives at the negative number by the requirement of the general execution [*allgemeinen Ausführung*] of subtraction; one further defines the rational [*gebrochene*] number perhaps [*etwa*] as a pair of numbers — then every linear function has a zero [*Nullstelle*] —, and finally the real number as a cut or a fundamental sequence — thereby obtaining that every whole rational indefinite [*indefinite*], and generally [*überhaupt*] every continuous indefinite function has a zero [*Nullstelle*]. We can call this method of introducing the concept of number the *genetic*, because the most general concept of real number is generated [*erzeugt*] by successive expansion [*Erweiterung*] of the simple concept of number.

One proceeds fundamentally differently with the structure [*? Aufbau*] of geometry. Here one tends [*pflegt*] to begin with the assumption of the existence of all elements, i.e., one presupposes at the outset three systems of things, namely the points, the lines, and the planes, and then brings these elements — essentially after the example of Euclid — in relation with each other [*miteinander in Beziehung*] by certain axioms, namely the axioms of concatenation [*? Verknüpfung*], of ordering [*Anordnung*], of congruency, and of continuity. Then the necessary task arises to show the *consistency* [*Widerspruchslosigkeit*] and *completeness* of these axioms, i.e., it must be proven that the application of the compiled [*aufgestellte*] axioms can never lead to contradictions, and moreover that the system of axioms suffices [*ausreicht*] to prove all geometric theorems. We want to call this chosen [*eingeschlagene*] procedure of investigation [*Untersuchungsverfahren*] the *axiomatic method.*

We raise the question, whether the genetic method is really the only pertinent for the study of the concept of number and the axiomatic method for the foundations of geometry. It also appears to be of interest to contrast both methods and to investigate which method is the most advantageous if one is concerned with [*es sich ... handelt*] the logical investigation of the foundations of mechanics or other physical disciplines.

My opinion is this: *Despite the great pedagogical and heuristic value of the genetic method, the axiomatic method deserves the preference [Vorzug] for the final representation and complete logical security [? Sicherung] of the content of our knowledge [Erkenntnis].*"

Already Peano developed number theory axiomatically[2] Hilbert now sets up [*stellt... auf*] an axiom system for analysis, by which the system of real

[2]Peano, G. "Arithmetices principia nova methodo exposita". (Torino 1889.) The introduction of recursive definitions is here not immaculate [*einwandfrei*]; the proof [*Nachweis*] of the solvability of the recursive equations is missing. Such a proof was provided already by Dedekind in his essay "Was sind und was sollen die Zahlen" (Braunschweig 1887). To introduce the recursive functions beginning with Peano's axioms it is best to proceed by first proving the solvability of the recursive equations for the sum after L. Kalmár by an induction conclusion [*? Induktionsschluß*] after the argument of the parameter [*? Parameterargument*], then defining the concept "less" with the help of the sum, and after this using Dedekind's consideration [*Überlegung*] for the general recursive definition. This procedure is displayed [*? dargestellt*] in Landau's textbook "Grundlagen der Analysis" (Leipzig 1930). Admittedly [*allerdings*] here the concept of function is used. If one wants to avoid it, the recursive equations of the sum and product have to be introduced as axioms. The proof of the general solvability of recursive equations follows then by a method [*Verfahren*] by K. Gödel (cf. "Über formal unentscheidbare Sätze ..." [Mh. Math. Physik, Bd. 38 Heft 1 (1931)], and also Hilbert-Bernays Grundlagen der Mathematik Bd. 1 (Berlin 1934) p. 412 ff.)

number is characterized as a real Archimedean field which cannot be extended to a more extensive [*umfassenderen*] field of the same kind.

A few exemplary [*beispielsweise*] remarks about dependencies follow the enumeration [*Aufzählung*] of the axioms. In particular it is mentioned that the law of commutativity of multiplication can be deduced from the remaining properties of a field [*Körpereigenschaften*] and the order properties [*Ordnungseigenschaften*] with the help of the Archimedean axiom, but not without it.

The requirement [*Forderung*] of the non-extendibility [*Nichterweiterbarkeit*] is formulated by the "axiom of completeness". This axiom has the advantage of conciseness [*Prägnanz*]; however, its logical structure is complicated. In addition it is not immediately apparent [*unmittelbar ersichtlich*] from it [*an ihm*] that it expresses a demand of continuity [*Stetigkeitsforderung*]. If one wants, instead of this axiom, such a one that clearly has the character of a demand of continuity [*Stetigkeitsforderung*] and on the other hand does not include the requirement of the Archimedean axiom, it is recommended to take Cantor's axiom of continuity, which says that if there is a series [*Folge*] of intervals such that every interval includes [*umschließt*] the following one, then there is a point which belongs [*angehört*] to every interval. (The formulation [*Aufstellung*] of this axiom requires the previous introduction of the concept of number series [*Zahlenfolge*])[3].

[3]Concerning the independence of the Archimedean axiom from the mentioned axiom of Cantor, cf. P. Hertz: "Sur les axiomes d'Archiméde et de Cantor". C. r. soc. de phys. et d'hist. natur. de Genéve Bd. 51 Nr. 2 (1934).

R. Baldus has pointed at [*hingewiesen*] latterly at Cantor's axiom. See his essay "Zur Axiomatik der Geometrie": :I. Über Hilberts Vollständigkeitsaxiom." Math. Ann. Bd. 100 (1928), :II. Vereinfachungen des Archimedischen und des Cantorschen Axioms", Atti

At the end of the essay [*Abhandlung*] the purpose [*Absicht*] which Hilbert pursues with the axiomatic version [*Fassung*] of analysis appears particularly clearly [*tritt... besonders deutlich... zutage*] in the following words:

"The objections [*Bedenken*] that have been raised [*geltend gemacht worden sind*] against the existence of the set [*? Inbegriff*] of all real numbers and infinite sets [*Mengen*] in general lose all their warranty [*Berechtigung*] with the view identified above [*oben gekennzeichneten Auffassung*]: we do not have to think the totality of all possible laws according to which the elements of a fundamental sequence can proceed as the set of real numbers, but rather — as has just been explained [*? dargelegt*] — a system of things whose relations between each other are given by the *finite and completed* [*abgeschlossen*] system of axioms I–IV, and about which new propositions [*Aussagen*] only are valid, if they can be deduced from those axioms in a finite number [*Anzahl*] of logical inferences [*Schlüßen*]."

