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The Philosophy of Mathematics and Hilbert's Proof Theory (1930)

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Comments:

The beginning of this text (until p. 24 in Abhandlungen) is based on Paolo Mancosu's translation. The rest is based on Ian Mueller's translation.—Revised by Steve Awodey, Bernd Buldt, Dirk Schlimm, and Wilfried Sieg. (7/17/02)

Part I: The Nature of Mathematical Knowledge

||^{A17} When we read and hear today about the foundational crisis in mathematics or of the dispute between "formalism" and "intuitionism," then those who are unfamiliar with the activity of mathematical science may think that this science is shaken to its very foundations. In reality, mathematics has

been moving for a long time in such quiet waters, that one rather senses a lack of stronger sensations, although there is no lack of significant systematic advances and brilliant achievements.

In fact, the current discussion of the foundations of mathematics does not spring at all from a predicament of mathematics itself. Mathematics is in a completely satisfactory state of methodical certainty. In particular, the concern caused by the paradoxes of set theory has long been overcome, ever since it was recognized that, for avoiding the known contradictions, it is sufficient to make restrictions that do not encroach in the least on the demands of mathematical theories on set theory.

The problems, the difficulties, and the differences of opinion really begin only when one inquires not just about the mathematical facts, but about the epistemological foundations and the demarcation of mathematics. These philosophical questions have become more urgent since the transformation, which the methodical approach of mathematics underwent towards the end of the nineteenth century.

The characteristic aspects of this transformation are: the emergence of the concept of set, by means of which the rigorous foundation of the infinitesimal calculus was achieved, and further the rise of existential axiomatics, that is, the method of developing a mathematical discipline as the theory of a system of things with certain relations whose properties constitute the content of the axioms. In addition, as a consequence of the two aforementioned ||[A18] aspects, a closer connection between mathematics and logic is established. ||[235]

This development confronted the philosophy of mathematics with a completely new situation and entirely new insights and problems. Since then no agreement has been reached in the discussion of the foundations of mathematics. The present stage of this discussion is centered around the struggle with the difficulties, which are caused by the role of the infinite in mathematics.

The problem of the infinite, however, is neither the first nor the most general question which one has to address in the philosophy of mathematics. Here, the first task is to gain clarity about what constitutes the specific nature of mathematical knowledge. We intend to concern ourselves with this question first and thereby recall the development of the different points of view, although only in terms of general features and not in exact chronological order.

§ 1. The Development of Conceptions of Mathematics

The older conception of mathematical knowledge proceeded from the division of mathematics into arithmetic and geometry; according to it mathematics was characterized as a theory of two particular kinds of domains, that of numbers and that of geometric figures. This division could no longer be maintained in the face of the rise to prominence of arithmetical methods in geometry. Also geometry was not restricted to the study of the properties of figures but was broadened to a general theory of manifolds. The completely changed situation of geometry found a particularly concise expression in Klein's Erlangen Program, which systematically summarized the various branches of geometry from a group-theoretical point of view.

In the light of this situation the possibility arose to incorporate geometry into arithmetic. And since the rigorous foundations of the infinitesimal calculus by Dedekind, Weierstrass, and Cantor reduced the more general concepts of number—as required by the mathematical theory of quantities (rational number, real number)—to the usual ("natural") numbers 1, 2, . . . , the conception emerged that the natural numbers constitute the true object of mathematics and that mathematics is precisely the *theory of numbers*.

$\|^{A19}$ This conception has many supporters. In its favor is the fact that all mathematical objects can be represented through numbers, or combinations of numbers, or through higher set formations obtained from the number sequence. From a foundational perspective the characterization of mathematics as a theory of numbers is already unsatisfactory, because it remains open what one considers here as essential to number. The question concerning the nature of mathematical knowledge is thereby shifted to the question concerning the nature of numbers.

This question, however, appears to be completely idle to the proponents of the conception of mathematics as the science of numbers. They proceed from the attitude common to mathematical thought, that numbers are a sort of things, which by their nature are completely familiar to us, so much so that an answer to the question concerning the nature of numbers could only consist in reducing something familiar to something less familiar. From this standpoint one sees the reason for the special status of numbers in the fact that numbers make up an essential component of the world order. This order is $\|^{236}$ comprehensible to us in a rigorous scientific way just to the extent to which it is governed by the aspect of number.

Opposing this view, according to which number is something completely absolute and final, there emerged soon, in the aforementioned epoch of the development of set theory and axiomatics, a completely different conception. This conception denies that mathematical knowledge is of a particular and characteristic kind and holds that mathematics is to be obtained *from pure logic*. One was led naturally to this conception through axiomatics, on the one hand, and through set theory, on the other.

The new methodical turn in axiomatics consisted in giving prominence to the fact that for the development of an axiomatic theory the epistemic character of its axioms is irrelevant. Rigorous axiomatics demands that in the proofs no other knowledge from the given subject be used than what is expressly formulated in the axioms. This is intended already in Euclid in his axiomatics, even though the program is not completely carried through at certain points.

According to this demand, the development of an axiomatic theory shows the logical dependence of the theorems on the axioms. But for this logical dependence it does not matter whether the axioms placed at the beginning are true sentences or not. It represents a $\|^{A20}$ purely hypothetical connection: If it is as the axioms say, then the theorems hold. Such a separation of deduction from asserting the truth of the initial statements is in no way idle hair splitting. On the contrary, an axiomatic development of theories, without regard to the truth of the fundamental sentences taken as starting points, can be of great value for our scientific knowledge: in this way, on the one hand, it is possible to test, in relation to the facts, assumptions of doubtful correctness by systematic development of their logical consequences; furthermore, the *possibilities of a priori theory construction* can be investigated mathematically from the point of view of systematic simplicity and, as it were, to develop a supply. With the development of such theories mathematics takes over the role of the discipline formerly called *mathematical natural philosophy*.

By completely ignoring the truth of the axioms of an axiom system, also the content of the basic concepts becomes irrelevant, and thus one is lead to completely *abstract from all intuitive content of the theory*. This abstraction is also supported by a second aspect, which comes as an addition to recent axiomatics, as it was developed in particular in Hilbert's *Foundations of Geometry*, and which is, in general, essential for the formation of recent mathematics, namely, the *existential* conception of the theory.

Whereas Euclid always thinks of the figures under consideration as constructed ones, contemporary axiomatics proceeds from the idea of a *system of objects*, which is fixed in advance. In geometry, for example, one conceives of the points, lines, and planes in their totality as such a system of things. Within this system one considers the relations of incidence (a point lies on a line, or in a plane), of betweenness (a point lies between two others), and of congruence as being determined from the outset. Now, regardless of their intuitive meaning, these relations can be characterized purely abstractly as certain *basic predicates*. (We will use the term "predicate" also in the case of a relation between several objects, so that we also speak of predicates with several subjects.)[1]

$\|^{A21}$ Thus, e.g., in Hilbert's system the Euclidean construction postulate,

[1]This terminology follows a suggestion of Hilbert. It has certain advantages over the usual distinction between "predicates" and "relations" for the conception of what is logical in principle and also agrees with the usual meaning of the word "predicate."

which demands the possibility of connecting two points with a line, is replaced by the existence axiom: For any two points there is always a straight line that belongs to each of the two points. "Belonging to" is here the abstract expression of incidence.

According to this conception of axiomatics, the axioms as well as the theorems of an axiomatic theory are statements about one or several predicates, which refer to the objects of an underlying system. And the knowledge provided to us by the proof of a theorem L, which is carried out by means of the axioms $A_1 \ldots A_k$—for the sake of simplicity we will assume that we are dealing here with only *one* predicate—consists in the realization that, if the statements $A_1 \ldots A_k$ hold of a predicate, then so does the statement L.

What we have before us is, however, a very general proposition about predicates, that is, a proposition of pure logic. In this way, the results of an axiomatic theory, according to the purely hypothetical and existential understanding of axiomatics, present themselves as *theorems of logic*.

These theorems, though, are only significant if the conditions formulated in the axioms can be satisfied at all by a system of objects together with certain predicates concerning them. If such a satisfaction is inconceivable, that is, logically impossible, then the axiom system does not lead to a theory at all, and the only logically important statement about the system is then the observation <*Feststellung*> that a contradiction results from the axioms. For this reason every axiomatic theory requires a proof of the *satisfiability*, that is, *consistency*, of its axioms.

Unless one can make do with direct finite model constructions, this proof is accomplished in general by means of the method of *reduction to arithmetic*, that is, by exhibiting objects and relations within the realm of arithmetic that satisfy the axioms to be investigated. As a result, one is again faced with the question of the epistemic character of arithmetic.

Even before this question became acute in connection with ||[A22] axiomatics, as just described, set theory and logistics had already taken a position on it in a novel way. Cantor showed that the number concept, both in the sense of cardinal number (Number <*Anzahl*>) and in the sense of ordinal number (order number <*Ordnungszahl*>), can be extended to infinite sets. The theory of natural numbers and the theory of positive real numbers <*Maßzahlen*> (analysis) were subsumed as parts under general set theory. Even if natural numbers lost an essential aspect of their distinguished role, nonetheless, from Cantor's standpoint, the number sequence still constitutes something immediately given, the examination of which was the starting point of set theory.

