Bernays Project: Text No. 5

Problems of Theoretical Logic (1927)

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(Probleme der theoretischen Logik, 1927)

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Comments:

Revised 6/18/02. Corrections by Bill Ewald and Wilfried Sieg from 4/3/03, 4/16/03, 5/8/03, 1/7/05.

||$^{369/A1}$ Lecture given at the 56th meeting of German philologists and teachers in Göttingen.

The topic of the lecture and its title have been chosen in the spirit of Hilbert. What is here called theoretical logic is usually referred to as symbolic logic, mathematical logic, algebra of logic, or logical calculus. The purpose of the following remarks is to present this research area in a way that justifies calling it theoretical logic.

Mathematical logic is in general not very popular. It is most often regarded as idle play that neither supports effectively practical reasoning nor contributes significantly to our logical insights.

To begin with, the charge of playfulness is only justified with regard to the early treatment of mathematical logic. The main emphasis was initially put on the formal analogy to algebra, and the pursuit of the latter was often considered as an end in itself. But this was the state of affairs decades ago, and today the problems of mathematical logic are inseparably intertwined with the questions concerning the foundations of the exact sciences, so that one can no longer speak of a merely playful character.

Secondly, concerning the application to practical inference, it has to be mentioned first that a symbolic calculus promises advantages only to someone who has sufficient practice in using it. But, in addition, one has to consider that—in contrast to most kinds of symbolisms which serve, after all, the purpose of abbreviating and contracting operations—it is the primary task of the logical calculus to decompose the inferences into their ultimate constituents and to make outwardly evident each individual step and bring it thereby into focus. The main interest connected with the application of the logical calculus is consequently not one of technique, but of theory and principle. $\|^{A2}$ This leads me to the third charge; namely that mathematical logic does not significantly further our logical insights. This opinion is connected with the view on logic expressed by Kant in the second preface to the *Critique of Pure Reason*, where he says: "It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine."[1]

It is my intention to show that this standpoint is erroneous. To be sure, Aristotle's formulation of the ultimate principles of inference and their im-

[1]Kant, CPR B viii=Kemp Smith, p. 17.

mediate consequences constitutes one of the most significant intellectual accomplishments; it is also one of the very few [accomplishments] which belong to the permanently secured part of the realm of philosophical knowledge. This fact will continue to receive its full due. But this does not prevent us from pointing out that traditional logic, in posing its problems, is essentially open-ended, and in arranging its facts it is insufficiently adapted to the needs of both a systematic overview and of methodical and epistemologico-critical insights. Only the newer logic, as it has developed under the name of algebra of logic or mathematical logic, introduced such concept formations and such an approach to formal logic as makes it possible to satisfy these needs of systematics and of philosophy.

The realm of logical laws, the world of abstract relations, has only thereby been revealed to us in its formal structure, and the relationship of mathematics and logic has been illuminated in a new way. I will try briefly to give an idea of this transformation and of the results it has brought to light.

In doing so I will not be concerned with presenting the historical development or the various forms in which mathematical logic has been pursued. Instead, I want to choose a presentation of the new logic that best facilitates relating and comparing it to traditional logic. As for logical symbols, I shall use the symbolism Hilbert employs now in his lectures and publications.

Traditional logic subdivides its problems into the investigation of concept formation, of judgment, and of inference. It is not advantageous to begin with concept formation, because its essential ||[3] forms are not elementary but are already based on judgments. Let us begin, therefore, with judgment.

Here, the newer logic immediately introduces an essentially new vantage point, replacing classifications by the search for elementary logical operations. One does not speak of the categorical or the hypothetical or the negative judgment, but of the categorical or hypothetical connexion, of negation as a logical operation. In the same way, one does not classify judgments into universal and particular ones but introduces logical operators for universality and particularity.

This approach is more appropriate than that of classification for the following reason. In judgments different logical processes generally occur in combination, so that a unique corresponding classification is not possible at all.

First let us consider the *categorical* relationship, i.e. that of subject and predicate. We have here an object and a proposition about it. The symbolic representation for this is

$$P(x),$$

to be read as:

"x has the property P."