But the methodical [*methodisch*] benefit [*Gewinn*] which is brought by this view requires more effort [*steht eine erhöhte Anforderung gegenüber*]: because the axiomatic formulation [*Fassung*] necessarily entails the task of proving the consistency for the axiom system in question [*das aufgestellte*].

Therefore, the problem of the proof of consistency for the arithmetical axioms was mentioned in the list of problems that Hilbert posed in his lecture in Paris "Mathematische Probleme"[4].

Congr. Int. Math. Bologna Bd. 4 (1928), "III. Über das Archimedische und das Cantorsche Axiom." S.-B. Heidelberg. Akad. Wiss. Math.-nat. Kl. 1930 Heft 5, as well as the following essay by A. Schmidt: "Die Stetigkeit in der absoluten Geometrie." S.-B. Heidelberg. Akad. Wiss. Math.-nat. Kl. 1931 Heft 5.

[4]Held at the International Congress of Mathematicians 1900 in Paris, published in

To accomplish the proof Hilbert thought to get by with a suitable modification of the methods used in the theory of real numbers.

But in the more detailed occupation [*genaueren Auseinandersetzung*] with the problem he was immediately confronted with the considerable difficulties that persist [*bestehen*] for this task. In addition, the set theoretic paradox that was discovered in the meantime by Russell and Zermelo prompted increased caution in the inference rules [*zu erhöhter Vorsicht in den Schlußweisen veranlaßte*]. Frege and Dedekind were forced to withdraw their investigations by which they thought to have provided unobjectionable foundations of number theory — Dedekind using the general concepts of set theory, Frege in the framework of pure logic[5] — since it resulted from that paradox that their considerations contained inadmissible inferences.

The talk[6] "Über die Grundlagen der Logik und der Arithmetik" held in 1904 shows us a completely novel aspect. Here at first the fundamental difference is pointed at, which holds for the problem of the consistency proof between arithmetic and geometry. The proof of consistency for the axioms of geometry uses an arithmetical interpretation of the geometric axiom system. However, for the proof of consistency of arithmetic "it seems that the appeal to another foundational discipline [*Grunddisziplin*] is not allowed".

But one could think of the reduction [*Zurückführung*] to logic. "But by attentive inspection [*Allein bei aufmerksamer Betrachtung*] we become aware that

Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. 1900, cf. also this volume essay no. 17.

[5]R. Dedekind: "Was sind und was sollen die Zahlen?" Braunschweig 1887. G. Frege: "Grundgesetze der Arithmetik" (Jena 1893).

[6]At the International Congress of Mathematicians in Heidelberg 1904, printed in "Grundlagen der Geometrie", 3-7 ed., as appendix VII.

certain artithmetical basic concepts [*Grundbegriffe*] are already used in the traditional [*hergebrachten*] formulation of the laws of logic, e,g., the concept of set, in part also the concept of number, in particular cardinal number [*Anzahl*]. So we get into a quandary [*Zwickmühle*] and to avoid paradoxes a partly simultaneous development of the laws of logic and arithmetic is required."

Hilbert now presents the plan of such a joint set-up [*gemeinsamen Aufbaus*] of logic and arithmetic. This plan contains already in great parts the leading viewpoints [*die leitenden Gesichtspunkte*] for proof theory, in particular the idea to transform the proof of consistency into a problem of elementary-arithmetic character by translating the mathematical proofs into the formula language [*Formelsprache*] of symbolic logic. Also rudiments [*Ansätze*] of the consistency proofs can be already found here.

But the execution remains still in its beginnings. In particular, the proof for the "existence of the infinite" is carried out only in the framework of a very restricted formalism.

The methodical standpoint of Hilbert's proof theory is also not yet developed to its full clarity in the Heidelberg talk. Some passages suggest that Hilbert want to avoid the intuitive idea of number [*anschauliche Zahlvorstellung*] and replace it with the axiomatic introduction of the concept of number. Such a procedure would lead to a circle in the proof theoretic considerations. Also the viewpoint of the restriction in the contentual [*inhaltlichen*] application of the forms of the existential and general judgment is not yet brought to bear explicitly and completely.

In this preliminary stadium Hilbert interrupted his investigations of the foundations of arithmetic for a long period of time[7]. Their resumption [*Wiederaufnahme*] is found announced in the 1917 talk[8] "Axiomatisches Denken".

This talk stands under the sign [*? steht unter dem Zeichen*] of the manifold successful axiomatic investigations that had been pursued by Hilbert himself and other researchers in the various fields of mathematics and physics. In particular in the field of the foundations of mathematics the axiomatic method had lead in two ways to an extensive systematic [*umfassenden Systematik*] of arithmetic and set theory. Zermelo formulated in 1907 his axiom system for set theory[9] by which the processes of set formation [*Mengenbil-*

[7]A continuation of the direction of research that was inspired by Hilbert's Heidelberg talk was carried out by J. König, who, in his book "Neue Grundlagen der Logik, Arithmetik und Mengenlehre" (Leipzig 1914), transcends the Heidelberg talk [*hinausgeht*] both by a more exact formulation and ... [*? einhergehende Darstellung*] presentation, and his realization [*Durchführung*]. Julius König died before finishing the book; it was edited by his son as a fragment. This work, which is a precursor of Hilbert's later proof theory, exerted no influence on Hilbert. But later J. v. Neumann followed the approach of König in his investigation "Zur Hilbertschen Beweistheorie" [Math. Z. Bd. 26 Heft 1 (1927)]

[8]At Naturvorscherversammlung Zürich, published in Math. Ann. Bd. 78 Heft 3/4; see also this volume essay no. 9.

[9]Zermelo E. "Untersuchungen über die Grundlagen der Mengenlehre I.: Math. Ann. Bd. 65. Different investigations followed in newer time this axiom system. A. Fraenkel added the axiom of replacement [*Ersetzungsaxiom*], an extension of the admissible formation of sets in the spirit of Cantor's set theory; J. v. Neumann added an axiom, which excludes that the process of going from a set to one of its elements is infinite for any set. Moreover, Th. Skolem, Fraenkel, and J. v. Neumann have made more precise, all in a different way, in the sense of a sharper implicit characterization of the concept of set the concept of "definite proposition" which was used by Zermelo in vague generality. The result of these efforts [*? Präzisierungen*] is presented in the most concise way in v. Neumann's axiomatic; namely it is achieved here, that all axioms are of the "first order"

dung] are delimited [*abgegrenzt*] in such a way that on the one hand the set theoretic paradoxes are avoided and on the other hand the set theoretic inferences that are customary in mathematics are retained [*erhalten*]. And Frege's project [*Unternehmen*] of a logical foundation of arithmetic — for which the method that Frege employed himself turned out to be faulty [*ungangbar*] — was ... [*? restituiert*] by Russell and Whitehead in their work "Principia Mathematica"[10].