This was not the end of the matter; rather, the logicians soon adopted the stronger claim: sets are nothing but extensions of concepts <*Begriffsumfänge*> and set theory is synonymous with the logic of extension <*Umfangslogik*>, and, in particular, the theory of numbers is to be derived from pure logic. With this thesis, that mathematics is to be obtained from pure logic, an old and favorite idea of rational philosophy, which had been opposed by the Kantian theory of pure intuition, was revived.

Now the development of mathematics and theoretical physics had already shown that the Kantian theory of experience, in any case, was in need of a fundamental revision. As to the radical opponents of Kant's philosophy the moment seemed to have arrived for refuting this philosophy in its very starting point, namely the claim that mathematics is synthetic in character.

This [refutation], however, was not completely successful. A first symptom that the situation was more difficult and complicated than the leaders of the logistic movement had thought became apparent in the discovery of the famous *set-theoretic paradoxes*. Historically, this discovery was the signal for the beginning of the critique. If today we want to discuss the situation philosophically it is more satisfactory to consider the matter directly without bringing in the dialectical argument involving the paradoxes.

§ 2. The mathematical element in logic—Frege's definitions of Number

In fact in order to see what is essential we need only to consider the new discipline of theoretical logic itself, the intellectual achievement <*Gedankenwerk*> of the great logicians Frege, Schröder, Peano, and Russell, ||[A23] and see what it teaches us about the relationship between mathematics and logic <*des Mathematischen zum Logischen*>.

One sees immediately a peculiar twosidedness in this relation which shows itself in a varying conception of the task of theoretical logic: Frege strives to subordinate mathematical concepts under the concept formations of logic, but Schröder, on the other hand, tries to bring to prominence the mathematical character of logical relations and develops his theory as an "algebra of logic."

But the difference here is only a matter of emphasis. In the different systems of logistic one never finds the specifically logical point of view dominating by itself, but rather mixed with a mathematical perspective everywhere from the start. Just as in the area of theoretical physics, the mathematical formalism and mathematical concept formation here prove to be the appropriate means to represent interconnections and to gain a systematic overview.

To be sure what is applied here is not the usual formalism of algebra and analysis, but a newly created calculus developed by theoretical logic on the basis of the formula language used to represent the logical connectives. No one familiar with this calculus and its theory will doubt its explicit mathematical character.

The first requirement that arises in connection with this situation is the delimitation of the concept of the mathematical independently of the actual situation in the mathematical disciplines by means of a principled characterization of the nature of mathematical knowledge [*Erkenntnisart*]. If we examine what is meant by the mathematical character of a consideration, it becomes apparent that the distinctive feature is a certain kind of abstraction that is involved. This abstraction, which may be called *formal* or *mathematical abstraction*, consists in emphasizing and taking exclusively into account the structural aspects of an object, that is, the manner of its composition from parts; "object" is understood here in its widest sense. One can, accordingly, define mathematical knowledge as resting on the *structural* consideration of objects.

The study of theoretical logic teaches us, furthermore, that in the relationship between mathematics and logic the mathematical point of view, in contrast to the contentual logical one, is $\|^{A24}$ under certain circumstances the more abstract one. The aforementioned analogy between theoretical logic and theoretical physics extends as follows: just as the mathematical laws of theoretical physics are contentually specialized by their physical interpretation, so the mathematical relationships of theoretical logic are also specialized through their contentual logical interpretation. *The laws of the logical relations appear here as a special model for a mathematical formalism.*

This peculiar relation between logic and mathematics—not only can mathematical judgments and inferences be subjected to logical abstraction, but also logical relationships can be subjected to mathematical abstraction—is based on the special role of the formal realm with respect to logic. Namely, whereas in logic one can usually abstract from the specifics of a given subject, this is not possible in the formal realm, because *formal elements enter essentially into logic itself.*

This holds in particular for *logical inference.* Theoretical logic teaches that logical proofs can be "formalized." The method of formalization consists first of all in representing the premises of the proof by specific formulas in the logical formula language, and furthermore in the replacement of the principles of logical inference by rules that specify determinate procedures, according to which one proceeds from given formulas to other formulas. The result of the proof is represented by an end formula, which, on the basis of the interpretation of the logical formula language, presents the proposition to be proved.

Here we use that all logical inference, considered as a process, is reducible to a limited number of logical elementary processes that can be exactly and completely enumerated. In this way it becomes possible to pursue questions of *provability* systematically. The result is a field of theoretical inquiry within which the theory of the different possible forms of categorical inference put forward in traditional logic deals with only a very specific special problem.

The typically mathematical character of the theory of provability reveals itself especially clearly, through the role of the logical *symbolism*. The symbolism is here the *means for carrying out the formal abstraction*. The transition from the point of view of logical content to the ||[A25] formal one takes place when one ignores the original meaning of the logical symbols and makes the symbols themselves representatives of formal objects and connections.

For example, if the hypothetical relation

"if A then B"

is represented symbolically by

$$A \rightarrow B,$$

then the transition to the formal standpoint consists in abstracting from all meaning of the symbol $\rightarrow$ and taking the connection by means of the "sign" $\rightarrow$ itself as the object to be considered. To be sure one has here a specialization in terms of figures instead of the original specialization of the connection in terms of content; this, however, is harmless insofar as it is easily recognized as an accidental ||[334] feature. Mathematical thought uses the symbolic figure to carry out the formal abstraction.

The method of formal consideration is not introduced here at all artificially; rather it is almost forced upon us when we inquire more closely into the effects of logical inference.

If we now consider why the investigation of logical inference is so much in need of the mathematical method, we discover the following fact. In proofs there are two essential features which work together: the elucidation of concepts, the feature of *reflection* and the mathematical feature of *combination*.

Insofar as inference rests only on elucidation of meanings, it is analytic in the narrowest sense; progress to something new comes about only through mathematical combination.

This combinatorial element can easily appear to be so obvious that it is not viewed as a separate factor at all. With regard to deductively obtained knowledge, philosophers especially were in the habit of considering only what is the precondition of proof as epistemologically problematic and in need of discussion, namely fundamental assumptions and rules of inference. This standpoint is, however, insufficient for the philosophical understanding of mathematics: for the typical effect of a mathematical proof is achieved only after the fundamental assumptions and rules of inference have been fixed. The remarkable character of mathematical results is not diminished when we modify the provable statements contentually by introducing the ultimate assumptions of the theory as premises and in addition explicitly state the rules of inference (in the sense of the formal standpoint).

To clarify the situation we can make-use of Weyl's comparison of a proof conducted in a purely formal way with a game of chess; the fundamental assumptions correspond to the initial position in the game, the rules of inference to the rules of the game. Let us assume that a bright chess master has for a certain initial position A discovered the possibility of checkmating his opponent in 10 moves. From the usual point of view we must then say that this possibility is *logically* determined by the initial position and the rules of the game. On the other hand, one can not maintain that the assertion of the possibility of a checkmate in 10 moves is implied by the specification of the initial position A and the rules of the game. The appearance of a contradiction between these claims disappears ‖[335] if we see clearly that the "logical" effect of the rules of the game depends upon *combination* and therefore does not come about just through analysis of meaning but only through genuine presentation.

Every mathematical proof is in this sense a presentation. We will show here by a simple special case how the combinatorial element comes into play in a proof.

We have the rule of inference: "if A and if A implies B, then B." In a formal translation of a proof this inference principle corresponds to the rule that the formula B can be obtained from the two formulas A and $A \to B$. Now let us apply this rule in a formal derivation, and we furthermore assume that A and $A \to B$ do not belong to the initial assumptions. Then we have a sequence of inferences S leading to A and a sequence T leading to $A \to B$ and according to the rule described the formulas A and $A \to B$ yield the formula B.

If we want to analyze what happens here, we must not prejudge the decisive point by the choice of notation. The endformula of the sequence of inferences T is initially only given as such, and it is epistemologically a new step to recognize that this formula coincides with the one which arises by connecting with a "$\to$" the formula A obtained in some other way and the formula B to be derived.

The determination of an identity is by no means always an identical or tautological determination. The coincidence to be noted in the present case can not be read off directly from the content of the formal rules of inference and the structure of the initial formulas; rather, it can be ‖[A27] read off only from the structure that is obtained by application of the rules of inference, that is to say by the carrying out of the inferences. Thus, a combinatorial element is here present in fact.[2]

If we become in this way clear about the role of the mathematical in logic, then it will not seem astonishing that arithmetic can be subsumed within the system of theoretical logic. But also from the standpoint we have now reached this subsumption loses its epistemological significance. For we know in advance that the formal element is not eliminated by the inclusion of arithmetic in the logical system. ||[336] But with respect to the formal we have found that the mathematical considerations represent a standpoint of higher abstraction than the conceptual logical ones. We therefore achieve no greater generality at all for mathematical knowledge as a result of its subsumption under logic; rather we achieve just the opposite; a *specialization by logical interpretation*, a kind of *logical clothing*.

A typical example of such logical clothing is the method by which Frege and, following him but with a certain modification, Russell defined the natural numbers.

Let us briefly recall the idea underlying Frege's theory. Frege introduces the numbers as cardinal numbers. His premises are as follows:

A cardinal number applies to a *predicate*. The concept of cardinal number arises from the concept of equinumerosity. Two predicates are called equinumerous if the things of which the one predicate holds can be corre-

[2]P. Hertz defended the claim that logical inference contains "synthetic elements" in his essay "Über das Denken" (1923). His grounds for this claim will be explained in an essay on the nature of logic, to appear shortly; they include the point developed here but rest in addition on still other considerations.

lated one-one with the things of which the other predicate holds.