The relation of the predicate to an object is here explicitly brought out by the variable. This is merely a clearer kind of notation; however, the remark that *several objects* can be subjects of a proposition is crucial. In that case one speaks of a *relation* between several objects. The notation for this is

$$R(x, y), \text{ or } R(x, y, z), \text{ etc.}$$

Cases and prepositions are used in ordinary language to indicate the different members of relations.

By taking into account relations, logic is extended in an essential way when compared with its traditional form. I shall speak about the significance of this extension when discussing the theory of inference.

The forms of universality and particularity are based on the categorical relationship. Universality is represented symbolically by

$$(x)P(x)$$

"all x have the property P."

The variable x appears here as a "bound variable;" the proposition does not depend on x—in the same way as the value of an integral does not depend on the variable of integration.

We sharpen the particular judgment first ||[4] by replacing the somewhat indefinite proposition, "some x have the property P," with the existential judgment:

"there is an x with the property P,"

written symbolically:

$$(Ex)P(x).$$

By adding *negation*, the four types of judgment are obtained which are denoted in Aristotelian logic by the letters "a, e, i, o".

If we represent negation by putting a bar over the expression to be negated, then we obtain the following representations for the four types of judgment:

$$\begin{array}{lll} \text{a:} & (x) & P(x) \\ \text{e:} & (x) & \overline{P(x)} \\ \text{i:} & (Ex) & P(x) \\ \text{o:} & (Ex) & \overline{P(x)}. \end{array}$$

Already here, in the doctrine of "oppositions," it proves useful for the comprehension of matters to separate the operations; thus we recognize, for example, that the difference between contradictory and contrary opposition lies in the fact that in the former case the whole proposition, e.g., $(x)P(x)$, is negated, whereas in the latter case only the predicate $P(x)$ is negated.

Let us now turn to the *hypothetical relationship.*

$$A \rightarrow B \quad \text{“if } A\text{, then } B\text{.”}$$

This includes a connexion of *two* propositions (predications). So the members of this connexion already have the form of propositions, and the hypothetical relationship applies to these propositions as *undivided units.* The latter already holds also for the negation $\overline{A}$.

There are still other such propositional connexions, in particular:
the fact that A *exists together* with B: A & B,
and further, the *disjunctive connexion*; there we have to distinguish between the exclusive "or," in the sense of the Latin "aut-aut", and the "or" in the sense of "vel". In accordance with Russell's notation this latter connexion is represented by $A \vee B$.

In ordinary language, such connexions are expressed with the help of conjunctions.

A consideration, analogous to that used in the doctrine of opposition, suggests itself here, namely to combine the binary propositional connexions with negation in one of two ways, either by negating the individual members of the connexion or ||[5] by negating the latter as a whole. And now, let us see what dependency relations result.

To indicate that two connexions have materially the same meaning (or are "equivalent"), I will write "eq" between them (though, "eq" is not a sign of our logical symbolism).

In particular the following connexions and equivalences result:

$$\begin{aligned}
\overline{A} \,\&\, \overline{B}&: \quad \text{"neither } A \text{ nor } B\text{"} \\
\overline{A \,\&\, B}&: \quad \text{"}A \text{ and } B \text{ exclude each other"} \\
\overline{A \,\&\, B} &\quad \text{eq } \overline{A} \vee \overline{B} \\
&\quad \text{eq } A \to \overline{B} \\
&\quad \text{eq } B \to \overline{A} \\
\overline{A} \to B &\quad \text{eq } A \vee B \\
\overline{\overline{B}} &\quad \text{eq } B
\end{aligned}$$

(double negation is equivalent to affirmation).