Hilbert says about this axiomatization of logic that one could "see the coronation of the work of axiomatization in general" in the completion of this enterprise [*Unternehmens*]. But praising recognition [*Anerkennung*] is immediately followed by the remark that the completion of the project "still needs new and versatile [*vielseitig*] work".

In fact, the viewpoint of Principia Mathematica contains an unsolved difficulty [*Problematik*]. What is supplied [*geliefert*] by this work is the elab-

[*ersten Stufe*] (in the sense of the terminology of symbolic logic). Zermelo rejects such a characterization [*? Präzisierungen*] of the concept of set, in particular in the light of the consequence that was first discovered by Skolem that such a sharper [*? verschärftes*] axiom system of set theory can be realized in the domain of individuals [*Individuenbereich*] of the whole numbers. — A presentation of these investigations up to the year 1928, with respective references [*? eingehenden Literaturangaben*], is contained in the textbook by A. Fraenkel: "Einleitung in die Mengenlehre", third edition (Berlin 1928). See also: J. v. Neumann "Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre" J. reine angew. Math. Bd. 160 (1929), Th. Skolem: "Über einige Grundlagenfragen der Mathematik" Skr. norske Vid.-Akad., Oslo. I. Mat. Nat. Kl. 1929 Nr. 4, E. Zermelo: "Über Grenzzahlen und Mengenbereiche." Fund. math. Bd. XVI, 1930.

[10]Russell, B., and Whitehead, A. N.: Principia Mathematica. Cambridge, vol. I 1910, vol. II 1912, vol. III 1913.

oration of a clear [*übersichtlich*] system of assumptions [*Voraussetzungen*] for a combined deductive set-up [*Aufbau*] of logic and mathematics, as well as the proof that this set-up in fact succeeds [*tatsächlich gelingt*]. Besides the contentual plausibility (which also from the point of view of Russell and Whitehead does not yield a guarantee [*Gewähr*] for the consistency), only the trial [*Erprobung*] in the deductive use is put forward [*geltend gemacht*] for the reliability [*Zuverlässigkeit*] of the assumptions. But also this trial provides us in regard [*inbetreff*] the consistency only an empirical [*erfahrungsmässig*] confidence [*Vertrauen*], not complete certainty. But the complete certainty [*völlige Gewißheit*] of the consistency is regarded by Hilbert as a requirement of mathematical rigor [*Strenge*].

Thus the task of providing a consistency proof remains also for those assumptions, according to Hilbert. To handle [*Behandlung*] this task as well as various further [*weitergehender*] fundamental questions, e.g., "the problem of the solvability in principle of every mathematical question" or "the questions of the relation between contentfulness [*? Inhaltlichkeit*] and formalism in mathematics and logic", Hilbert thinks it necessary [*hält es für erforderlich*] to make "the concept of the specific mathematical proof itself the object of an investigation".

In the following years, in particular since 1920, Hilbert devoted himself especially [*vornehmlich*] to the hereby anew plan [*? dem hiermit von neuem gefaßten Plan*] of a proof theory[11]. He gained a reinforced impetus [*verstärkter Antrieb*] from the opposition which Weyl and Brouwer directed at the usual

[11] To collaborate on this enterprise Hilbert then invited P. Bernays with whom he has regularly [*ständig*] discussed his investigations since then.

proceeding [*Verfahren*] in analysis and set theory[12].

Thus Hilbert begins his first communication about his "Neubegründung der Mathematik"[13] by discussing [*? auseinandersetzt*] the objections of Weyl and Brouwer. It is noteworthy in this dispute that Hilbert, despite his energetic rejection of the objections that have been raised against analysis and despite his advocacy for the warranty [*Berechtigung*] of the usual inferences [*Schlußweisen*], agrees with the opposing standpoint that the usual proceeding [*Verfahren*] of analysis is not offhandly acceptable [*? ohne Weiteres einsichtig*] and does not conform to the requirements of mathematical rigor. From his point of view, he "warranty" [*Berechtigung*] that Hilbert awards [*zuerkennt*] the usual method [*Verfahren*] is not based on evidence, but on the reliability [*Zuverlässigkeit*] of the axiomatic method, of which Hilbert explains that if it is appropriate anywhere at all, then it is here. From this view [*Auffassung*] the problem of a proof of consistency for the assumptions of analysis arises.

Moreover, regarding the methodical attitude [*Einstellung*] which Hilbert bases his proof theory upon and which he explains using [*an Hand*] the in-

[12]H. Weyl: "Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis" (Leipzig 1918). — "Der circulus vitiosus in der heutigen Begründung der Analysis" Jber. dtsch. Math.-Ver. Bd. 28 (1919). — "Über die neue Grundlagenkrise der Mathematik" Math. Z. Bd. 10 (1921). L. E. J. Brouwer: "Intuitionisme en formalisme." Inaugural address. Groningen 1912. — "Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten." I and II. Verh. d. Kgl. Akad. d. Wiss. Amsterdam, 1. Sekt., part XII no. 5 and 7 (1918/19). — "Intuitionistische Mengenlehre." Jber. dtsch. Math.-Ver. Bd. 28 (1919). — "Besitzt jede reelle Zahl eine Dezimalbruchentwicklung?" Math. Ann. Bd. 83 (1921).

[13]Talk, given in Hamburg 1922, published in Abh. math. Semin. Hamburg. Univ. Bd. 1 Heft 2, see also this volume essay no. 10.

tuitive [*anschaulich*] treatment of number theory, there is a great approximation [*weitgehende Annäherung*] to the standpoint of Kronecker[14] — regardless of Hilbert's comment [*Stellungnahme*] against Kronecker. This consists in particular in the application of the intuitive [*anschaulich*] concept of number and also that the intuitive form of complete induction, i.e., the inference which is based on the intuitive idea [*anschauliche Vorstellung*] of the "setup" of the numerals [*Ziffern*], is regarded as acceptable [*einsichtig*] and as not requiring any further reduction [*Zurückführung*]. By deciding to adopt this methodical assumption Hilbert also got rid of the reason of the objection that Poincaré had raised at that time against Hilbert's enterprise [*unternehmen*] of the foundation of arithmetic based on the exposition in the talk in Heidelberg.