If the predicates are divided into classes by reference to equinumerosity in such a way that all the predicates of a class are equinumerous with one another and predicates of different classes are not equinumerous, then every class represents the *cardinal number* which applies to the predicates belonging to it.

In the sense of this general definition of cardinal number, the particular finite numbers like 0, 1, 2, 3 are defined as follows: ||[A28]

0 is the class of predicates which hold of no thing. 1 is the class of "one-numbered" predicates; and a predicate P is called one-numbered if there is a thing x of which P holds and no other thing different from x of which P holds. Similarly, a predicate P is called two-numbered if there is a thing x and a thing y different from it such that P holds of x and y and if there is no thing different from x and y of which P holds. 2 is the class of two-numbered predicates. The numbers 3, 4, 5 etc. are to be explained as classes in an analogous way. After he has introduced the concept of a number immediately following a number, Frege defines the general concept of finite number in the following way: a number n is called finite if every predicate holds of n, which holds of 0 and which, if it holds of a number a holds of the immediately following number.

||[337] The concept of a number belonging to the series of numbers from 0 to n is explained in a similar way. The formulation of these concepts is followed by the derivation of the principles of number theory from the concept of finite number.

We now want to consider in particular Frege's definition of the individual finite numbers. Let us take the definition of the number 2, which is explained as the class of two-numbered predicates. It may be objected to this explanation that the belonging of a predicate to the class of two-numbered predicates depends upon extralogical conditions and the class therefore constitutes no logical object whatsoever.

This objection is, however, eliminated if we adopt the standpoint of Russell's theory with respect to the understanding of classes (sets or extensions of concepts). According to it classes (extensions of concepts) are not actual objects at all; rather they function only as dependent terms within a reformulated sentence. If, for example, K is the class of things with the property E, i.e. the extension of the concept E, then, according to Russell, the assertion that an thing a belongs to the class K is to be viewed only as a reformulation of the assertion that the thing a has the property E.

If we combine this conception with Frege's definition of cardinal number, we arrive at the idea that the number 2 is to be defined not in terms of the class of two-numbered predicates but in terms of the concept the extension of which constitutes this class. The number 2 is then identified with the *property of two-numberedness* for predicates, i.e. with the ||[A29] property of a predicate of holding of an thing x and of an thing y different[3] from x but of no thing different from x and y.

For the evaluation of this definition it is essential to know how the process of defining is understood here and what claims are involved in it. What will be shown here is that this definition is not a correct reproduction of the

[3]For the sake of simplicity we shall skip the considerations regarding the concept of difference, resp. its contradictory concept of identity.

true meaning of the cardinal number concept “two” by means of which this concept is revealed in its logical purity freed from all inessential features. Rather it will be shown that it is exactly the specifically logical element in the definition that is an inessential addition.

The two-numberedness of a predicate P means nothing else but that there are two things of which the predicate P holds. Here three distinct conceptual features are present: the concept “two things,” the existential feature, and the fact that the predicate P holds. The content ||[338] of the concept “two things” here does not depend on the meaning of either of the other two concepts. “Two things” means something already without the assertion of the existence of two things and also without reference to a predicate which holds of two things; it means simply: “one thing and one more thing.”

In this simple definition the concept of cardinal number shows itself to be an elementary *structural concept*. The appearance that this concept is reached from the elements of logic results, in the case of the logical definition of cardinal number under consideration, only from the fact that the concept is conjoined with logical elements, namely the existential form and the subject-predicate relation, which are in themselves inessential for the concept of cardinal number. Therefore we [will] have here in fact a a formal concept in *logical clothing*.[4]

The result of these considerations is that the claim of the logicists that mathematics is a purely logical field of knowledge shows itself to be imprecise and misleading when theoretical logic is examined more closely. That claim is sound only if the concept of the mathematical is taken in the sense of its

[4]Editorial remark: Check with original article.

historical demarcation and the concept of the logical is systematically broadened. But such a determination of concepts hides what is epistemologically essential and ignores the special nature of mathematics. ||[A30]

§ 3. Formal abstraction

We have determined that formal abstraction, i.e. the focusing on the structural side of objects, is the characteristic feature of mathematical reasoning and have thus demarcated the field of the mathematical in a fundamental way. If we want likewise to gain an epistemological understanding of the concept of the logical, then we are led to separate from the entire domain of the theory of concepts, judgments, and inferences, which is commonly called logic, a narrower subdomain, that of *reflective or philosophical logic*. This is the domain of knowledge which is *analytic in the genuine sense* and which stems from a pure *awareness of meaning*. This philosophical logic is the starting point of systematic logic, which takes its initial elements and its principles from the results of philosophical logic and, using mathematical methods, develops from them a theory.

In this way the extent of genuinely analytic knowledge is separated clearly from that of mathematical knowledge, and it becomes apparent what is justified in *Kant's theory of pure intuition* on the one hand and in the claim ||[339] of the logicists on the other. We can distinguish Kant's fundamental idea that mathematical knowledge and also the *successful application of logical inference rest on an intuitive evidence* from the particular form that Kant gave to this idea in his theory of space and time. By doing this we also arrive at the possibility of doing justice to both the very elementary character of mathematical evidence and to the high degree of abstraction of the mathematical point of view, emphasized in the claim about the logical character of mathematics.

Our conception also gives a simple account of the role of number in mathematics: we have explained mathematics as the knowledge which rests upon the formal (structural) consideration of objects. However, the numbers constitute as cardinal numbers *the simplest formal determinates* and as ordinal numbers the *simplest formal objects*.

Cardinality concepts present a special difficulty for philosophical explication because of their special categorial position, which also makes itself felt in language in the need for a unique species of number words. We do not have to bother here with more detailed explication, but we do have to observe that the determination of cardinal number involves the putting together of a complex given or imagined totality out of components, which is just what constitutes the structural side of an object. And indeed it is the most elementary structural characteristics that are conveyed by cardinal numbers. Thus cardinal numbers play a role in all domains to which formal consideration are applicable; in particular we encounter cardinal number within theoretical logic in a wide variety of ways: for example, as cardinal number of the subjects of a predicate (or as one says, as cardinal number of the arguments of a logical function); as cardinal number of the variable predicates involved in a logical sentence; as cardinal number of the applications of a logical operation involved an a concept-formation or sentence; as cardinal number of the sentences involved in a mode of inference; as the type-number of a logical expression, i.e. the highest number of successive subject-predicate relations involved in the expression (in the sense of the ascent from the objects of a theory to the predicates, from the predicates to the predicates of the predicates, from these latter to their predicates, and so on).

Cardinal numbers, however, provide us only with formal *determinations* and not yet with formal objects. For example, in the conception of the cardinality three there ||[340] is still no unification of three things into one object. The bringing together of several things into one object requires some kind of ordering. The simplest kind of order is that of mere succession, which leads to the concept of *ordinal number*. An ordinal number in itself is also not determined as an object; it is merely a place marker. We can, however, standardize it as an object, by choosing as *place markers the simplest structures deriving from the form of succession*. Corresponding to the two possibilities of beginning the sequence of numbers with 1 or with 0, two kinds of standardization can be considered. The first is based on a sort of things and a form of adjoining a thing; the objects are figures which begin and end with a thing of the sort under consideration, and each thing, which is not yet the end of the figure, is followed by an adjoined thing of that sort. In the second kind of standardization we have an initial thing and a process; the objects are then the initial thing itself and in addition the figures that are obtained by beginning with the initial thing and applying the process one or more times.

If we want to have the ordinal numbers, according to either standardization, as unique objects free from all inessential features, then we must take in each case as object the bare schema of the respective figure of repetition [*Wiederholungsfigur*] which are obtained by repetition; this requires a very high degree of abstraction. $\|^{A32}$ However, we are free to represent these purely formal objects by concrete objects ("number signs" or "numerals"); these then possess inessential arbitrarily added characteristics, which, however, can be immediately recognized as such. This procedure is based on a certain agreement, which must be kept throughout one and the same consideration.[5] Such an agreement, according to the first standardization, is the representation of the first ordinal numbers by the figures $1, 11, 111, 1111$. According to an agreement corresponding to the second standardization, the first ordinal numbers are represented by the figures 0, $0'$, $0''$, $0'''$, $0''''$.

Having found a simple access to the numbers in this way by regarding them structurally, our conception of the $\|^{341}$ character of mathematical knowledge receives a new confirmation. For, the dominant role of number in mathematics becomes clear on the basis of this conception; and our characterization of mathematics as a theory of structures seems to be an appropriate extension of the view mentioned at the beginning of this essay that numbers constitute the real object of mathematics.

The satisfactory features of the standpoint we have reached must not mislead us into thinking that we have already obtained all the fundamental insights required for the problem of the grounding of mathematics. In fact, until now we have only dealt with the preliminary question that we wanted

[5]Philosophers are inclined to treat this relation of representation as a connection of meaning. One must notice, however, that there is an essential difference here from the usual relation of word and meaning; namely the representing thing contains in its constitution the essential properties of the object represented, so that the relationships to be investigated among the represented objects can also be found among the representatives and can be determined by consideration of the latter.

to clarify first, namely, what is the specific character of mathematical knowledge? Now, however, we must turn to the problem that raises the main difficulties in grounding mathematics, the problem of the infinite.

Part II: The problem of the infinite and the formation of mathematical concepts.

