Bernays Project: Text No. 2

On Hilbert's thoughts concerning the founding of arithmetic (1922)

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Über Hilberts Gedanken zur Grundlegung der Arithmetik, JDMV 31, 1922, pp. 10-19.

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Comments:

1st pass in proofreading by Bernd & Dirk; 2nd pass done by the CMU Bernays Task Force; 3rd pass by Bill Tait; 4th pass CMU BTF.
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Lecture delivered at the Mathematikertagung in Jena, September, 1921.

Hilbert's new methodological approach for the founding of arithmetic, which I will discuss, is a modified and more definite version of the plan that Hilbert already had in mind for a long time and that he first expressed in his Heidelberg lecture. The previous, quite obscure hints have been replaced by a sharply outlined and comprehensible program, the beginnings of which have already been carried out.

The problem whose solution we are seeking here is that of proving the consistency of arithmetic. First we have to recall how one arrives at the formulation of this problem.

The development of arithmetic (in the wider sense, thus encompassing analysis and set theory), as it has been done since the introduction of rigorous methods, is an *axiomatic* one. This means that, as in the axiomatic grounding of geometry, one begins by assuming a *system of things* with certain relational properties [*Verknüpfungseigenschaften*]. In Dedekind's foundations of analysis the basis is taken to be the system of elements of the continuum, and in Zermelo's construction of set theory, it is the domain $\mathfrak{B}$ [*Operationsbereich*]. Also in the grounding of analysis that starts by considering number sequences, the number series is conceived of as a closed, surveyable system, akin to an infinite piano keyboard.

In the assumption of such a system with certain relational properties lies something, as it were, *transcendent* for mathematics, and the question arises as to which fundamental position one should take regrading it.

An appeal to an intuitive grasp of the number series as well as to the manifold of magnitudes is certainly to be considered. ||[11] But, in any case, this should not be a question of intuition in the primitive sense; for, in any case, no infinite manifolds are given to us in the primitive mode of intuitive representation. And even though it might be quite rash to contest any farther-reaching kind of intuitive evidence from the outset, we will nevertheless respect that tendency of exact science that aims to eliminate the finer organs of cognition as far as possible and to use only the most primitive means of cognition.

According to this viewpoint we will consider whether it is possible to ground those transcendent assumptions in such a way that only *primitive intuitive cognitions are applied*. Because of this restricted use of cognitive means, on the other hand, we cannot demand of this grounding that it allows us to recognize as truths (in the philosophical sense) the assumptions that are to be grounded. Rather, we will be content if one succeeds in showing the arithmetic built on those assumptions to be a possible (i.e., consistent) system of thought.

Hereby we have already arrived at the Hilbertian formulation of the problem. But before we look at how the problem must be tackled, we first have to ask ourselves whether there is not a different and perhaps more natural position to take on the transcendent assumptions.

In fact two different attempts suggest themselves and have also been undertaken. The one attempt aims likewise at a demonstration of consistency, not by the means of primitive intuition, but rather with the help of *logic*.

One will recall that the consistency of Euclidian geometry was already proved by Hilbert using the method of reduction to arithmetic. That is why it now seems appropriate to prove also the consistency of arithmetic by reduction to logic.

Frege and Russell, in particular, vigorously tackled the problem of the logical grounding of arithmetic.

In regard to the original goal, the result was negative.

First of all it became obvious with the famous *paradoxes of set theory* that no greater certainty with respect to the usual mathematical methods is achieved by a reduction to [*Zurückgehen auf*] logic. The contradictions of naive set theory could be seen [*ließen sich wenden*] logically as well as set theoretically. Also the control of inferences through the logical calculus, which had been developed further ||[12] precisely for securing mathematical reasoning, did not help in avoiding the contradictions.

When Russell then introduced the very cautious procedure of the [ramified] higher order predicate calculus, it turned out that analysis and set theory in their usual form could not be obtained in this way. Thus, Russell and Whitehead, in *Principia Mathematica*, were forced to introduce an assumption about the system of "first order" predicates, the so-called *axiom of reducibility*.

But one thereby returned completely to the axiomatic standpoint and gave up the goal of the logical grounding.

By the way, the difficulty appears already within the theory of whole numbers. Here, to be sure, one succeeds—by defining the numbers [*Anzahlen*] logically according to Frege's fundamental idea—in proving the laws of addition and multiplication and also the individual numerical equations as logical theorems. However, by this procedure one does not obtain the usual number theory, since one cannot prove that for every number there exists a greater one—unless one expressly introduces some kind of axiom of infinity.