From this it furthermore follows:

$$\begin{aligned}
A \to B &\quad \text{eq } \overline{A \,\&\, \overline{B}} \\
&\quad \text{eq } \overline{A} \vee B \\
\overline{A \vee B} &\quad \text{eq } \overline{\overline{A} \to B} \\
&\quad \text{eq } \overline{A} \,\&\, \overline{B}.
\end{aligned}$$

On the basis of these equivalences it is possible to express some of the logical connexions

$$\overline{}, \to, \&, \vee$$

by means of others. In fact, according to the above equivalences one can express

$$\begin{array}{ccl} \rightarrow & \text{by} & \vee \text{ and } \overline{} \\ \vee & \text{by} & \& \text{ and } \overline{} \\ \& & \text{by} & \rightarrow \text{ and } \overline{} \end{array}$$

so that each of

$$\begin{array}{cl} & \& \text{ and } \overline{} \\ \text{or} & \vee \text{ and } \overline{} \\ \text{or} & \rightarrow \text{ and } \overline{} \end{array}$$

alone suffice as basic connexions. One can get along even with a single basic connexion, but, to be sure, not with one of those for which we already have a sign. If we introduce for the connexion of mutual exclusion $\overline{A \& B}$ the sign

$$A|B$$

then the following equivalences obtain:

$$\begin{array}{rcl} A|A & \text{eq} & \overline{A} \\ A|\overline{B} & \text{eq} & \overline{A \& \overline{B}} \\ & \text{eq} & A \rightarrow B. \end{array}$$

||[6] This shows that with the aid of this connexion one can represent negation as well as $\rightarrow$ and, consequently, the remaining connexions. Just like the relation of mutual exclusion also the connexion

$$\text{“neither — nor”} \quad \overline{A} \& \overline{B}$$

can be taken as the only basic connexion. If for this connexion we write

$$A \parallel B,$$

then we have

$$\begin{array}{lll} A \parallel A & \text{eq} & \overline{A} \\ \overline{A} \parallel \overline{B} & \text{eq} & A \mathbin{\&} B; \end{array}$$

thus, negation as well as & is expressible by means of this connexion.

These reflections already border somewhat on the playful. Nevertheless, it is remarkable that the discovery of such a simple fact as that of reducing all propositional connexions to a single one was reserved for the 20th century. The equivalences between propositional connexions were not at all systematically investigated in the old logic.[2] There one finds only occasional remarks about, for example, the equivalence of

$$A \to \overline{B} \text{ and } B \to \overline{A}$$

on which the inference by "contraposition" is based. The systematic search for equivalences is, however, all the more rewarding as one reaches here a

[2]Today these historical remarks stand in need of correction. In the first place, the reducibility of all propositional connexions to a single one was already discovered in the 19th century by Charles S. Peirce—to be sure, a fact which became more generally known only with the publication of his collected works in 1933. Further, it is not correct that the equivalences between propositional connectives were not considered systematically in the old logic—to be sure, not in Aristotelian logic, but in other Greek schools of philosophy. (On this topic see Bochenski's book *Formal Logic* [Editorial footnote: REFERENCE?].)

Remark: This footnote, as well as the next three, are subsequent additions occasioned by the republication of this lecture.

self-contained and entirely surveyable part of logic, the so-called *propositional calculus*. I will explain in some detail the value of this calculus for reasoning.

Let us reflect on what the sense of equivalence is. When I say

$$\overline{A \mathbin{\&} B} \text{ eq } \overline{A} \vee \overline{B},$$

I do not claim that the two complex propositions have the same sense but only that they *have the same truth value*. That is, no matter how the individual propositions A, B are chosen, $\overline{A} \vee \overline{B}$ and $\overline{A \mathbin{\&} B}$ are always simultaneously true or false, and consequently these two expressions can represent each other with respect to truth.

Indeed, any complex proposition A and B can be viewed as a mathematical function assigning to each pair of propositions A, B one of the values "true" or "false." The actual content of the propositions A, B does not matter at all. Rather, what matters is only whether A is true or false and whether B is true or false. So we are dealing with *truth functions*: To a pair of truth values another truth value is assigned.

Each such function can be given by a schema in such a way, that the four possible connections of two truth values (corresponding to the propositions A, B) are represented by four cells, and in each of these the corresponding truth value of the function ("true" or "false") is written down.

The schemata for $A \mathbin{\&} B, A \vee B, A \rightarrow B$ are specified here.