The rudiment of proof theory, as it is expressed [*niedergelegt*] in the first communication, already contains the detailed formulation [*genauere Ausgestaltung*] of the formalism. In contrast to the Heidelberg talk the sharp separation of the logical-methematical formalism and the contentful "metamathematical" consideration is therein prominent, which is expressed [*ausprägt*] in particular by the distinction of signs "for communication" and symbols and variables of the formalism.

But, the formal restriction of the negation to inequalities [*Ungleichungen*] appears as a remnant of the stadium when this separation had not been per-

[14]In a later talk "Die Grundlegung der elementaren Zahlenlehre" (held in Hamburg 1931. Math. Ann. Bd. 104 Heft 4, an excerpt of it in this volume no. 12) Hilbert has spoken more clearly about this. After mentioning Dedekind's investigation "Was sind und was sollen die Zahlen?" he explains: "Around the same time, thus already for more than a human's life [*Menschenalter*], Kronecker has articulated a view clearly and exemplified it by many examples which today in essence coincides with our finite attitude [*Einstellung*].

formed yet, while a restriction is really only needed in the metamathematical application of negation.

As a characteristic of Hilbert's approach the formalization of the "tertium non datur" by transfinite functions appears already in the first communication. In particular the "tertium non datur" for the whole numbers is formalized with the function function [*Funktionenfunktion*] $\chi(f)$, whose argument is a number theoretic function, and which has the value 0 if $f(a)$ has the value 1 for all number values [*Zahlwerte*] a, but otherwise represents the smallest number value [*Zahlwert*] a for which $f(a)$ has a value different than 1.

The leading idea [*Leitgedanke*] for the proof of the consistency of the transfinite functions (i.e., of their axioms), which Hilbert had ready already then, is not presented in this communication. A proof of consistency is rather provided here only for a certain part of the formalism [*Teilformalismus*]; but this proof has only importance [*Bedeutung*]as an example for a metamathematical proof[15] [*Beweisführung*].

The Leipzig talk "Die logischen Grundlagen der Mathematik"[16] which followed closely after the first communication, we find the approach [*Ansatz*] and realization of proof theory developed further in various respects. I want to mention briefly [*Es seien kurz ... genannt*] the main points in which the presentation [*Ausführungen*] of the Leipzig talk goes beyond those of the first

[15]The method of proof rests here mainly on the fact that the elementary inference rules [*Schlußregeln*] for the implication, which are formalized by the "Axioms of logical inference [*Schlißens*]" (numbered 10 through 13), are not included in the part of the formalism under consideration.

[16]Held at the Deutschen Naturforscher-Kongreß 1922. Math. Ann. Bd. 88 Heft 1/2, this volume essay no. 11.

communication:

1. The reason for the transgression of the intuitive approach [*anschauliche Betrachtungsweise*] by common mathematics, which consists in the unrestricted use of the concepts "all", "there exists" to infinite totalities, is pointed out [*wird aufgezeigt*] and the concept of "finite logic" is elaborated. Also a comparison between the role of "transfinite" formulas and that of ideal elements is carried out [*angestellt*] here for the first time.

2. The formalism is freed from unnecessary restrictions (in particular the avoidance of negation).

3. The formalization of the "tertium non datur" and also of the principle of choice [*Auswahlprinzip*] using transfinite functions is simplified.

4. The main features [*Grundzüge*] of the formalism of analysis are developed.

5. The proof of consistency is provided for the elementary number theoretic formalism, which results from the exclusion of the bound variables. The task of proving the consistency of number theory and analysis is then focused [*konzentriert sich damit*] to the treatment of the "transfinite axiom"

   $$A(\tau(A)) \to A(a),$$

   which is employed in two ways, since the argument of A is related on the one hand to the domain of ordinary [*gewöhnlichen*] numbers and on the other hand to the number series [*Zahlenfolgen*] (functions).

6. A method is stated for the treatment of the "transfinite axiom" in the consistency proof which is successful at least in the simplest cases [*welches jedenfalls in den einfachsten Fällen zum Ziel führt*].

The basic form of its structure was reached with the formulation [*Gestaltung*] of proof theory as presented in the Leipzig talk.

The two publications of Hilbert on proof theory following next, the Münster talk "Über das Unendliche"[17] and the (second) talk in Hamburg "Die Grundlagen der Mathematik"[18], in which the basic idea and the formal approach of proof theory is presented anew and in more detail, still show various changes and extensions in the formalism. However, they serve only in smaller part the original goal of proof theory; they are brought up mainly with respect to the plan to solve Cantor's continuum problem, i.e., the proof of the theorem that the continuum (the set of real numbers) has the same cardinality [*Mächtigkeit*] as the set of numbers of the second number class.

Hilbert had the idea to order the number theoretic functions, i.e., the functions that map every natural number to another such — (the elements of the continuum surely can be represented by such functions) — after the kind [*Gattung*] of the variables which are needed for their definition, and to attain a mapping of the continuum to the set of numbers of the second number class on the basis of the rise [*Aufstieg*] of the variable kinds [*Variablen-Gattungen*], which is analog to that of the transfinite ordinal numbers [*? Ordnungszahlen*]. But the pursuit of this goal did not get beyond a sketch [*Entwurf*], and Hilbert

[17]Presented in 1925 on the occasion of a meeting organized in honor of the memory of Weierstrass, published in Math. Ann. vol. 95.

[18]Presented in 1927, published in Abh. math. Semin. Hamburg. Univ. Bd. IV Heft 1/2.

has therefore left out the parts which refer to the continuum problem in the reprints of both mentioned talks in "Grundlagen der Geometrie"[19].

Hilbert's considerations about the treatment of the continuum problem have produced various fruitful suggestions [*Anregung*] and viewpoints [*Gesichtspunkte*] anyhow [*Gleichwohl*].

Thus W. Ackermann has been inspired to his investigation "Zum Hilbertschen Aufbau der reellen Zahlen"[20] by the considerations regarding the recursive definitions. Hilbert lectures [*referiert*] in his talk in Münster on the question and the result of this paper (which had not been published at the time): "Consider the function

$$a + b;$$

by iterating n times and equating it follows from this:

$$a + a + \ldots + a = a \cdot n.$$

Likewise one arrives from

$$a \cdot b \text{ to } a \cdot a \cdot \ldots a = a^n,$$

further from

$$a^n \text{ to } a^{(a^a)}, a^{(a^{(a^a)})}, \ldots.$$

So we successively obtain the functions

$$a + b \quad = \quad \varphi_1(a, b),$$

[19]Both talks are included in the seventh edition of "Grundlagen der Geometrie" as appendix VIII and IX. Other than the omissions also small editorial changes have been done, in particular with respect to the notation of the formulas.

