Predicative Categories of Classes

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October 28, 2004

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Abstract

In this thesis the tools of category theory and categorical logic are employed in order to study predicative set theories. Specifically, we introduce two constructive set theories **BCST** and **CST** and prove that they are sound and complete with respect to models in categories with certain structure. Specifically, *basic categories of classes* and *categories of classes* are axiomatized and shown to provide models of the aforementioned set theories.

We then show that given any Heyting pretopos $\mathcal{E}$ there exists a subcategory $\mathbf{Idl}(\mathcal{E})$ of sheaves on $\mathcal{E}$, called the *ideal completion of* $\mathcal{E}$, which is such a category of classes. Specifically, we construct fixed points for the powerobject functor $\mathcal{P}_s(-) : \mathbf{Idl}(\mathcal{E}) \longrightarrow \mathbf{Idl}(\mathcal{E})$ in order to build models of the untyped set theory **BCST**. Furthermore, if $\mathcal{E}$ is a locally cartesian closed pretopos, then the construction yields models of **CST** inside $\mathbf{Idl}(\mathcal{E})$. Finally, it is a consequence of this work that the set theories in question are sound and complete with respect to such models in categories of ideals. This embedding results serves to establish, in effect, the conservativity of the set theory **CST** over a form of dependent type theory. Additional minor results which may (or may not) be of independent interest are also obtained regarding the categories in question.

Acknowledgments

I am extremely grateful to Steve Awodey for both the excellent guidance he has provided me and the kindness he has shown to me. I have also been lucky enough to have benefitted from discussions of the subject matter of this thesis with Nicola Gambino and Alex Simpson. Henrik Forssell, Ivar Rummelhoff and I were all at Carnegie Mellon University learning about algebraic set theory at the same time and I have had many useful conversations about the subject with both of them. I have also received useful input from Jeremy Avigad, Carsten Butz, Nikolaj Jang Pedersen, Dana Scott, Wilfried Sieg and Thomas Streicher. Finally, this thesis would certainly not exist were it not for the patience and support of Laura, Carol and Richard.

Introduction

This thesis is about three areas in the foundations of mathematics and their interrelations: set theory, type theory and category theory. Set theory is the superficial topic; type theory is the more subtle topic; and category theory is the topic by which the others are bound together and organized. That these three topics, which some have found to be at odds with one another, should appear together is really not surprising at all. To begin with, a category may be regarded, via its *internal language*, as a type theory and, similarly, type theories give rise to categories (e.g., their *syntactic categories*, among others). Moreover, a set theory may be regarded at least intuitively as a type theory in which all of the types have been lumped together or 'summed'. Indeed, the 'stage' heuristic which is commonly given to motivate the axioms of Zermelo-Fraenkel set theory is related to type theoretic concerns. Finally, the elegant methods of category theory are as applicable to logic and set theory as they are to algebraic geometry or algebraic topology.

To be specific, though, the main question which it is the aim of this thesis to elucidate and answer is: *How can we axiomatize, in a first-order set theory, the 'sets' constructed in a predicative type theory? That is, given a predicative type theory T, what is the first-order set theory such that the category $\mathcal{S}$ of its sets satisfies exactly the axioms of T?* More generally, the methods of category theory and categorical logic are employed in this thesis in order to provide models of certain *constructive* or *predicative* set theories.[1]

Before delving into a description of the specific contents and technical results contained in this thesis it will be instructive to briefly review some of the history of constructive mathematics with special attention given to the rôle of predicativity in constructive mathematics. This discussion should

[1]Throughout this introduction 'constructive' is taken in a rather broad sense which may include impredicative intuitionistic views and theories. However, it is conventional to refer to predicative set theories as 'constructive set theories' and we follow this usage (we also discuss Russell's ramified theory of types which is, alas, based on classical logic).

serve to provide those readers unfamiliar with what may appear to be rather exotic theories and concepts with at least enough background to appreciate some of the technical results contained herein. Following this historical aside the reader will find a short summary of the thesis including descriptions of the main results.

Predicativity and Constructive Mathematics

One may make a fairly convincing case that prior to the 19th Century all of mathematics was, essentially, constructive and that the advent of non-constructive mathematics arose simultaneously with the introduction of what might be called 'abstract' methods in mathematics.[2] Yet, the history of constructivism posterior to the adoption of essentially non-constructive methods by the likes of Cantor and others is by no means the linear and rigid tale which one sometimes encounters (namely, that the 'mystic' Brouwer was unhappy with non-constructive mathematics and therefore developed his philosophy of intuitionism, for which only the Dutch seem to retain some predilection). On the contrary, once placed in the proper historical context the development of constructivism appears as dynamic and, indeed, natural as does that of mathematics simpliciter (the historical appendix of Beeson's [9] is a good place to begin looking for such context). One particularly interesting facet of this development is associated with the notion of *predicativity* which originally arose in connection with the antinomies of naïve set theory.[3]

Anyone with elementary training in logic is aware that the antinomies result from admitting the extension $\{x|\varphi\}$ of every formula φ as a set; for given such freedom one easily concludes that such curiosities as $R := \{x|x \notin x\}$ are sets. Russell and Poincaré both attributed the antinomies to the presence of 'vicious circles' lurking within the suspect definitions and collections. For instance, in the case of R this circularity is to be found in the fact that R itself may occur as a value of the bound variable x in the expression $\{x|x \notin x\}$. The solution proposed by Russell and Poincaré is captured by the so-called *vicious circle principle* which, roughly, states that, 'whatever contains an apparent variable must not be a possible value of that variable [30].' For Russell, a propositional function $\varphi(x)$ is *predicative* precisely when $\{x|\varphi(x)\}$ defines a class and *impredicative* when it does not (i.e., when

[2]That this is so has led some to mistakenly conflate modern (abstract) mathematics and non-constructive mathematics.