Even though the development of mathematical logic did not in principle lead beyond the axiomatic standpoint, an impressive systematic construction of all of arithmetic, equal in rank to the system of Zermelo, has nonetheless emerged in this way.

Moreover, symbolic logic has taken us further in methodical knowledge: Whereas one previously only took account of the *assumptions* of the mathematical theories, now also the *inferences* are made precise. And it turns out that one can replace mathematical reasoning—in so far as only its outcomes matter—by a purely formal manipulation (according to determinate rules) in which actual thinking is completely eliminated.

However, as already said, mathematical logic does not achieve the goal of a logical grounding of arithmetic. And it is not to be assumed that the reason for this failure lies in the particular form of the Fregean approach. It seems rather to be the case that the problem of reducing mathematics to logic is completely ill-posed, namely, because mathematics and logic do not stand at all to each other in the relationship of particular and general.

Mathematics and logic are based on two different directions of abstraction. While logic deals with the *contentually* most ||[13] general [*das inhaltlich Allgemeinste*], (pure) mathematics is the general theory of the *formal* relations and properties. Such that, on the one hand each mathematical consideration is subject to the logical laws, while, on the other hand each logical figure-of-thought falls into the domain of mathematical consideration because of the external structure that necessarily comes with it.

In view of this situation one is impelled to an attempt that is, in a certain way, opposed to that of the logical grounding of arithmetic. Because one fails to establish as logically necessary the mathematically transcendent basic assumptions, the question arises whether these assumptions cannot be dispensed with at all.

In fact, one possibility for eliminating the axiomatic basic assumptions seems to consist of elimination entirely the existential form of the axioms and replacing the existential assumptions by *construction postulates*.

Such a replacement procedure is not new to the mathematician; especially in elementary geometry the constructive version of the axioms is often applied. For example, instead of laying down the axiom that any two points determine a line, one postulates the connection of two points by a line as a possible construction.

Likewise, one can now proceed with the arithmetical axioms. For example, instead of saying "each number has a successor," one introduces progression by one or the attachment of +1 as a basic operation.

One thus arrives at the attempt of a *purely constructive development of arithmetic*. And indeed this goal for mathematical thought is a very tempting one: Pure mathematics should be the carpenter of its own house and not be dependent on the assumption of a certain system of things.

This constructive tendency, which was first brought forcefully into prominence by Kronecker, and later, in a less radical form, by Poincaré, is currently pursued by Brouwer and Weyl in their new founding of arithmetic.

Weyl first checks the higher modes of inference in regard to the possibility of a constructive reinterpretation; that is, he investigates whether or not the methods of analysis as well as those of Zermelo's set theory can be interpreted constructively. He finds this impossible, for in the attempt to thoroughly carry out a replacement of the existential axioms by constructive methods, one constantly falls into logical circles. ||[14]

From this Weyl draws the conclusion that the modes of inference of analysis and set theory have to be restricted to such an extent that in their constructive interpretation no logical circles arise. In particular, he feels compelled to give up the theorem of the existence of the upper bound.

Brouwer goes even further in this direction by also applying the constructive principle to large numbers. If one wants, as Brouwer does, to avoid the assumption of a closed given totality of all numbers and to take as a foundation only the act of progressing by one, performable without bound, then statements of the form "There are numbers of such and such a type . . . " do not have a *prima facie* meaning. Thus, one is also not justified in generally putting forward, for each number theoretical statement, the alternative that either the statement holds for all numbers or that there is a number (respectively, a pair of numbers, a triple of numbers, . . .) by which it is refuted. This way of applying the "tertium non datur" is then at least questionable.

Thereby we find ourselves in a great predicament: The most successful, elegant, and time-tested modes of inference ought to be abandoned just because, from a specific standpoint, one has no justification for them.

The unsatisfactoriness of such a procedure can not be overcome by the considerations, by which Weyl tries to show that the concept formation of the mathematical continuum, as it is fundamental for ordinary analysis, does not correspond to the visual [*bildlich*] representation of continuity. <*des Stetigen*>. For, an exact analogy to the content of perception is not at all necessary for the applicability and the fruitfulness of analysis; rather, it is perfectly sufficient that the method of idealization and conceptual interpolation used in analysis be consistently applicable. As far as pure mathematics is concerned, it only matters whether the usual, axiomatically characterized mathematical continuum is in itself a possible, that is, a consistent, structure [*Gebilde*].