[20]Math. Ann. Bd. 29 Heft 1/2 (1928).

$$
\begin{aligned}
a \cdot b &= \varphi_2(a, b),\\
a^b &= \varphi_3(a, b).
\end{aligned}
$$

$\varphi_4(a, b)$ is the b^{th} value in the series:

$$a, a^a, a^{(a^a)}, a^{(a^{(a^a)})}, \ldots .$$

In analogous way one obtains $\varphi_5(a, b), \varphi_6(a, b)$ etc.

It would now be possible to define $\varphi_n(a, b)$ for variable n by substitution [*Einsetzung*] and recursion; but these recursions would not be ordinary successive ones, but rather one would be lead to a crossed [*? verschränkte*] recursion of different variables at the same time (simultaneous), and to resolve this into ordinary successive recursions is only possible by using the concept of a function variable [*Funktionsvariablen*]: the function $\varphi_a(a, a)$ is an example for a function of the number variable a, which can not be defined by substitution [*Einsetzung*] and ordinary successive recursion alone, if one allows only for number variables [*Zahlenvariable*][21]. How the function $\varphi_a(a, a)$ can be defined using function variables [*Funktionsvariablen*] is shown by the following formulas:

$$
\begin{aligned}
\iota(f, a, 1) &= a,\\
\iota(f, a, n+1) &= f(a, \iota(f, a, n));\\
\varphi_1(a, b) &= a + b\\
\varphi_{n+1}(a, b) &= \iota(\varphi_n, a, b).
\end{aligned}
$$

Here ι stands for an individual function with two arguments, of which the first one is itself a function of two ordinary number variables [*Zahlenvariablen*]."

[21]W. Ackermann has provided a proof for this claim. (Footnote [*Anmerkung*] in Hilbert's text.)

The investigation of recursive definitions has been newly continued by Rozsa Péter. She proved that all recursive definitions which proceed [*fortschreiten*] only after the values of *one* variable and which do not require any other sort of variables [*Variablenart*] than the free number variables [*Zahlenvariablen*], can be reduced to the simplest recursion schema. Using this result she also simplified substantially the proof of the paper of Ackermann just mentioned[22].

These results concern the use of recursive definitions to obtain [*zur Gewinnung*] number theoretic functions. In the plan of Hilbert's proof [*Beweisplan*] recursive definition also occurs in a different way, namely as method [*Verfahren*] to form [*Bildung*] numbers of the second number class and also sorts of variables [*? Variablengattungen*]. Hereby certain general formations of concepts [*Begriffsbildungen*] concerning the sorts of variables [*Variablenarten*] are assumed, of which Hilbert gives the following short summary in the talk "Die Grundlagen der Mathematik":

"The *mathematical variables* are of two kinds:

1. the *basic variables* [*Grundvariablen*],

2. the *sorts of variables* [*? Variablengattungen*].

1. While one gets by with the ordinary whole number as the only basic variable in all of arithmetic and analysis, now a basic variable for each one of Cantor's transfinite number class is added, which is able to assume [*? anzunehmen*] the ordinal numbers belonging to this class. Each basic variable

[22] See R. Péter: "Über den Zusammenhang der verschiedenen Begriffe der rekursiven Funktionen." Math. Ann. Bd. 111 Heft 4 (1935).

corresponds accordingly to a proposition [*Aussage*], that is characterized by it as such; this is defined implicitly by axioms.

To each basic variable belongs a kind of recursion, which is used to define functions whose argument is such a basic variable. The recursion belonging to the number variable is the "usual recursion" by which a function of a number variable n is defined by specifying which value it has for $n = 0$ and how the value for n' is obtained from the value at n[23]. The generalization of the usual recursion is the transfinite recursion, whose general principle is to determine the value of the function for a value of the variable using the previous values of the function.

2. We derive further kinds of variables from the basic variables by applying logical connectives [*Verknüpfungen*] to the propositions for the basic variables, e.g., to Z[24]. The so defined variables are called sorts of variables [*? Variablengattungen*], the so defined propositions are called propositions about sorts [*? Gattungsaussagen*]; each time a new symbol for individuals [*Individualzeichen*] is introduced for them. Thus the formula

$$\Phi(f) \sim (x)(Z(x) \to Z(f(x)))$$

yields the simplest example of a sort of variables [*? Variablengattung*]; this formula defines the sort of function variables (being a function [*Funktionsein*]). A further example is the formula

$$\Psi(g) \sim (f)(\Psi(f) \to Z(g(f)));$$

it defines the "being a function function" [*Funktionenfunktion-sein*]; the argument g is the new function function variable [*Funktionenfunktionsvariable*].

[23]Here n' is the formal expression for "the number following n".

[24]The formula $Z(a)$ corresponds to the proposition "a is an ordinary whole number".

For the construction [*Herstellung*] of higher variable sorts [*Variablengattungen*] the propositions about sorts [*? Gattungsaussagen*] have to be equipped with indices which enables a method of recursion [*Rekursionsverfahren*]."

These concept formations [*Begriffsbildungen*] are applied in particular in the theory of numbers of the second number class. Here a new suggestion [*Anregung*] emerged from Hilbert's conjecture that every number of the second number class can be defined without transfinite recursion, but using ordinary recursion alone — assuming a basic element [*Ausgangselement*] 0, the operation of progression [*Fortschreitens*] by one ("Stroke-function" [*Strichfunktion*]) and the limit process [*Limesprozess*], furthermore the number variable [*Zahlenvariable*] and the basic variable of the second number class —.

The first examples of such definitions that go beyond the most elementary cases, namely the definition of the first ε-number (in Cantor's terminology) and the first critical ε-number[25], have already been given by P. Bernays and J. v. Neumann. Hereby already recursively defined sorts of variables [*Variablengattungen*] are used[26].

[25]An ε-number is a transfinite ordinal number [*Ordnungszahl*] α with the property $\alpha = \omega^\alpha$. The first ε-number is the limit of the series

$$\alpha_0, \alpha_1, \alpha_2, \dots,$$

where $\alpha_0 = 1$, $\alpha_{n+1} = \omega^{\alpha_n}$. the first critical ε-number is the limit of the series

$$\beta_0, \beta_1, \beta_2, \dots,$$

where $\beta_0 = 1$, β_{n+1} is the β_n-th ε-number.