§ 1. The postulates of the theory of the infinite. The impossibility of its grounding by intuition.—The finitist standpoint

The mathematical theory of the infinite is analysis (infinitesimal calculus) and its extension by general set theory. We can restrict ourselves here to consideration of the infinitesimal calculus because the step from it to general set theory requires only additional assumptions, but no fundamental change of philosophical conception.

The foundation given to the infinitesimal calculus by Cantor, Dedekind, and Weierstraß shows that a rigorous development of this theory succeeds if two things are added to the elementary inferences of mathematics:

1. the application of the method of existential inference to the integers, i.e. the assumption of the *system* of integers in the manner of a domain of objects of an axiomatic theory, as is explicitly done in *Peano's axioms for number theory*.

2. the conception of the totality of all sets of integers as a combinatorially surveyable manifold. A set of integers is determined by a distribution of the values 0 and 1 to ||[342] the positions in the number series. The number n belongs to the set or not depending on whether the nth position in the distribution is 1 or 0. Just as the totality of possible distributions of the values

0, 1 over a finite number of positions, e.g. over five positions, is completely surveyable, by analogy the same is assumed also for the entire number series.

From this analogy follows in particular also the validity of *Zermelo's principle of choice* for collections of sets of numbers. However, for the time being we will put aside the discussion of this principle, it will fit in naturally at a later point.

If we now consider these requirements from the standpoint of our general characterization of mathematical knowledge, it seems at first that there is no fundamental difficulty in justifying them on that basis. For both in the case of the number series and in that of the sets derived from it, one deals with *structures*, which differ from those treated in elementary mathematics only in being structures of infinite manifolds [*Mannigfaltigkeiten*]. The existential inference applied to numbers also seems to be justified by their objective character as formal objects the existence of which can not depend on accidental facts about people's conceptions of numbers.

Against this argumentation it is to be remarked, however, that it is premature to conclude from the character of formal objects, i.e. from their being free of accidental empirical features, that formal entities must be related to a domain of existing formal things. As an argument against this conception we could put forward the set-theoretic paradoxes; but it is simpler to point out directly that primitive mathematical evidence does not assume such a domain of existing formal entities and that, in contrast, the connection with to what is actually imagined [*das Vorgestellte*] is essential as a starting point for formal abstraction. In this sense the Kantian assertion that pure intuition is the form of empirical intuition is valid.

Correspondingly, existence assertions in disciplines that rest on elementary mathematical evidence do not have a proper meaning. In particular, in elementary number theory we only deal with existence assertions that refer to an explicit totality of numbers that can be exhibited, or to an explicit process that can be executed intuitively, or to both together, i.e. to a totality of numbers that can be obtained by such a process.

||[343] Examples of such existence claims are: "There is a prime number between 5 and 10," namely 7 is a prime number.

"For every number there is a greater one," namely if n is a number, then construct $n + 1$. This number is greater than n.

"For every prime number there is a greater one," namely if a prime number p is given, then construct the product of this number and all smaller prime numbers and add 1. If k is the number obtained in this way, then there must be a prime number among the numbers between $p + 1$ and k.

In each of these cases the existence assertion is made more precise by a further specification; the existence claim is restricted to explicit processes that can be carried out in intuition and makes no reference to a totality of all numbers. Following Hilbert, we will call this elementary point of view, restricted by the requirements imposed by intuitability in principle, the *finitist* standpoint; and in the same sense we will speak of finitist methods, finitist considerations, and finitist inferences.

It is now easy to see that existential reasoning goes beyond the finitist standpoint. This transcending of the finitist standpoint takes place already when any existence assertion is made without a more exact determination of the existence claim, as for example when asserting that there is at least one prime number in every infinite arithmetic sequence

$$a \cdot n + b \quad (n = 0, 1, 2, 3, ...)$$

if a, b are relatively prime numbers.

An especially common and important case of transcending the finitist standpoint is the inference from the failure of an assertion to hold universally (for all numbers) to the existence of a counterexample or, in other words, the principle according to which the following alternative holds for every number predicate $P(n)$: either the universal assertion that $P(n)$ holds of all numbers is valid, or there is a number n of which $P(n)$ does not hold. From the standpoint of existential reasoning this principle results as a direct application of the law of the excluded middle, i.e. from the meaning of negation. This logical consequence fails to hold for the finitist standpoint, because the assertion that $P(n)$ holds for all numbers has here the purely hypothetical sense that the predicate holds for any given number, and thus the negation of this claim does not have the positive meaning of an existence assertion.—

But, this does not yet close the discussion of the possibilities of a **convincing/evident** mathematical foundation for the assumptions of analysis. It has to be admitted that the assumption of a totality of formal objects does not correspond to the standpoint of primitive mathematical evidence, but the $\|^{344}$ demands of the infinitesimal calculus can be motivated by the observation that the totalities of numbers and number sets one deals with are *structures of infinite sets*. In particular, the application of existential reasoning on number would thus not be inferred from the idea of the concept of numbers in the realm of formal objects, but rather from considering the structure of the number sequence in which the individual numbers occur as elements. Indeed we have not yet considered the argument already mentioned that mathematical knowledge can also **concern/be about** structures of infinite manifolds.

Herewith we come to the question of the *actual infinite*. For the infinite insofar as infinite manifolds are concerned, is the true actual infinite in contrast to the "potential infinite;" by the latter is meant not an infinite object but merely the unboundedness of the progression from something finite to something that is again finite. The unboundedness holds, for example, also from the finitist standpoint for numbers, since for every number a greater one can be constructed.

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The question about the actual infinite which we have to ask first is whether the actual infinite is given to us as an object of intuitive-mathematical knowledge.

One could, in harmony with what we have determined up to this point, be of the opinion that we really are capable of an intuitive knowledge of the actual infinite, For even if it is certain that we have a concrete conception only of finite objects, nevertheless the result of formal abstraction could be the following: the elimination of the restriction to what is finite and the passing to the limit in the case of certain indefinitely continuable processes. In particular one is tempted to point to geometric intuition and to introduce examples of intuitively given infinite manifolds from the domain of geometric objects.

Now in the first place geometric examples are never conclusive. One is easily deceived here because he interprets the spatially intuitive on the basis of an existential conception. For example, a lane is not intuitively given as an ordered manifold of points but as a unitary whole, although, to be sure, an extended whole within which *positions* are distinguishable. The conception of one position on the line is intuitive, positions but the totality of *all* positions on the line is merely a mental schematization. From an intuitive point of view we here reach only the potential ‖[345] infinite in which every position on the line corresponds to a division into two shorter lines each of which is in turn divisible into shorter lines yet.

Furthermore, one cannot point to infinitely extended configurations like infinite lines, infinite planes, or infinite space as objects which are intuitively conceived. In particular, space as a whole is not given to us in intuition. We do indeed represent every spatial figure as situated in space. But this relationship of individual objects to the whole of space is objectively given in intuition only to the extent that a spatial environment is represented along with every spatial object. Above and beyond this representation the position in the whole of spice is conceivable *only as an idea*. (Contrary to Kant we must maintain this view.)

The main argument which Kant introduced in favor of the intuitive character of our representation of space as a whole in fact proves only that one cannot attain the concept of an inclusive space via simple the assertion generalizing abstraction. But this says nothing at all about the merely mental accessibility of our conception of the whole of space, the assertion that we are here dealing with what might be called a merely general conception.

However, we have in mind a more complicated situation; in the representation of the whole of space there are, two different kinds of concept both of which go beyond the standpoint of intuition and of reflective (or philosophical) logic. One rests upon the idea of the connection of things with the universe and therefore stems from our notion of reality. The other is a *mathematical idea* which is, to be sure, connected with intuition but is not restricted to the domain of the intuitively representable; it is the representation of space as a manifold of points subject to the laws of geometry.

In neither of these ways of representing space as a whole is the whole apprehended as present; rather it is only put forward for the purpose of investigation. The representation of the whole of physical ||[346] is problematic in a fundamental way; for exactly from the standpoint of contemporary physics there is always the possibility of giving this very vague idea a more narrow and precise formulation by means of which inquiry becomes possible and systematically significant. Geometrical notions of spatial totalities are to be sure precise, but they require a proof that they are consistent.

Thus we have no basis for the assumption that we have an intuitive representation of space as a whole. We can not point directly to such a representation; nor is there any necessity to introduce that assumption as an explanatory principle. If, however, we deny the intuitiveness of the whole of space, we must also deny that infinitely extended spatial configurations ca1 be represented intuitively.

It must also be noted that the natural intuitive conception of elementary Euclidean geometry does not in the least require a representation of infinite figures. in elementary geometry one deals only with finitely extended figures, Also infinite manifolds of points are never involved since there are no underlying general existential assumptions; rather every existential assertion consists in the affirmation of a possible geometric construction. For example, from this standpoint the assertion that every distance has a midpoint says only that for every distance a midpoint can be constructed.[6]

Thus the apparent possibility of showing an actual infinity in the domain of the objects of geometrical intuition is misleading. We can, however, also show in a general way that there is no point in speaking about an elimination of the condition of finitude via formal abstraction in the sense in which such an elimination is required for an intuition of the actually infinite. The requirement of finitude is no accidental empirical limitation but an essential characteristic of a formal object.

Limitation to the empirical still lies within the domain of the finite, but formal abstraction must enable us to go beyond the boundaries of our actual power of representation0 A clear example of this requirement is the unlimited divisibility of a distance. Our actual ability to represent division breaks down when the division reaches a certain degree of fineness. From a physical point of view the boundary here is accidental ||[347] and it can be overcome with the help of optical equipment. But all optical equipment becomes useless after, a certain smallness, and finally our metrical representations of space become entirely meaningless physically. In representing unlimited divisibility we already abstract from the requirements of actual representation as well as from the requirements of physical reality.