At best, this question could be rejected if, instead of the hitherto prevailing mathematical continuum, we had at our disposal a simpler and more perspicuous conception that would supersede it. But if one examines more closely the new approaches by Weyl and Brouwer, one notices that a gain in simplicity can not be hoped for here; rather, the complications required in the concept formations and modes of inference are only increased instead of decreased.

Thus, it is not justified to dismiss the question concerning the consistency ||[15] of the usual axiom system for arithmetic. And what we are to draw from Weyl's and Brouwer's investigations is the result that a demonstration of consistency is not possible by means of replacing existential axioms by construction postulates.

So we come back to Hilbert's idea of a theory of consistency based on a primitive-intuitive foundation. And now I would like to describe the plan, according to which Hilbert intends to develop such a theory, and the leading principles to which he adheres in doing so.

Hilbert takes over what is positively fruitful from each of the two attempts at grounding [mathematics] discussed above. From the logical theory he takes the method of the rigorous formalization of inference. That this formalization is necessary follows directly from the way the task is formulated. For the mathematical proofs are to be made the object of a concrete-intuitive form of view. To this end, however, it is necessary that they are projected, as it were, into the domain of the formal. Accordingly, in Hilbert's theory we have to distinguish sharply between the formal image [*Abbild*] of the arithmetical statements and proofs as the *object* of the theory, on the one hand, and the contentual thought about this formalism, as the *content* of the theory, on the other hand. The formalization is done in such a way that formulas take the place of contentual mathematical statements, and that a sequence of formulas, following each other according to certain rules, takes the place of an inference. But one does not attach any meaning to the formulas; the formula does not count as the expression of a thought, but it corresponds to a contentual judgment only insofar as it plays, within the formalism, a role analogous to that which the judgment plays within the contentual consideration.

More basic than this connection to symbolic logic is the contiguity of Hilbert's approach with the constructive theories of Weyl and Brouwer. For Hilbert in no way wants to abandon the constructive tendency that aims at the autonomy of mathematics; rather, he is especially eager to bring it to bear in the strongest way. In light of what we stated with respect to the constructive method, this appears at first to be incompatible with the goal to demonstrate the consistency proof of arithmetic. In fact, however, the obstacle to the unification of both goals lies only in a preconceived opinion from which the advocates of the constructive tendency have always proceeded until now, namely, that within the domain of arithmetic every construction must indeed be a *number construction* (respectively set construction). ||[16] Hilbert considers this view to be a prejudice. A constructive reinterpretation of the existential axioms is possible not only in such a way that one transforms them into generating principles for the construction of numbers; rather, the mode of inference made possible by such an axiom can, as a whole, be replaced by a formal procedure in a such a way that certain signs replace general concepts like number, function, etc.

Whenever concepts are missing, a sign will be readily available. This is the methodical principle of Hilbert's theory. An example should explain what is meant. The existential axiom "Each number has a successor" holds in number theory. In keeping with the restriction to what is concretely intuitive, the general concept of number as well as the existential form of the statement must now be avoided.

As mentioned above, the usual constructive reinterpretation consists in this case in replacing the existential axiom by the procedure of progression by one. This is a procedure of *number* construction. Hilbert, on the contrary, replaces the concept of number by a symbol Z and lays down the formula:

$$Z(a) \to Z(a+1).$$

Here a is a variable for which any mathematical expression can be substituted, and the sign $\to$ represents the hypothetical propositional connective "if—then," that is, the following rule holds: if two formulas $\mathfrak{A}$ and $\mathfrak{A} \to \mathfrak{B}$ are written down, then $\mathfrak{B}$ can also be written down.

On the basis of these stipulations, the mentioned formula accomplishes, within the framework of the formalism, exactly what is otherwise accomplished by the existential axiom for contentual argumentation [*Beweisführung*].

Here we see how Hilbert utilizes the method of formalizing inferences according to the constructive tendency; for him it is in no way merely a tool for the demonstration of consistency. Rather, it is, at the same time, also the way to a *rigorous constructive development* of arithmetic. Moreover, the methodical idea of construction is here conceived of so broadly, that also all higher mathematical modes of inference can be incorporated in the constructive development.

After having characterized the goal of Hilbert's theory, I would now like to outline the basic features of the theory. The following three questions are to be answered:

1. The constructive development should represent the formal image [*Abbild*] of the system of arithmetic and at the same time it should ||[17] the object for the intuitive theory of consistency. How does such a development take shape?

2. How is the consistency statement to be formulated?

3. What are the means of the contentual consideration by which the demonstration of consistency [*Widerspruchslosigkeit*] is to be carried out?