[26]Cf. the statement [*Angabe*] in Hilbert's talk "Die Grundlagen der Mathematik" ("Grundlagen der Geometrie", 7 ed. appendix IX, p. 308. — The examples mentioned have not been published yet.

But these various considerations, which refer to the recursive definitions, already transgress the narrower domain of proof theoretic questions [*Problemstellung*]. Since Hilbert's Leipzig talk it was the task of this narrower field of investigation [*Problemgebiet*] of proof theory to prove the consistency according to Hilbert's approach, including the transfinite axiom. Shortly after the talk in Leipzig the transfinite axiom was brought into the form of the logical "ε-axiom"

$$A(a) \to A(\varepsilon_x A(x))$$

by the introduction of the choice function [*Auswahlfunktion*] $\varepsilon(A)$ (in detail: $\varepsilon_x A(x)$) replacing the earlier function $\tau(A)$. The role of this ε-axiom is explained by Hilbert in his talk in Hamburg in with following words:

"The ε-function is applicable in the formalism in three ways, namely as follows:

a) It is possible to define "all" and "there exists" with the help of ε, namely as follows[27]:

$$(x)A(x) \sim A(\varepsilon_x \overline{A(x)}),$$
$$(Ex)A(x) \sim A(\varepsilon_x A(x)).$$

Based on this definition the ε axioms yields the valid logical notations [*Bezeichnungen*] for the 'for all' and 'there exists' symbols [*All- und Seinszeichen*], like

$$(x)A(x) \to A(a) \quad \text{(Aristotelian axiom)},$$
$$\overline{(x)}A(x) \to (Ex)\overline{A(x)} \quad \text{(Tertium non datur)}.$$

[27] Instead of the double arrow used by Hilbert the symbol of equivalence $\sim$ is applied in both following formulas; the remarks on the introduction of the symbol $\sim$ in Hilbert's text are thus dispensable.

b) If a proposition [*Aussage*] $\mathfrak{A}$ is true of one and only one thing, then $\varepsilon(frakA)$ is *that thing* for which $\mathfrak{A}$ holds.

Thus, the ε-function allows to resolve [*auflösen*] such a proposition $\mathfrak{A}$ that holds of only one thing into the form

$$a = \varepsilon(\mathfrak{A}).$$

c) Moreover, the ε plays the role of a choice function, i.e., in the case that $\mathfrak{A}$ holds of more than one thing, $\varepsilon(\mathfrak{A})$ is *any* of the things a of which $\mathfrak{A}$ holds."

The ε-axiom can be applied to different sorts of variables [*Variablengattungen*]. For a formalization of number theory the application to number variables suffices, i.e., the sort [*Gattung*] of natural numbers. In this case the number theoretic axioms

$$a' \neq 0,$$
$$a' = b' \rightarrow a = b,$$

moreover the recursive equations for addition and multiplication[28] and the principle of inference [*Schlußprinzip*] of complete induction, have to be added to the the logical formalism and the axioms of equality. This principle of inference can be formalized using the ε symbol by the formula

$$\varepsilon_x A(x) = b' \rightarrow \overline{A(b)}$$

in connection with the elementary formula

$$a \neq 0 \rightarrow a = (\delta(a))'.$$

[28]Cf. footnote 1 on p, 197 of this report for this.

The additional formula for the ε symbols corresponds to a part of the statement [*Teilaussage*] of the least number principle [29] and the added elementary formula represents the statement that for every number different than 0 there is a preceeding one.

For the formalization of analysis one has to apply the ε-axiom also to a higher sort of variables [*Gattung von Variablen*]. Different alternatives are possible here, depending on whether one prefers the general concept of predicate, set or of function. Hilbert chooses the sort of function variables [*Funktionsvariablen*], i.e., more precisely, of the variable number theoretic function of one argument.

The introduction of higher sorts of variables [*Variablengattung*] allows for the replacement of the inference principle [*Schlußprinzip*] of complete induction by a definition of the concept of natural number after the method of Dedekind.

The fundamental moment of the extension of this formalism is based on the connection between the ε-axiom and the insertion rule [*Einsetzungsregel*] for the function variable, whereby the "impredicative definitions" of functions, i.e., the definitions of functions in reference to the totality of functions, are incorporated into the formalism.

The task of proving the consistency for the number theoretic formalism and for analysis is hereby mathematically sharply delimited [*umgrenzte*]. For its treatment one had Hilbert's approach at one's disposal, and at the beginning it seemed that only an insightful and extensive effort [*verständnisvollen und eingehenden Bemühung*] was needed to develop this approach to a complete

[29] I.e., the principle of the existence of a least number in every nonempty set of numbers.

proof.

However, this vision [*Vorstellung*] has been proved mistaken. In spite of intensive efforts and a manifold of contributed proof ideas [*Beweisgedanken*] the desired goal has not been achieved. The expectations that had been entertained [*die gehegten Erwartungen*] have been disappointed step by step, whereby it also became apparent [*geltend machte*] that the danger of mistake is particularly great in the domain of metamathematical considerations.

At first the proof for the consistency of analysis seemed to succeed [*gelingen*], but this appearance soon revealed itself [*erweisen*] as an illusion. Hereafter one believed to have reached the goal at least for the number theoretic formalism. Hilbert's talk in Hamburg "Die Grundlagen der Mathematik" falls into this stadium, where he brings a report on a consistency proof by Ackermann at the end, as well as the talk "Probleme der Grundlegung der Mathematik"[30], held 1928 in Bologna, where Hilbert gave an overview of the situation of the problem in proof theory of at that time [*damaligen Problemstand*] and presented in part problems of consistency and in part problems of completeness.

Here Hilbert connects all problems of consistency to the ε-axiom, presenting the mathematical domains that are encompassed in place of the various formalisms.

The view that was shared at that time by all parties [*allen Beteiligten*], that the proof for the consistency of the formalism of number theory has been given already by the investigations of Ackermann and v. Neumann, is expressed in this presentation.

[30]Math. Ann. Bd. 102 Heft 1.

That in fact this goal had not been achieved yet was only realized when it became dubious, based on a general theorem of K. Gödel, whether it was at all possible to provide a proof for the consistency of the number theoretic formalism with elementary combinatorial methods in the sense of the "finite standpoint".