The situation is analogous in the case of the representation in number

[6]In Euclid's axiomatization this standpoint is not carried out consistently since one finds here the idea of an *arbitrarily great extension* of a distance. This idea can in fact be avoided; one need only formulate the axiom of parallels differently.

theory of unlimited addition. Here too there are limits to the realizability of repetitions both in the sense of actual representability and in the sense of physical accomplishment. We consider as an example the number $10^{(10^{1000})}$. We can reach this number in a finitist way as follows: we start from the number 10, which, in accordance with. the standardization given earlier, we represent by the figure

1111111111.

Let z be an arbitrary number represented by an analogous figure. If in the representation of 10 we replace each with the figure z, there results, as we can see intuitively, another number-figure which for purposes of communication is called "$10 \times z$."[7] In this way we obtain the process of multiplication by 10. Then we get the process transforming a number a into 10^a by letting the first 1 in a correspond to the number 10 and every subsequent 1 to the process of multiplication by 10 until the end of the figure a is reached. We designate the number which is obtained by this process of multiplying by 10 10^a.

From an intuitive viewpoint this procedure offers no fundamental difficulty. If however we want to realize the process in detail our representation breaks down already in the case of rather small numbers. We can get some further help from instruments or by bringing in from nature objects with respect to which determination of very large numbers are involved. But nevertheless we soon reach a limit: it is easy for us to represent the number 20; $10^2$0 far transcends our actual power of presentation but is definitely within the domain of physical realizability but it is ultimately very questionable

[7]Here we use"meaningful" symbols.

whether the number $10^{(10^{2}0)}$ occurs in any way in physical reality either as a relation between magnitudes or as a cardinal number.

But intuitive abstraction does not depend on such limits on the possibility of realization. For from the formal standpoint these limits are ||[348] accidental. Formal abstraction, so to speak, finds no place to make a distinction in principle until it reaches the difference between finite and infinite.

This difference is in fact fundamental. If we think more precisely about how an infinite manifold as such can be characterized at all, then we realize that such characterization is not at all possible by means of some intuitive indication; rather it is possible only by means of the assertion (or assumption or postulation) of a lawlike connection. Infinite manifolds are therefore only accessible to us *in thought*, Such thought ||[A40] is indeed a kind of representation, but in it lie manifold is not represented as an object, but rather conditions are represented which some manifold satisfies or has to satisfy.

The essential connection of formal abstraction with finitude is made especially clear by the fact that the property of,finitude is not a special limiting characteristic from the standpoint of intuitive evidence in the case of considerations of totalities and of figures. From this standpoint limitation to the finite is direct and, so to speak, *tacit*. In this case we do not need a special definition of finitude because the finitude of objects is for formal abstraction completely self-evident. So, for example, the intuitive or structural introduction of numbers is acceptable only in the case of *finite* numbers. From the intuitive or formal standpoint "repetition" is *eo ipso* finite repetition.

This representation of the finite, which is implicitly given together with the formal point of view, contains the epistemological justification for the principle of complete induction and for the admissibility of recursive definition. (Both procedures are here construed in their elementary form as "finite induction" and "finite recursion.")

Admittedly this introduction of the representation of the finite no longer belongs to what must enter into logical inference from the domain of intuitive evidence, This introduction rather corresponds to the standpoint from which one already *reflects* on the general characteristics of intuitive objects. And also application of the intuitive representation of the finite can be avoided in number theory if one refuses to treat this theory in an elementary way. But the intuitive representation of finitude forces itself upon a person when he makes a formalism the object of consideration, in particular therefore in the case of the systematic ||[349] consideration of logical inference. For in this case it becomes explicit that finitude is an essential feature in the construction of every formalism. However, the limits of a formalism are nothing other than the general representability of intuitive connections.

Thus our answer to the question whether the actually infinite is intuitively knowable turns out to be negative. A further result is that the finitary method from the standpoint of intuitive mathematical knowledge.

||[A41] But in this way we can not verify the presuppositions of the infinitesimal calculus which we have already mentioned.

§ 2. Intuitionism—Arithmetic as a theoretical framework

How should we direct ourselves to this situation? The positions in relation to this question are divided. There are here conflicting views similar to the conflict we found in the case of the question of the characterization of mathematical knowledge. The defenders of the standpoint of primitive intuition infer directly from the fact that the postulates of analysis and set theory transcend the finitary standpoint the conclusion that these mathematical theories must be abandoned in their present form and revised from the ground up. On the other hand those who subscribe to the standpoint of theoretical logic try to provide a foundation in logic for those postulates of the theory of the infinite while attributing no fundamental significance whatsoever to the difference between finite and infinite.

The first conception was already defended by Kronecker at the time of the emergence of the method of existential inference; he was probably the first person to pay close attention to the methodological standpoint which we call finitary and to make its importance explicit. Nevertheless his attempts to satisfy this methodological requirement in the field of analysis remain fragmentary; and he failed to give a more precise philosophical presentation of his standpoint. Thus Kronecker's frequently cited assertion that God has created the integers but everything else is the work of man is particularly inappropriate for motivating the demands put forward by Kronecker:[8] if God created the integers, then one ||[350] would suppose that application of existential inference to numbers is permissible, but Kronecker directly excludes the existential point of view already in the case of the integers.

Brouwer has extended Kronecker's standpoint in two directions: on the one hand with respect to philosophical motivation ||[A42] by putting forward his theory of "intuitionism,"[9] and on the other by showing how one can apply

[8]The methodological standpoint which suits this assertion is the one adopted by Weyl in his work *Das Kontinuum* (1918).

[9]For the sake of clarity in the discussion it seems to me advisable to use the expression

the finitary standpoint in the domain of analysis and set theory and can found these theories, or at least a considerable portion of them, in a finitary way by fundamentally revising the ways of forming concepts and making inferences.

Admittedly the upshot of his investigation has its negative side; for it turns out that in the process of treating analysis and set theory finitistically one must put up with not only great complications but also serious losses from the point of view of systematization.

The complications appear already in connection with the first concepts of the infinitesimal calculus such as boundedness, convergence of a sequence of numbers, or the difference between rational and irrational. We take as an example the concept of the boundedness a sequence of integers. According to the usual conception the following alternative holds: either the sequence exceeds every bound and the sequence is unbounded or every number in the sequence is less than some given bound and the sequence is bounded. In order to determine the concept involved, here finitarily we must make the definition of boundedness and of unboundedness precise in the following way: a sequence is called bounded if we can indicate a bound for the numbers in the sequence either directly or by giving a procedure (for producing it); the sequence is called unbounded if there is a law according to which every bound is necessary exceeded by the sequence, if, therefore, the assumption that the sequence has a limit leads to an absurdity.

When the concept is construed in this way the definition certainly attains a finitary character, but there is no longer a complete disjunction between

"intuitlonism" to designate a philosophical view and to differentiate this expression from the term "finitary," which refers to a specific way of inferring and forming concepts.

boundedness and unboundedness We can therefore not infer from a proof refutinging as impossible the assumption that a sequence is unbounded the conclusion that the sequence is bounded; likewise we cannot consider ||[351] an assertion to be established when it is proved under the assumption that a certain sequence of numbers is bounded and under the assumption that it is unbounded.

||[A43] In addition to complications of this kind which permeate the whole theory there is a yet more essential disadvantage: the general theorems, which are the source of the systematic perspicuousness of mathematics, become unacceptable for the most part. So, for example, not even the assertion that every continuous function has a maximum value in a finite closed interval holds in Brouwerian analysis.

The philosophical demand that mathematics should give up its simpler and more fruitful methods in favor of a more cumbersome method which is inferior from a systematic point of view and without being forced to do so by an internal necessity seems unjustified. This demand makes the intuitionistic standpoint suspect to us.

Let us see what are the main points of the philosophical view developed by Brouwer. It includes first of all a characterization of mathematical evidence. Our earlier treatment of formal abstraction agrees in essential points with this characterization, particularly in with Kant's theory of pure intuition.

Admittedly there is a divergence insofar as according to Brouwer's conception the feature of temporality belongs essentially to mathematical reality. Here we do not need to enter upon a discussion of this point since it has no influence on the position we have taken with respect to the question of mathematical methodology: the conclusion which Brouwer draws from the connection of mathematics with time is nothing but what we conclude from the connection of formal abstraction with its concrete, intuitive starting point, namely the methodological restriction of finitary procedure.

The decisive consequences of intuitionism first result from the further assertion that all mathematical thinking which is to be able to claim scientific validity must be carried out on the basis of mathematical evidence, that therefore the limits of mathematical evidence are at the same time limits for mathematical thought in general.

The demand that mathematical thought be limited to the intuitively evident appears at first to be completely justified. Indeed it corresponds to the conception of mathematical certainty with which we are familiar. We must however remember that this customary conception of ||[352] mathematical certainty originally went together with a philosophical view according to which there was no question of the intuitive evidence of the ||[A44] foundation of the infinitesimal calculus. However we have departed from such a view since we found that intuition can not verify the postulates of analysis; the representation of infinite totalities which is made fundamental in analysis cannot be grasped in intuition but only as an *ideal construction*.