First, as far as the constructive development is concerned, it is accomplished in the following way. Above all, the different kinds of signs are introduced, and at the same time the substitution rules are determined. Furthermore, certain formulas are layed down as basic formulas. And now "proofs" are to be formed.

What counts here as a proof is a concretely written-down sequence of formulas where for each formula the following alternative holds: Either the formula is identical to a basic formula or to a preceding formula, or it results from such a formula by a valid substitution; or, it constitutes the end formula in an "inference," that is, in a sequence of formulas of the type

$$\frac{\begin{array}{c}\mathfrak{A}\\ \mathfrak{A}\rightarrow\mathfrak{B}\end{array}}{\mathfrak{B}.}$$

Hence a "proof" is nothing else than a figure with certain concrete properties and such figures constitute the formal image [*Abbild*] of arithmetic.

This answer to the first question makes the urgency of the second especially evident. For what should the statement of consistency mean in regard to the pure formalism? Isn't it impossible that mere formulas can contradict themselves?

The simple reply to this is: The contradiction is formalized just as well. Faithful to his principle Hilbert introduces the letter Ω for the contradiction; and the role of this letter within the formalism is determined by laying down basic formulas in such a way that from any two formulas—to which contrary statements correspond—Ω can be deduced. More precisely, by adding two such formulas to the basic formulas, a proof with Ω as end formula can be constructed.

In particular the basic formula

$$a = b \rightarrow (a \neq b \rightarrow \Omega)$$

serves us here, where $\neq$ is the usual sign of inequality. (The relation of inequality is taken by Hilbert as a genuine arithmetical relation, just as equality is, but not as the logical negation of ||[18] equality. Hilbert does not introduce a sign for negation at all.)

Now, the statement of consistency is simply formulated as follows: Ω can not be obtained as the end formula of a proof.

Hence, this claim is in need of a demonstration.

Now the only remaining question concerns the means by which this demonstration should be carried out. In principle this question is already settled.

For our whole problem originates from the demand of taking only what is concretely intuitive as a basis for mathematical considerations. Thus the matter is simply to realize which tools are available to us from the concrete-intuitive point of view. <*Betrachtungsweise*>

This much is certain: We are justified in using, to the full extent, the elementary ideas of succession and order as well as the usual counting. (For example, we can see whether there are three, or fewer, occurrences of the sign $\rightarrow$ in a formula.)

However, we cannot get by in this way alone; rather, it is absolutely necessary to apply certain forms of complete induction. Yet, in doing so we do not go beyond the domain of what is concretely intuitive.

To wit, two types of complete induction are to be distinguished: the narrower form of induction, which applies only to something completely and concretely given, and the wider form of induction, which uses in an essential manner either the general concept of whole number or the operating with variables.

Whereas this wider form of complete induction is a higher mode of inference which is to be grounded by Hilbert's theory, the narrower form of inference is part of primitive intuitive knowledge and can therefore be used as a tool of contentual argumentation.

As typical examples of the narrower form of complete induction, as it is used in the argumentations of Hilbert's theory, the following two inferences can be adduced:

1. If the sign $+$ occurs at all in a concretely given proof, then, in reading through the proof, one finds a place where it occurs for the first time.

2. If one has a general procedure for eliminating, from a proof with a certain concretely describable property $\mathfrak{E}$, the first occurrence of the sign Z, without the proof ||[19] losing the property $\mathfrak{E}$ in the process, then one can, by repeated application of the procedure, completely remove the sign Z from such a proof, without its losing the property $\mathfrak{E}$.

(Notice, that here it is exclusively a question of formalized proofs, i.e., proofs in the sense of the definition given above.)

The method which the theory of consistency must follow is hereby set forth in its essentials. Currently the development of this theory is still in its early infancy; most of it has yet to be accomplished. In any case though, the possibility in principle and the feasibility in practice of the required point of view can already be recognized from what has been achieved so far; and one also sees that the considerations to be employed here are *mathematical* in the very genuine sense.

The great advantage of Hilbert's procedure is just this: The problems and difficulties that present themselves in the founding of mathematics are transferred from the epistemologico-philosophical domain into the realm of what is properly mathematical.

Mathematics creates here a court of arbitration for itself, before which all fundamental questions can be settled in a specifically mathematical way, without having to rack one's brain about subtle questions of logical scruples [*Gewissensfragen*] such as whether judgments of a certain form make sense or not.

Hence we can also expect that the enterprise of Hilbert's new theory will soon meet with approval and support within mathematical circles.

(Received October 13, 1921.)