$A \mathbin{\&} B$:

B \ A	true	false
true	true	false
false	false	false

$A \lor B$:

B \ A	true	false
true	true	true
false	true	false

$A \to B$:

B \ A	true	false
true	true	true
false	false	true

‖[8] One can easily calculate that there are exactly 16 different such functions. The number of different functions of n truth values

$$A_1, A_2, \ldots, A_n$$

is, correspondingly, $2^{(2^n)}$.

To each function of two or more truth values corresponds a class of intersubstitutable[3] connexions of propositions. Among these one class is distinguished, namely the class formed by those connexions that are always true.

These connexions represent all logical sentences that hold generally and in which individual propositions occur only as undivided units.[4] We will call the expressions representing sentences that hold generally *valid formulas.*

We master propositional logic, if we know the valid formulas (among the propositions of connexions), or if we can decide for a given propositional

[3]Editorial footnote: "Um uns kurz ausdrücken zu können wollen wir zwei Aussagenverknüpfungen durch einander 'ersetzbar' nennen, wenn sie dieselbe Wahrheitsfunktion darstellen." HB I, pp. 47–48.

[4]Editorial footnote: concerning allgemein gültig versus allgemeingültig

connexion whether or not it is valid. After all, the task for reasoning in propositional logic is formulated as follows:

Certain connexions

$$V_1, V_2, \ldots, V_k,$$

are given; they are built up from elementary propositions $A, B, \ldots$, and represent true sentences for a certain interpretation of the elementary propositions. The question is whether another given connexion D of these elementary propositions follows logically from the validity of $V_1, V_2, \ldots, V_k$, without regard to the more precise content of the propositions $A, B, \ldots$.

The answer to this question is "yes," if and only if

$$(V_1 \ \& \ V_2 \ \& \ \ldots \ \& \ V_k) \rightarrow D,$$

composed from $A, B, \ldots$, represents a valid formula.

The decision concerning the validity of a propositional connexion can in principle always be reached by trying out all relevant truth values. The method of considering equivalences, however, provides a more convenient procedure. That is to say, by means of equivalent transformations each formula can be put into a certain *normal form* in which only the logical symbols $\&, \vee, ^{-}$ occur, and from this normal form one can read off directly whether or not the formula is valid.

The rules of transformation are also very simple. One can in particular calculate with & and $\vee$ in complete analogy to $+$ and $\cdot$ in algebra. Indeed, matters are here even simpler, as & and $\vee$ can be treated in a completely symmetrical way.

‖[9] By considering the equivalences, we entered, as already mentioned, the domain of inferences. But here we carried out the inferences, as it were, in a naive way, on the basis of the meaning of the logical connexions, and we turned the task of making inferences into a decision problem.

But for logic there remains the task of *systematically* presenting the rules of inference.

Aristotelian logic lays down the following principles of inference:

1. Rule of categorical inference: the "dictum de omni et nullo": what holds universally, holds in each particular instance.

2. Rule of hypothetical inference: if the antecedent is given, then the consequent is given, i.e. if A and if $A \to B$, then B.

3. Laws of negation: law of contradiction and law of excluded middle: A and $\overline{A}$ can not both hold, and, at least one of the two propositions must hold.

4. Rule of disjunctive inference: if at least one of A or B holds and if $A \to C$ as well as $B \to C$, then C holds.

One can say that each of these laws represents the implicit definition for a logical process: 1. for universality, 2. for the hypothetical connexion, 3. for negation, 4. for disjunction ($\vee$).

These laws contain indeed the essence of what is expressed when inferences are being made. But for a complete analysis of inferences this does not suffice. For this we demand that nothing needs to be reflected upon, once the principles of inference have been spelled out. The rules of inference must be constituted in such a way that they eliminate logical thinking. Otherwise we would have to have other logical rules which specify how to apply those rules.

This demand to exorcise the mind can indeed be met. The development of the doctrine of inferences obtained in this way is analogous to the axiomatic development of a theory. Certain logical laws written down as formulas correspond here to the axioms, and operating [on formulas] externally according to fixed rules, that lead from the initial formulas to further ones, corresponds to contentual reasoning that usually leads from axioms to theorems.