Now we can not expect that this new view of the limits of intuitive evidence to fit directly with the old conception of the epistemological character of mathematics; rather on the basis of what we have determined it seems likely that the customary conception of mathematics represents the situation too simply and that we can not do justice to what goes on in mathematics from the standpoint of evidence alone; we must acknowledge that *thought* *has its own distinctive role.*

Thus we arrive at a distinction between the standpoint of elementary mathematics and a systematic standpoint which goes beyond it, This distinction is by no means artificial or merely *ad hoc*; rather it expresses the two starting points from which one is led to arithmetic: on the one hand combinatorial consideration of relations between discrete entities, and on the other the theoretical demand placed on mathematics by geometry and physics.[10] The system of arithmetic by no means springs only from an activity of constructing and considering intuitively; in larger part it springs from the task of grasping conceptually and exactly and mastering theoretically the geometric and physical representations of quantity, area, motion, velocity, and so on. The method of arithmetization is a means to ||A45 this end. But in order to serve this purpose arithmetic must *broaden* its methodological standpoint from the original elementary ||353 standpoint of number theory *to a systematic attitude* with respect to its postulates.

The arithmetic constituting the great frame within which the geometric and physical disciplines are ordered does not consist simply in the elementary, intuitive treatment of numbers; rather it itself has the character of a *theory*

[10]It is remarkable that Fries, who still ascribed to mathematical evidence a domain going far beyond the finitary (in particular, according to his view "the continuous series of larger and smaller" is given in pure intuition), already made a methodological distinctiqn between "arithmetic as a theory" which conceptualises and scientifically develops the intuitive representation of magnitudes and "combinatory theory or syntactic" which rests only on the postulate of the possibility of providing an arbitrary ordering for given elements and of repeating it arbitrarily often and which reeds no axioms since its operations are "immediately comprehensible in themselves." (Compare Fries, *Mathematische Naturphilosophie* l822.)

in that it has as a basis the representation of the totality of numbers as a system of things and of the totality, of sets of numbers. This systematic arithmetic performs its task very well; and there is no basis in its method for objection so long as we are clear that we are not here dealing with the standpoint of elementary intuitiveness but of a *thought construction*, in other words with that standpoint which Hilbert calls *axiomatic*.

The reproach of arbitrariness against this axiomatic procedure is also not justified; for in the case of the foundation of systematic arithmetic we are not dealing with an arbitrary axiom system put together as occasion demands but with a *natural systematic extrapolation from elementary number theory*; and the analysis and set theory developed on this foundation constitutes a theory which is *marked out pure intellect* and which it is arppropriate to take as the theory κατ' ἐξοχὴν; in it we arrange the systems and theoretical statements of geometry and physics.

Thus we cannot accept the veto which intuitionism directs against the methodology of analysis. The point, on which we agree with intuitionism that the infinite is not given to us intuitively does indeed require us to modify our philosophical conception of mathematics but not to transform mathematics itself.

To be sure, the problem of the infinite returns again. For in taking a thought construction as starting point for arithmetic we have introduced something problematic. Even if a thought construction is very plausible and natural from a systematic point of view it contains in itself no guarantee that it can be carried out consistently. While we do apprehend the idea of the infinite totality of numbers and of sets of numbers, we do not thereby preclude the possibility that this idea leads to a contradiction in its consequences. Thus it remains it to investigate the question of ||^{A46} the freedom from contradiction or the consistency[11] of the system of arithmetic.

Intuitionism would spare us this task since it limits mathematics to the domain of finitary considerations; but this elimination of what is problematic costs too much: the problem disappears, but the systematic simplicity and clarity of analysis is lost.

§ 3. Difficulties in logicism—The value of the logistic reduction of arithmetic

The representatives of the logistic standpoint think they are able to resolve our problem in a completely different way. In discussing this standpoint we start from our earlier reflections on logistic. There it was determined that intuitive evidence already plays a role in deductive logic the logical definition of cardinal number does not establish the specifically logical nature of the concept of cardinal number (as a concept of pure reflection) but rather is only a logical normalization of elementary structural concepts.

These considerations are relevant to the distinction between the logical in the narrow sense and the formal. But the recognition of the formal element in logic does not by any means resolve the methodological questions about logistic. Logistic is not simply concerned with the development of the theory of inference; but, as already explained, it takes as a task the reduction of all arithmetic to the formalism of logic. This reduction takes place via first

[11] It might be appropriate to suggest here that this expression ("Konsistenz") which was used by Cantor specifically with respect to construction of sets be applied generally with respect to any theory which is put forward.

the introduction of cardinal numbers as properties of predicates in the way we have already described and then (in a way which will not be described more precisely here) the expression in terms of the logical formalism of the construction of sets of numbers; here one replaces each set with a defining predicate, and thus the totality of predicates of numbers replaces the totality of sets of numbers.

In this way there results in fact a correlation of every arithmetic sentence with a sentence from the domain of theoretical logic, a sentence in which ||[47] except for "logical constants," i.e. fundamental logical operations like conjunction, negation, generality, etc., only variables occur.

||[355] Now it is clear that the problem of the infinite is not solved just by this translation of arithmetic into the formalism of. logic. If theoretical logical can obtain the system of arithmetic deductively, then its procedures must include either explicit or hidden presuppositions which make possible the introduction of the actually infinite.

The fundamental weak point of logicism is the account which is given of these presuppositions and the position which is adopted with respect to them. Thus the precision and rigor of the thought and methods of proof of Frege and Dedekind excellent; but they were not at all critical about what they took as a presumably self-evident principle to be the basis of the standpoint of general logic; that is to say, they were not critical about the conception of a closed totality of all conceivable objects whatsoever.

Admittedly, if this conception were tenable, it would be more satisfactory from a systematic point of view than the more specialized postulates of arithmetic. As is known, the presupposition must be dropped because of the contradictions to which it leads. Since dropping the presupposition logicism has forgone proving the existence of an infinite totality and has rather explicitly postulated an *axiom infinity*.

However, the axiom of infinity is not a sufficient presupposition for obtaining arithmetic as logically construed. With it we can only obtain what follows from our first postulate, the admissibility of existential inference with respect to the integers. Conformity with our second postulate requires a further assumption: the application of existential inference *with respect to predicates*. The justification of this application can initially seem logically self-evident; and in fact such a justification is out of the question, given the conception which lies at the basis of the work of Frege and Dedekind. But once the representation of the totality of all logical objects is given up, the representation of the totality of all predicates becomes problematic as well; and in the latter case closer inspection shows a particular, fundamental difficulty.

What is involved in the genuine logisticist standpoint is the following: we construe the totality of predicates as a totality which essentially first comes into existence ||[A48] in the frame of the system of logic; and it comes into existence when logical constructions are applied to certain initial, prelogical predicates, e.g. predicates taken from intuition. ||[356] Further predicates are now obtained by reference to the totality of predicates. Frege's definition of finite number constitutes an example of this procedure: "a number n is called finite if every predicate holds of it which holds of the number 0 and which, if it holds of a number a holds of the following number." In this case the predicate of finiteness is defined by reference to the totality of all predicates.

Definitions of this kind are called "impredicative;"[12] they occur everywhere in the foundation of arithmetic and in essential places.

Now in itself there is nothing at all to be said against the specification of a member of a totality by means of a property which refers to this totality. For example, a particular number in the totality of numbers is defined by the property of being the greatest prime number which multiplied by 1000 is larger than the result of multiplying the preceding prime number by 1001.[13]

But in such cases it is presupposed that the totality in question is determined *independently* of the definition referring to it; other wise we are involved in.a vicious circle.

However, this precondition can not be considered to be satisfied immediately in the case of the totality of predicates and the impredicative definitions which refer to it; for according to the conception expounded here the totality of predicates is determined by the ||[49] laws of logical construction, and these laws include impredicative definitions.

It would suffice to avoid the vicious circle to show that every predicate introduced by an impredicative definition can also be defined "predicatively."

[12]The term comes from Poincaré, who (as opposed to the other critics of set theory who all concerned themselves only with the axiom of choice) brought the aspect of impredicative definitions into the discussions and laid weight on it. His criticism is nevertheless vulnerable since he represented the use of impredicative definitions as a novelty introduced by set theory. But Zermelo could reply to him that impredicative definitions are among the customary modes of inference in analysis which Poincaré accepted.

Since then Russell and Weyl have explained the role of impredicative definition in analysis thoroughly and made it completely clear.

[13]The example is chosen so that the reference to the totality of numbers can not be directly eliminated as is the case in most of the simpler examples.

Indeed, one could even get by with a weaker result since in the logical foundation of arithmetic a predicate is considered only with respect to its extension, i.e. only with respect to the set of things of which it holds; thus we only need to know that every predicate introduced by an impredicative definition *has the same extension* as a predicatively defined predicate.

Russell, who recognized with total clarity the difficulty involved in impredicative definition, assumed this postulate together with the axiom of infinity as the "axiom of reducibility."

If the axiom of reducibility is the expression of a logical law, then its validity must be independent of the domain of prelogical initial predicates which is presupposed—at least if we assume that that the domain satisfies the axiom of infinity. But to say this is to say that the domain of predicates of an axiomatic theory in which universal and existential judgments (existential inference) are applied only to objects and not to predicates is not enlarged by the introduction of impredicative definitions provided only that the satisfaction of the axiom system requires ab infinite system of objects.

But there is no point in wondering about the correctness of such an assertion; it is easy to construct examples which refute it.