The theorem mentioned is one of the different important results of Gödel's paper [*Abhandlung*] "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I"[31], which has brought fundamental enlightenment [*Aufklärung*] with regard to the relation between contentfulness [*Inhaltlichkeit*] and formalism — whose investigation has been mentioned by Hilbert in "Axiomatisches Denken" as one of the aims [*Zwecke*] of proof theory —.

The basic message [*Aussage*] of the theorem is that a proof for the consistency of a consistent formalism, which encompasses the usual logical calculus [*Logikkalkül*] and number theory, cannot be represented in this formalism itself, more precisely: it is not possible to deduce the elementary arithmetical theorem which represents the claim of the consistency of the formalism — based on a certain kind of enumeration of the symbols and variables and an enumeration of the formulas and of the finite series of formulas derivated from it — in the formalism itself.

To be sure, nothing is said hereby directly about the possibility of finite consistency proofs; but a criterion follows, which every proof of the consistency for a formalism of number theory or a more comprehensive [*umfassend*] formalism has to meet: a consideration must occur in the proof which can not

[31]Mh. Math. Physik Bd. 38 Heft 1 (1931).

be represented — based on the arithmetical translation — in the formalism mentioned.

By means of this criterion one became aware that the existing consistency proofs were not yet sufficient for the full formalism of number theory[32].

Moreover, the conjecture was excited [*erweckt*] that it was in general impossible to provide a proof for the consistency of the number theoretic formalism within the framework of the elementary intuitive [*anschaulichen*] considerations that conformed [*entsprachen*] to the "finite standpoint" which Hilbert had based proof theory upon.

This conjecture has not been disproved yet[33]. However, K. Gödel and G. Gentzen have noticed[34] that it is rather easy to prove the consistency of the usual formalism of number theory[35] assuming the consistency of intuitionistic arithmetic[36] as formalized by A. Heyting.

From the standpoint of Brouwer's Intuitionism the proof of the consistency of the formalism of number theory has hereby been achieved. But this does not disprove the conjecture mentioned above, since intuitionistic arith-

[32]V. Neumann's proof referred to a narrower formalism from the outset; but it appeared that the extension to the entire formalism of number theory would be without difficulties.

[33]But see postscriptum [*Nachtrag*] on p. 216.

[34]K. Gödel: "Zur intuitionistischen Arithmetik und Zahlentheorie". Erg. math. Kolloqu. Wien 1933 Heft 4. G. Gentzen has withdrawn his paper about the subject matter which was already in print because of the publication of Gödel's note.

[35]Namely it is possible to show that every formula that is deducible in the usual formalism of number theory, which does not contain any formula variable, disjunction, or existential quantifier [*Seinszeichen*], can be deduced also in Heyting's formalism.

[36]A. Heyting: "Die formalen Regeln der intuitionistischen Logik" and "Die formalen Regeln der intutionistischen Mathematik", S.-B. preuß. Akad. Wiss. Phys.-math. Klasse 1930 II.

metic goes beyond the realm of intuitive, finite considerations by having also contentful proofs as objects [*das inhaltliche Beweisen zum Gegenstand macht*] besides the proper mathematical objects, and therefore needing the abstract general concept [*Allgemeinbegriff*] of an insightful implication [*einsichtige Folgerung*]. —

A brief compilation of various finite consistency proofs for formalisms of parts of number theory that have been given will be presented here. Let the formalism which is obtained from the logical calculus [*Logikkalkül*] (of first order [*der ersten Stufe*]) by adding axioms for equality and number theory, but where the application of complete induction is restricted to formulas without bound variables, be denoted by F_1; with F_2 we denote the formalism that results from F_1 by adding the ε-symbol and the ε-axiom, — whereby the formulas and schemata for the universal and existential quantifiers [*Allzeichen und Seinzeichen*] can be replaced by explicit definitions of the universal and existential quantifiers [*Allzeichen und Seinzeichen*][37]. A consistency proof for F_2 immediately results in the consistency of F_1.

The consistency of F_2 is shown:

1. by a proof of W. Ackermann, which starts with the approach presented in Hilbert's Leipzig talk "Die logischen Grundlagen der Mathematik"[38];

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[37]See in this paper p. 209. — With regard to the axioms of equality it is to note that they appear in the formalism in the more general form

$$a = a, a = b \to (A(a) \to A(b))$$

so that in particular the formula

$$a = b \to \varepsilon_x A(x, a) = \varepsilon_x A(x, b)$$

can be deduced. In the formalism F_1 the formula

$$a = b \to (A(a) \to A(b))$$

can be replaced [*vertreten*] by the more special axioms

$$a = b \to (a = c \to b = c), a = b \to a' = b'.$$

[38]The end of the proof is not yet carried out in detail in Ackermann's dissertation "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie der Widerspruchsfreiheit" [Math. Ann. Bd. 93 (1924)]. Later Ackermann provided a complete and at the same time more simple proof [*Beweisführung*]. This definite version of Ackermann's proof has not been published yet, but only Hilbert's already mentioned report in his second talk in Hamburg "Die Grundlagen der Mathematik" and the more detailed "Appendix" [*Zuatz*] by P. Bernays which appeared with the talk in Abh. math. Semin. Hamburg Univ. Bd. 6 (1928) are available. (The remark at the end the appendix with regard to the inclusion of complete induction has to be abandoned.

2. by a proof by J. v. Neumann, who starts from the same assumptions[39];

3. using a second so far unpublished approach of Hilbert by Ackermann; the idea behind this approach consists in applying a disjunctive rule of inference to eliminate the ε symbol instead of replacing the ε by number values [*Zahlenwerte*][40].

The consistency of F_1 is shown:

1. by a proof of J. Herbrand which rests on a general theorem about the logical calculus that has been stated for the first time and proved by Herbrand in his thesis "Recherches sur la théorie de la démonstration"[41];

2. by a proof of G.Gentzen, which results from a sharpening [*Verschärfung*] and extension of Herbrand's theorem mentioned above found by Gentzen[42].

For the time being [*einstweilen*] one has not gone beyond these results, which are important mainly for theoretical logic and elementary axiomatic, and the uncovering mentioned before of the relation between the usual number theoretic formalism and that of intuitionistic arithmetic.

But all problems of completeness which Hilbert posed in his talk "Probleme der Grundlegung der Mathematik" have been treated in various directions.