Each formula that can be derived in such a way represents a valid logical proposition.

Here it is once again advisable to separate out *propositional logic*, which rests on the ‖[10] principles 2., 3., and 4. We need only the following rules; we represent the elementary propositions by variables

$$X, Y, \ldots$$

The first rule now states: any propositional connexion can be substituted for such variables (substitution rule).

The second rule is the inference schema

$$\frac{\begin{array}{c}\mathfrak{S} \\ \mathfrak{S} \to \mathfrak{T}\end{array}}{\mathfrak{T}}$$

according to which the formula $\mathfrak{T}$ is obtained from two formulas $\mathfrak{S}, \mathfrak{S} \to \mathfrak{T}$.

The choice of the initial formulas can be made in quite different ways. One has taken great pains, in particular, to get by with the smallest possible number of axioms, and in this respect the limit of what is possible has indeed been reached. The purpose of logical investigations is better served, however, when we separate, as in the axiomatics for geometry, various *groups of axioms* from one another, such that each group gives expression to the role of one logical operation. The following list then emerges:

I		Axioms of implication
II	a)	Axioms for &
II	b)	Axioms for $\vee$
III		Axioms of negation.

This system of axioms[5] generates through application of the rules *all* valid formulas of propositional logic.[6] This *completeness* of the axiom system can be characterized even more sharply by the following facts: if we add any underivable formula to the axioms, then we can deduce with the help of the rules arbitrary propositional formulas.

The division of the axioms into groups has a particular advantage, as it allows one to separate out *positive logic*. We understand this to be the system of those propositional connexions that are valid without assuming that an opposite exists.[7] For example:

$$(A \;\&\; B) \to A$$
$$(A \;\&\; (A \to B)) \to B.$$

[5]Editorial footnote: fixing the axioms as in HB I, p. 65; already formulated in the early twenties As to completeness, cf. the Habilitationsschrift of Bernays written in 1918.

[6]We refer here only to those formulas that can be built up with the operations $\to, \&, \vee$ and with negation. If further operation symbols are added, then they can be introduced by replacement rules. To be sure, one is not forced to distinguish the four mentioned operations in this particular way.

[7]Editorial footnote: "Die "*positive Logik*" ... d.h. die Formalisierung derjenigen logischen Schlüsse, welche unabhängig sind von der Voraussetzung, daß zu jeder Aussage ein Gegenteil existiert." HB I, p. 67.

‖[11] The system of these formulas presents itself in our axiomatics as the totality of those formulas that are derivable without using axiom group III. This system is far less perspicuous than the full system of valid formulas. Also, no decision procedure is known by which one can determine, in accordance with a definite rule, whether a formula belongs to this system.[8] It is not the case that, for instance, every formula expressible in terms of $\rightarrow, \&, \vee$, which is valid and therefore derivable on the basis of I–III, is already derivable from I–II. One can rigorously prove that this is not the case.

An example is provided by the formula

$$A \vee (A \rightarrow B).$$

Representing $\rightarrow$ by $\vee$ and $^{-}$ this formula turns into

$$A \vee (\overline{A} \vee B),$$

and this representation allows one immediately to recognize the formula as valid. However, it can be shown that the formula is not derivable within positive logic, i.e., on the basis of axioms I–II. Hence, it does not represent a law of positive logic.

We recognize here quite clearly that negation plays the role of an *ideal element* whose introduction aims at rounding off the logical system to a totality with a simpler structure, just as the system of real numbers is extended to a more perspicuous totality by the introduction of imaginary numbers, and just as the ordinary plane is completed to a manifold with a simpler projective structure by the addition of points at infinity. Thus this method of ideal

[8]Since then decision procedures for positive logic have been given by Gerhard Gentzen and Mordechaj Wajsberg.

elements, fundamental to science, is already encountered here in logic, even if we are usually not aware of its significance.