Dedekind's introduction of the concept of number consttitutes an example. Dedekind starts from a system in which an object 0 is distinguished and for which a one-one mapping onto a subset not containing 0 is possible. Suppose we represent this mapping by a predicate with two subjects and formulate the required property of this predicate as an axioms; then we get an elementary axiom system which contains in its axioms no reference $\|^{A50}$ to the totality of predicates and which, moreover, can be satisfied only by an infinite system of objects. We consider now Dedekind's concept of number; if we translate his definition from the language of set theory into that of the theory of predicates, the definition can be formulated completely analogously to that of Frege: "an object n of our system is a number if every predicate holds of n which holds of 0 and which, if it holds of a thing a in our system, holds also of the thing to which a is correlated by the one-one mapping." This definition is impredicative; and one can convince himself that it is not possible from the basic elements of the theory to obtain via a predicative definition a predicate with the same extension as the concept of being a number which is defined here.[14] ||[358]

Thus we can consider only the second interpretation of the axiom of reducibility according to which it is the expression of a *requirement for the initial domain of prelogical predicates.*

But if one introduces such a presupposition he abandons the conception of the domain of predicates as produced by logical processes. The goal of a genuinely logical theory of predicates is hereby given up.

Once one accepts the point of view developed here it seems more natural and appropriate to return to the conception of a *logical function* corresponding to Schroeder's standpoint: one construes a logical function as an assignment of the values "true" and "false" to the objects of the domain of individuals. Every predicate defines such an assignment; but, in analogy with the finite, the totality of assignments of values is a *combinatorial manifold which exists independently of conceptual definitions.*

[14] Waismann has given another example in a note on "Die Natur des Reduzibilitäts-Axioms" (1928). This example, however, needs some modification.

This conception does away with the circularity of the impredicative definitions of theoretical logic; all we have to do is replace every expression referring to the totality of predicates by the corresponding expression referring to the totality of logical functions. In this way the axiom of reducibility can be dispensed with.

This step was actually taken by the logicistic school under the influence of Wittgenstein and Ramsey; these two made it particularly clear that for avoiding the paradoxes connected with the concept of the set of all mathematical ‖[A51] objects it is not necessary to divide according to the way in which they are defined, as Whitehead and Russell had done in *Principia Mathematica*; rather it suffices to delimit clearly the domain of definition of predicates in such a way as to distinguish between predicates of individuals, predicates of predicates, predicates of predicates of predicates, and so on.

In this way people have returned from the type theory of *Principia Mathematica* to the simpler conceptions of Cantor and Schroeder.

One might now mistakenly suppose that as a result of these considerations one has not gone very far from the standpoint of logical self-evidence. The presuppositions which are now the basis of theoretical logic ‖[359] are in principle of quite the same kind as the fundamental postulates of analysis and in content completely analogous to them: the representation of the sequence of numbers corresponds to the axiom of infinity in logical theory; and instead of the concept of all sets of numbers logical theory postulates the concept of all logical functions referring to the "domain of individuals" or to a determinate domain of predicates.

There is then no economizing with respect to presuppositions in the reduction of arithmetic to the system of theoretical logic. Contrary to what one might at first think, this reduction has by no means the significance of a derivation of the postulates of arithmetic from more minimal assumptions; the value of this reduction lies rather in the fact that by being united with the formalism of logic mathematical theory is placed on a broader basis.

Thus logic receives methodological distinction in a higher degree; it shows not only that its presuppositions are obtained from intuitive number theory by a natural extrapolation but at the same time that it is subordinate to number theory since we *extrapolate the logic of extensions* by expanding it to cover infinite totalities,

Moreover, the union of arithmetic with theoretical logic makes possible an insight into the connection of the process of set construction with the fundamental operations of logic; as a result the logical structure of the formation of concepts and of inference becomes clearer.

In particular the meaning of the *axiom of choice* becomes completely comprehensible in relation to the formalism of logic. We can express this principle In the following way: if $B(x, y)$ is a predicate of two subjects (defined in a definite domain) and $\|^{A52}$ if for every object x in the domain there is at least one object y in the domain for which $B(x, y)$ holds, then there is at least one function $y = f(x)$ such that for every object x in the domain of definition of $B(x, y)$ the value $f(x)$ is an object in this domain such that $B(x, f(x))$ holds.

We consider first what this assertion says in the particular case of a domain of two objects, objects which we can represent by the numbers 0, 1; in this case there are (extensionally) only four different functions $y = f(x)$ to consider; in this case the assertion is a simple application of the *distributive law* governing the relation between conjunction and disjunction, i.e. the following theorem of elementary logic: "if A is the ||[360] case and if either B or C is the case, then either A and B is the case or A and C is the case."[15]

And in the case of any subject domain consisting of a determinate, finite number of objects the truth of the axiom of choice follows from this distributive law. The general assertion of the axiom of choice is therefore nothing but the extension of an elementary law for conjunction and disjunction to infinite totalities; thus the axiom of choice constitutes a generalization of the logical rules governing universal and existential judgment, i.e. the rules of existential inference; but application of these rules to infinite totalities means in any case that certain elementary laws for conjunction and disjunction are carried over into the infinite.

The axiom of choice has a distinctive position with respect to these rules only insofar as its .formulation requires the *concept of function*, a concept which, in its turn, gets a satisfactory implicit characterization only by means of the axiom of choice.

This concept of function corresponds to the concept of logical function; the only difference is that the values of the former are not taken to be "true" and "false" but the objects of the subject domain. The totality of all func tions which is in question here is therefore the totality of all "images" of the subject domain,

||[A53] In the sense of this concept of function the existence of a function

[15]'Or' is in both cases meant not in the sense of the exclusive 'or' but in the sense of the latin "vel." But in any case the theorem also holds for the exclusive 'or'.

with the property E does not at all mean that there exists a way of constructing concepts which uniquely determines a definite function with the property E. Consideration of this situation renders the usual objections to the axiom of choice trivial; most of these objects stem from the fact that one is induced by the name "axiom of choice" to the opinion that this principle asserts the possibility of choosing.

At the same time we acknowledge that the presupposition which finds expression in the axiom of choice is fundamentally no extension of the conception which we already had to consider fundamental in the methodology of theoretical logic in order to be able to designate free from circularity without introducing the axiom of reducibility.

To be sure, we can also give an opposite formulation to what we have said: the controversy over the axiom of choice, the exposition of which is to be found in the thorough development of the standpoint of theoretical ||[361] logic, makes us especially conscious of what is problematic about this standpoint.

The consideration of the logistic foundation of arithmetic has also lead us to the same result: the process of reducing arithmetic to theoretical logic does indeed create a broader foundation for arithmetical theory and provides a motivation for its presuppositions in terms of content; but it does not go beyond the methodological standpoint of idealization, i.e. beyond the standpoint of axiomatics.

The problem of the infinite is indeed formulated in this way, but it is not solved. For there remains the question (raised even within the logicistic program) whether the analogy between the finite and the infinite, which is postulated as a presupposition in the construction of analysis and set theory, is admissible, i.e. whether the idea can be carried out consistently.

Intuitionism chooses to sidestep this question by excluding the problematic presuppositions; most logicists dispute the propriety of the question since they refuse to acknowledge a fundamental difference between the finite and the infinite; the question is approached in a positive way by *Hilbert's proof theory.*

§ 4. Hilbert's proof theory

In order to grasp the main ideas of proof theory better we will recall once again the nature of the problem which is ||[54] to be solved here. It is a question of showing the consistency of the ideas on which the theory of arithmetic rests.

Philosophers have often raised the question whether a proof of consistency alone constitutes a justification of these ideas. This way of putting the question is misleading; it does not take into account the fact that the scientific motivation for the theoretical approach to arithmetic has essentially a been provided by science and that the proof of consistency is the only *desideratum* which has not yet been fulfilled,

The system of arithmetic is built on a foundation of conceptions which have decisive significance for scientific systematization in general: namely the principle of *conservation* (or "*permanence*") *of laws*, which occurs in this connection in the postulate of the unlimited applicability of the usual logical forms of judgment and inference, and the demand for a purely *objective* conception of the theory so that it is freed from all connection with our *knowing.* ||[362]

The *inner* motivation and distinctive character of the treatment of arithmetic theory lies in the fundamental methodological significance of these requirements,

In addition to this inner motivation the striking features of the conceptual system of arithmetic, its deductive fruitfulness, its systematic success, and the illuminating character of its consequences, are relevant. This conceptual system is clearly suited for understanding the relations of cardinal numbers and of magnitudes. There is nothing equal to the systematic character of this magnificent theory which results from the unification of the theory of functions with number theory and algebra. And as an all inclusive conceptual apparatus for the construction of scientific theories arithmetic shows itself to be suited not only for the formulation and development of laws; it has also been used with great success and to an extent which had not been anticipated in the search for laws.

The illuminating character of its consequences is best proved by the intensive theoretical development of analysis and its many numerical applications.

All that is lacking is a genuine understanding of the ||[A55] consistency, i.e. of the complete agreement of its results with one another, of arithmetic theory to replace the merely empirical trust in its consistency which has been gained by making many tests; the task of a proof of consistency is to produce this understanding.

The situation then is not that the conceptual system of arithmetic must first be established by means of a proof of consistency; rather the purpose of this proof is to provide for this conceptual system,which is already motivated by the internal ground of systematization and is as secure as possible for its use as an intellectual tool, the complete and comprehensive certainty that it can not be ruined by the failure of its consequences to agree with one another.