One of these problems deals with the proof of the completeness of the system of logical rules which are formalized in the logical calculus (of first

[39]J. v. Neumann: "Zur Hilbertschen Beweistheorie." Math. Z. Bd. 26 (1927).

[40]Cf. the statement [*Angabe*] in the talk "Methoden des Nachweises von Widerspruchsfreiheit und ihre Grenzen" Verh. d. int. Math.-Kongr. Zürich 1932, second volume, by P. Bernays.

[41]Thèse de l'Univ. de Paris 1930, published in Travaux de la Soc. Sci Varsovie 1930.

[42]G. Gentzen: "Untersuchung über das logische Schließen" Math. Z. Bd. 39 Heft 2 u. 3 (1934).

order [*Stufe*]). This proof has been given by K. Gödel in the sense that he showed[43]: if it can be shown that a formula of the first order logical calculus is not deducible, then it is possible to give a counterexample to the universal validity [*Allgemeingültigkeit*] of that formula in the framework of number theory[44].

Another problem of completeness regards the axioms of number theory; it should be shown: "If the consistency with the axioms of number theory can be shown for a sentence[45] $\mathfrak{S}$, then the consistency with those axioms can not be also shown for the sentence $\overline{\mathfrak{S}}$ (the opposite of $\mathfrak{S}$)."

This problem contains insofar an indeterminateness, as it is not specified on which formalism of logical inference it should be based. However, it was shown that the claim of completeness is justified for all logical formalisms, as long as one maintains the requirement of the rigorous formalization of the proofs.

This result stems again from K. Gödel, who proved the following general theorem in the paper "Über formal unendscheidbare Sätze der Principia Mathematica und verwandter Systeme I" mentioned already before: If a formalism $\mathfrak{F}$ is consistent in the sense that it is impossible to deduce the negation of a formula $(x)\mathfrak{A}(x)$ if the formula $\mathfrak{A}(\mathfrak{z})$ can be deduced in $\mathfrak{F}$ for all numerals $\mathfrak{z}$ [*Ziffer*], and the formalism is sufficiently extensive [*umfassend*] to contain the formalism of number theory (or an equivalent formalism), then

[43]K. Gödel: "Die Vollständigkeit der Axiome des logischen Funktionenkalküls." Mh. Math. Pysik Bd. 37 Heft 2 (1930).

[44]using the "tertium non datur", in particular in the form of the least number principle

[45]A sentence is meant which can be represented in the formalism of number theory without free variables.

it is possible to state a formula with the property that neither itself nor its negation is deducible[46]. Thus, under the conditions mentioned above, the formalism $\mathfrak{F}$ does not have the property of deductive completeness (in the sense of Hilbert's claim for the case of number theory)[47].

Even before this result of Gödel was known Hilbert already had given up the original form of his problem of completeness. In his talk "Die Grundlegung der elementaren Zahlenlehre"[48] he treated the problem for the special

[46]Moreover this formula has the special form

$$(x)(\varphi(x) \neq 0),$$

where $\varphi(x)$ is a function defined by elementary recursion, and the non-deducibility of this formula as well as the correctness [*Richtigkeit*] and deduciblity of the formula $\varphi(\mathfrak{z}) \neq 0$) follows already from the consistency in the ordinary sense without the more restricted requirement mentioned above.

[47]A different kind of incompleteness has been shown recently by Th. Skolem for the formalism of number theory ("Über die Unmöglichkeit einer vollständigen Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems." Nordk. Mat. Forenings Skifter, Ser. II Nr. 1–12 1933). The formalism is not "categorical" (the term is used in analogy to the denotation of O. Veblen), as it is possible to state an interpretation of the relations $=, <$ and of the functions $a', a+b, a \cdot b$ in relation to a system of things (they are number theoretic functions) — using the "tertium non datur" contentfully [*inhaltlich*] for whole numbers —, such that on the one hand every number theoretic theorem that can be deduced in the formalism of number theory remains true also for that interpretation, but on the other hand that the system is by no means isomorphic to the number sequence (with regard to the relations under consideration), but that it contains in addition to the subset that is isomorphic to the number sequence also elements that are *greater* (in the sense of the interpretation) than all elements of that subset.

[48]Held 1930 in Hamburg, published in Math. Ann. Bd. 104 Heft 4, this collection essay no. 12.

case of formulas of the form $(x)\mathfrak{A}(x)$, which do not contain any bound variables other than x. He modified the task by adding an inference rule which says that a formula $(x)\mathfrak{A}(x)$ of the kind under consideration can be always taken as a basic formula [*? Ausgangsformel*] if it is possible to show that the formula $\mathfrak{A}(\mathfrak{z})$ represents a true statement (according to the elementary arithmetic interpretation [*Deutung*]) for all numerals [*Ziffer*] $\mathfrak{z}$.

With the addition of this rule the result follows very easily from the fact that if a formula of the special form under consideration is consistent is also true [*zutreffend*] under the contentful interpretation[49].

The method by which Hilbert enforces, so to speak, the positive solution of the completeness problem (for the special case that he considers) means a deviation [*bedeutet ein Abgehen*] from the previous program of proof theory. In fact, the requirement for a complete [*restlos*] formalization of the rules of inference [*Schlüsse*] is abandoned by the introduction of the additional inference rule.

One does not have to regard this step as final. But one will consider [*ins Auge fassen*] the possibility of extending the hitherto methodological framework of the metamathematical considerations in the light of the difficulties that have come up with the problem of consistency.

This previous framework is not explicitly required by the basic ideas [*Grundgedanken*] of Hilbert's proof theory. It will be crucial [*darauf ankom-*

[49] Hilbert had already mentioned earlier this fact in his second Hamburg talk "Die Grundlagen der Mathematik". There he used it to show that the finite consistency proof for a formalism also yields a general method for obtaining a finite proof from a proof of an elementary arithmetical theorem in the formalism, for example of the character of Fermat's theorem.

men] for the further development of proof theory if one succeeds to develop the finite standpoint appropriately [*in sachgemäßer Weise*], such that the main goal, the proof of the consistency of usual analysis, remains achievable — regardless of the restrictions of the goals of proof theory that follow from Gödel's results —.

During the printing [*? Drucklegung*] of this report [*? Referat*] the proof for the consistency of the full number theoretic formalism has been presented by G. Gentzen[50], using a method that conforms to the fundamental demands [*Anforderung*] of the finite standpoint. Thereby the mentioned conjecture of the range [*Reichweite*] of the finite methods (p. 212) is disproved.

[50]This proof will be published soon in the Math. Ann.