A special part of positive logic is constituted by the doctrine of *chain inferences* that was discussed already in Aristotelian logic. In this area there are also natural problems and simple results, not known to traditional logic and again requiring that specifically mathematical considerations be brought to bear. I have in mind Paul Hertz's investigations of sentence-systems. —[9]

The axiomatics we have considered up to now refers to those inferences which depend solely on the rules of the hypothetical and disjunctive inference, ‖[12] and of negation. Now we still have the task of incorporating *categorical reasoning* into our axiomatics. How this is done I will only describe briefly here.

Of the *dictum de omni et nullo* we need also the converse: "what holds in each particular instance, also holds generally." Furthermore, we have to take into account the particular judgment. It holds analogously:

"If a proposition $A(x)$ is true of some object x, then there is an object of which it is true, and vice versa."

Thus we obtain four principles of reasoning that are represented in the axiomatics by two new initial formulas and two rules. A substitution rule for the individual variables $x, y, \ldots$ is also added.

Moreover, the substitution rule concerning propositional variables $X, Y, \ldots$ has to be extended in such a way now that the formulas of propositional logic can be applied also to expressions containing individual variables.

Let us now see how the typical Aristotelian inferences are worked out

[9]Editorial footnote: ref. to HB I, p. 84, but also Keynes, p. 300.

from this standpoint. For that it is necessary to say first something about the interpretation of the universal judgment "all S are P".

According to the Aristotelian view, such a judgment presupposes that there are certain objects with property S, and it is then claimed that all these objects have property P. This interpretation of the universal judgment, to which Franz Brentano in particular objected from the side of philosophy, is admittedly quite correct. But it is suited neither for the purposes of theoretical science nor for the formalization of logic, since the implicit presupposition brings with it unnecessary complications. Therefore we shall restrict the content of the judgment, "all S are P," to the assertion, "an object having property S has also property P."

Accordingly, such a judgment is simultaneously universal and hypothetical. It is represented in the form

$$(x)(S(x) \rightarrow P(x)).$$

The so-called categorical inferences contain consequently a combination of categorical and hypothetical inferences. I want to illustrate this by a classical example:

"All men are mortal, Cajus is a man, therefore Cajus is mortal."

If we represent "x is ‖[13] human" and "x is mortal" in our notation by $H(x)$ and $M(x)$ respectively, then the premises are

$$(x)(H(x) \rightarrow M(x)),$$
$$H(Cajus),$$

and the conclusion is: $M(Cajus)$.

The derivation proceeds, first, according to the inference from the general to the particular, by deducing from

$$(x)(H(x) \to M(x))$$

the formula

$$H(Cajus) \to M(Cajus).$$

And this proposition together with

$$H(Cajus)$$

yields according to the schema of the hypothetical inference:

$$M(Cajus).$$

It is characteristic for this representation of the inference that one refrains from giving a quantitative interpretation of the categorical judgment (in the sense of subsumption). Here one recognizes particularly clearly that mathematical logic does not depend in the least upon being a logic of extensions.

Our rules and initial formulas permit us now to derive all the familiar Aristotelian inferences as long as they agree with our interpretation of the universal judgment—that leaves just 15. In doing so one realizes that there are actually only very few genuinely different kinds of inferences. Furthermore, one gets the impression that the underlying problem is delimited in a quite arbitrary way.

A more general problem, which is also solved in mathematical logic, consists in finding a decision procedure that allows one to determine whether a predicate formula is valid or not. In this way, one masters reasoning in the domain of predicates, just as one masters propositional logic with the decision procedure mentioned earlier.

But our rules of inference extend much farther. The actual wealth of logical connections is revealed only when we consider *relations* (predicates with several subjects). Only then does it become possible to capture *mathematical proofs* in a fully logical way.

However, here one is induced to add various *extensions* which are suggested to us also by ordinary language.

The first extension consists in introducing a formal ||[14] expression for "x is the same object as y," or "an object different from y." For this purpose the "*identity* of x and y" has to be formally represented as a particular relation, the properties of which are to be formulated as axioms.

Second, we need a symbolic representation of the logical relation we express linguistically with the aid of the genitive or the relative pronoun in such phrases as "the son of Mr. X" or "the object that." This relation forms the basis of the *concept of a function* in mathematics. It matters here that an object, having uniquely a certain property or satisfying a certain relation to particular objects, is characterized by this property or relation.