If this proof succeeds, we will know that the idea of the closed infinite can be developed in a consistent way. And we can depend upon the results of applying the fundamental postulates of arithmetic just as if we were in the position of verifying these postulates intuitively. For insofar as we recognize the consistency of applying these postulates we also recognize that an intuitive assertion, i.e. one having meaning the finitary sense, which follows from then can never contradict an intuitively perceived fact. But in the case of a finitary assertion ||[363] the determination that it is not refutable is equivalent to the determination that it is true.

From this consideration of the demand for and the goal of a proof of consistency it results in particular that the upshot of such a proof is nothing else but an understanding of the consistency of arithmetic theory in the literal sense of the word: *the impossibility its being refuted from within*.

The neu feature of Hilbert's procedure is that he limit himself to the consideration of this problem; formerly the proof of the consistency of an axiomatic theory was always carried out in such a way that at the same time the satisfaction of the axions by certain objects was exhibited in a positive way. In the case of arithmetic this method of exhibition provides no solution; in particular, Frege's idea of taking the objects to be exhibited from the domain of logic does not lend to the goal because, as we have made clear, the application of ordinary logic to the infinite is just as problematic as the theory which is to be shown consistent. Indeed the fundamental postulates of arithmetic theory involve exactly the extension of the application of the ordinary forms of judgment and inference.

In recalling this situation, we are led $\|^{A56}$ directly to the *first leading principle of Hilbert's proof theory*: it says that in the proof of the consistency of arithmetic we must include in the domain of what is to be shown consistent the laws of logic as applied in arithmetic; thus the proof of consistency is extended to cover *logic and arithmetic together.*

The first essential step in carrying out this idea is already taken in the reduction of arithmetic to the system of theoretical logic. On the basis of this reduction the task of proving the consistency of arithmetic is accomplished by establishing the consistency of theoretical logical, or, in other words, determining the consistency of the axiom of infinity, impredicative definition, and the axiom of choice.

It is convenient in this connection to replace Russell's axiom of infinity with Dedekind's characterization of the infinite. Russell's axiom of infinity postulates the existence of an n-numbered predicate for every finite number n (in the sense of Frege's definition of finite cardinal number) and thus implicitly postulates the infinity of the domain of individuals, the fundamental domain of objects. Now it is an unnecessary complication which is objectionable from the foundational standpoint that in this axiom three infinities of different types are $\|^{364}$ mixed together: that of the infinitely many objects in the domain of individuals, then that of the infinitely many predicates, and then that of the infinitely many cardinal numbers (defined as predicates of predicates) which results from the preceding domain.

We can avoid this complexity by fixing the infinity of the domain of individuals with a single predicate of two subjects in place of an infinite series of predicates of one subject; any predicate will do if it provides a one-one mapping of the domain of individuals onto a proper subdomain, i.e. a domain which does not include at least one object. The introduction of Dedekind's characterization of the infinite takes its most simple and elementary form if we do not postulate the one-one mapping by means of an existence postulate hut introduce it explicitly by taking as fundamental elements an initial element and a fundamental process.

The result of this procedure is that the numbers occur as objects in the domain of individuals rather than as predicates of predicates. But this discussion already concerns the particular method of carrying out the construction of a system, and there are several possible ways of doing this. However, we must ||[A57] orient ourselves more generally with respect to the question how a proof of consistency in the intended sense can be carried out at all. The possibility of such a proof is not directly self-evident. For how can one survey all possible consequences of the presuppositions of arithmetic or of theoretical logic?

Here the investigation of mathematical proof with the help of the logical calculus comes into play in a decisive way. This investigation has shown that the methods of forming concepts and making inferences which are employed in analysis and set theory are reducible to a limited number of processes and rules; thus it is possible to formalize these theories completely in the framework of an exactly specified symbolism.

This formalization was done originally only for the sake of a more precise analysis of proof; Hilbert inferred from the possibility of this formalization the *second leading principle of his proof theory*: the task of proving the consistency of arithmetic is a *finitary problem*.

An inconsistency in the original theory must show itself in the formalization by the derivability according to the rules of the formalism of two formulas of which one results from the other by means of the process which is the formal image of negation. The assertion of consistency is therefore equivalent to the assertion that two formulas standing in the above relation can not be derived by means of the rules of the formalism. ||[365] But this assertion has fundamentally the same character as any general statement of finitary number theory, e.g. the statement that it is impossible to produce three integers a, b, c (different from 0) such that $a^3 + b^3 = c^3$.

Thus the proof of consistency for arithmetic becomes in fact a finitary problem of the theory of inference. Hilbert call the finitary investigation which ich has the formalized theories of mathematics as its object *metamathematics*. The task falling to metamathematics vis-à-vis mathematics is analogous to the one which Kant ascribed to the critique of reason vis-à-vis the system of philosophy.

In terms of this methodological program proof theory has already been developed to a considerable extent;[16] but there are still ||[58] considerable

[16]Already in 1904 in his Heidelberg lecture "On the foundations of logic and arithmetic" Hilbert gave a first sketch of proof theory. Here the first leading idea of treating logic and arithmetic together is explicitly formulated; the methodological principle of the finitary standpoint is also implicit but not explicitly expressed. The investigation of Julius Koenig, "New foundations for logic, arithmetic, and set theory" (1914) falls between the lecture and Hilbert's more recent publications on proof theory; Koenig conies very close to Hilbert's standpoint and gives a proof of consistency which is entirely within the domain of proof

mathematical difficulties to be overcome. The proofs of Ackermann and von Neumann guarantee the consistency of the first postulate of arithmetic the applicability of existential inference the integers. For the further her problem of the consistency of the general concept of a set or function of numbers including the axiom of choice associated with it there is under consideration an approach due to Ackerman, which has been carried quite far.

If this problem were solved, then almost the whole domain of mathematical theories now in existence would be shown consistent.[17] In particular, this proof would be sufficient to establish the consistency of geometric and physical theories.

One can also go further into the problem and investigate the consistency of more inclusive theories, axiomatic set theory for example. Axiomatic set theory, as first put forward by Zermelo and enlarged and ||[366] extended by Fraenkel and von Neumann, goes in its constructions far beyond what is actually used in mathematics; and the determination that it is consistent would also show the consistency of the system of theoretical logic.

But an absolute end to the construction of concepts would not be reached here. For formalized set theory makes possible metamathematical considerations which take the formal constructions of set theory as an object and therefore go beyond these constructions.[18]

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theory. This proof involves a very narrow domain of formal operation and is therefore only of methodological significance.

[17]Cantor's theory of numbers in the second number class is also included here.

[18]The more detailed explanation of this situation is connected with the Richard paradox, of which Skolem has recentl given a more precise formulation. These considerations are not conclusive since they are made in the framework of a non-finitary metamathematics. A final clarification of the question discussed here would result if one succeeded in producing

||^{A59} In spite of this possibility of enlarging the construction of concepts a formalized theory can be closed in character if no new results in the domain of the laws formulated in terms of the concepts of the theory come about by means of the enlargement of the construction of concepts.

This condition is fulfilled whenever the theory is *deductively closed*, in other words if it is impossible to add a new axiom expressible in terms of the concepts of the theory but not already derivable without producing a contradiction,—or, what amounts to the same thing: if every statement formulable within the framework of the theory is either provable or refutable.[19]

We believe that number theory as determined by Peano's axioms with the addition of definition by recursion is deductively closed in this sense; but the task of giving a real proof of this belief is still completely unsolved. The question becomes even more difficult if we go beyond the domain of number theory to analysis and further set theoretic ways of constructing concepts.

In the region of these and related questions there lies a considerable field of open problems. But these problems are not of such a kind that they represent an objection to the standpoint we have adopted. We ||^{367} must only keep in mind that the formalism of statements and proofs which we use to represent our ideas is not identical with the structure which we have in mind in our conceptual thinking. The formalism suffices for formulating our

in a finitary, way a set of numbers which could be shown not to occur in axiomatic set theory.

[19]Notice that the requirement of being deductively closed is not as strong as the requirement that every question of the theory be *decidable*. The latter says that there should be a procedure for deciding for any pair of contradictory assertions belonging to the theory which of the two is provable ("correct").

ideas about infinite manifolds and for drawing out the logical consequences of these ideas; but in general it is not able, so to speak, to produce the manifold out of itself combinatorially.

||[A60] The view of the theory of the infinite which we have reached can be considered a form of the philosophy of the "as if." However, it differs fundamentally from the philosophy of Vaihinger designated by this phrase: our view places weight on the consistency and reliability of ideas; but Vaihinger considers the demand for consistency to be a prejudice and says that the contradictions in the infinitesimal calculus are "not simply not to be disavowed, but ... (are) precisely the means by which progress was attained."[20]

Vaihinger's discussion falls completely within tile domain of scientific *heuristic*. He recognizes only "fictions" which occur simply as temporary aids for thinking; in introducing these fictions thought puts restrictions upon itself; and, if it is a question of "genuine fictions," their contradictory character is rendered harmless by a suitable compensation for the contradictions.

Ideas in our sense are an enduring possession of the spirit. They are distinctive forms of systematic extrapolation and of idealizing approximation to reality. They are by no means arbitrary or forced upon thought; on the contrary, they constitute a world in which our thought feels at home and from which the human spirit, absorbed in this world, creates satisfaction and joy.

The *Nachtrag* in the *Abhandlungen* is not translated by Mueller.

[20]Vaihinger, *Die Philosophie des Als ob*, second edition, ch. XII.