The most significant extension, however, is brought about by the circumstance that we are led to consider predicates and relations themselves as objects, just as we do in ordinary language when we say, for example, "patience is a virtue." We can state properties of predicates and relations, and furthermore, [second order] relations between predicates and also between relations. Likewise, the forms of universality and particularity can be applied with respect to predicates and relations. In this way we arrive at a logic of "*second order*;" for its formal implementation the laws of categorical reasoning have to be extended appropriately to the domain of predicates and relations.

The solution of the decision problem—which, incidentally, is here automatically subsumed under a more general problem—presents an enormous task for this enlarged range of logical relation resulting from the inclusion of relations and the other extensions mentioned. Its solution would mean that we have a method that permits us, at least in principle, to decide for any given mathematical proposition whether or not it is provable from a given list of axioms. As a matter of fact, we are far from having a solution of this problem. Nevertheless, several important results of a very general character have been obtained in this area through the investigations of Löwenheim and Behmann; in particular one succeeded in completely solving the decision problem for *predicate logic* also in the case of second order logic.[10]

‖[15] Here we see that the traditional doctrine of inferences comprises only a minute part of what really belongs to the domain of logical inference.

As yet I have not even mentioned *concept formation*. And, for lack of time, I cannot consider it in detail. I will just say this much: a truly penetrating logical analysis of concept formation becomes possible only on the basis of the theory of relations. Only by means of this theory one realizes what kind of complicated combinations of logical expressions (relations, existential propositions, etc.) are concealed by short expressions of ordinary language.

[10]Notice that one speaks here of "predicate logic" in the sense of the distinction between predicates and relations. Thus, what is meant here by "predicate logic" is what currently is mostly called the logic of monadic predicates. The logic of polyadic predicates is already generally undecidable for the first order case, as was shown by Alonzo Church.

Such an analysis of concept formation has been initiated to a large extent, especially by Bertrand Russell, and it has led to knowledge about general logical processes of concept formation. The methodical understanding of science is being furthered considerably through their clarification.

I now come to the end of my remarks. I have tried to show that logic, that is to say the correct old logic as it was always intended, obtains its genuine rounding off, its proper development and systematic completion, only through its mathematical treatment. The mathematical mode of consideration is introduced here not artificially, but rather arises in an entirely natural way, in the further pursuit of [actual] problems.

The resistance to mathematical logic is widespread, particularly among philosophers; it has—apart from the reasons mentioned at the beginning—also a principled one. Many approve of having mathematics absorbed into logic. But here one realizes the opposite, namely, that the system of logic is absorbed into mathematics. With respect to the mathematical formalism logic appears here as a specific interpretation and application, perfectly resembling the relation between, for example, the theory of electricity and mathematical analysis, when the former is treated according to Maxwell's theory.

That does not contradict the generality of logic, but rather the view that this generality is superordinate to that of mathematics. Logic is about certain contents that find application to any subject matter whatsoever, insofar as it is thought about. Mathematics, on the other hand, is about the most general laws of any ||[16] combination whatsoever. This is also a kind of highest generality, namely, in the direction towards the *formal*. Just as every thought, including the mathematical ones, is subordinate to the laws of logic, each structure, each manifold however primitive—and thus also the manifold given by the combination of sentences or parts of sentences—must be subject to mathematical laws.

If we wanted a logic free of mathematics, no theory at all would be left, but only pure reflection on the most simple connections of meaning. Such purely contentual considerations—which can be comprised under the name "philosophical logic"—are, in fact, indispensable and decisive as a starting point for the logical theory; just as the purely physical considerations, serving as the starting point for a physical theory, constitute the fundamental intellectual achievement for that theory. But such considerations do not constitute fully the theory itself. Its development requires the mathematical formalism. Exact systematic theory of a subject is, for sure, mathematical treatment, and it is in this sense that Hilbert's dictum holds: "Anything at all that can be the object of scientific thought, as soon as it is ripe for the formation of a theory... will be part of mathematics."[11] Even logic can not escape this fate.

[11] Editorial footnote on Hilbert source.