[3]For a more detailed account of the history of predicativity the reader is directed to Feferman's excellent survey [13].

$\{x|\varphi(x)\}$ does not signify anything). The rôle of the vicious circle principle is then to provide us with a means for distinguishing between those propositional functions that are predicative and those that are impredicative. Intuitively, $\varphi(x)$ is impredicative when $\{x|\varphi(x)\}$ would be contained in the range of x or any of the bound variables which occur in φ. Given this intuitive understanding of predicativity the question becomes: How are we to implement this intuitive notion in a formal logical system? Before answering this question we will briefly review Russell's simple type theory (not to be confused with Church's simply typed lambda calculus) which, although impredicative, affords a reasonable starting point for understanding Russell's implementation of the vicious circle principle in his ramified theory of types.

If we are interested merely in preventing the formation of extensions of those propositional functions $\varphi(x)$ which may themselves occur as values of the bound variable x in $\{x|\varphi(x)\}$, then it seems that extensions $\{x|\varphi(x)\}$ may only be formed when the range of the bound variable x is fixed in advance or 'constructed' prior to $\{x|\varphi(x)\}$.[4] Russell's simple theory of types provides one way of capturing this intuitive idea and key to this theory is the concept of *type*.

From the perspective of natural language the notion of type is well understood. For instance consider the following English locution:

_____ solves the problem on the blackboard.

The blank may only be filled in certain ways for the locution to become a well-formed sentence of the English language. For instance, 'Jones' and 'She' may be reasonably substituted in place of the blank; whereas no plural may (cf. Russell's discussion of type in appendix B of [29]).[5]

In a modern form, the language of the theory of types contains variables of different types where there is a type for each natural number n (cf. [15]). E.g., x_n, y_3, z_{15}, and so forth (with the subscript indicating the type of the variable). An atomic formula $\xi \in \zeta$ (with ξ and ζ meta-variables ranging over variables of arbitrary types) is then well-formed if and only if ζ is of type $n+1$ if ξ is of type n. Similarly, $\xi = \zeta$ is well formed if and only if both variables are of the same type. The following axiom schemata specifying the behavior of the membership relation are also adopted:

[4]The reader should note that this proscription differs from the vicious circle principle in that here $\{x|\varphi(x)\}$ may now occur as a value of variables bound in $\varphi(x)$.

[5]Other languages, e.g. Latin or German, provide even better examples of type distinctions due to the systematic use of cases and genders.

Extensionality: $\forall x^{n+1}.\forall y^{n+1}((\forall z^n.z^n \in x^{n+1} \Leftrightarrow z^n \in y^{n+1}) \Rightarrow x^{n+1} = y^{n+1})$.

Comprehension: $\exists y^{n+1}.\forall x^n(x^n \in y^{n+1} \Leftrightarrow \varphi(x^n))$.

One easily verifies that R cannot be a set in the theory since a variable of the same type occurs on both sides of the membership relation. The stratification to which the type constraints give rise prevents the derivation of the usual antinomies (and also leads to some curiosities for which we refer the reader to [15]). As noted above, the simple theory of types is impredicative in the sense that it does not adhere to the vicious circle principle. However, Russell's *ramified theory of types*, which is based on a similar notion of type to that of the simple theory, is predicative.

If we are to take the vicious circle principle seriously, then it seems that in order for $\varphi(x)$ to have a genuine extension it must be that not only is the range of the variable x in $\{x|\varphi(x)\}$ fixed, but also the values of all bound variables occurring in $\varphi(x)$ must be fixed in advance. That is, the values of x and all bound variables in $\varphi(x)$ must be constructed prior to $\{x|\varphi(x)\}$. This is made precise in the ramified theory of types by now considering *orders* of variables as well as types. The language of the ramified theory consists of variables of *type* 0 or 'individuals' x_0, y_0, et cetera and, for all natural numbers n and m greater than 1, a stock of variables of *type* n and *order* m: $x_n^m, y_n^m, \ldots$, where the superscript indicates the order of the variable and the subscript indicates its type. Then $\xi = \zeta$ is well formed if and only if ξ and ζ have the same order and the same type and $\xi \in \zeta$ is well formed if and only if the type of ζ is $n+1$ if the type of ξ is n. The comprehension axiom for the ramified theory is then (for this modern treatment of the ramified theory see [15]):

Comprehension: $\exists z_{n+1}^{m+1}.\forall x_n^l(x_n^l \in z_{n+1}^{m+1} \Leftrightarrow \varphi(x_n^l))$, where m is the maximum order of any bound variable of type $n+1$ occurring in $\varphi(x_n^l)$.

Intuitively, a variable x_0 is to be thought of as a usual variable and a variable x_1^1 is to be thought of the extension of a propositional function that takes individuals as arguments and in which there occur either no bound variables at all or only bound variables ranging over individuals (such a propositional function is said to be of *first-order*). A variable x_1^2 would then be the extension of a propositional function which takes individuals as arguments and in which there occurs at least one bound variable ranging over first-order functions and no bound variables ranging over neither first-order functions nor individuals (such a function is of *second-order*). Thus, whereas the simple theory restricts comprehension only on the basis of the argument of a propositional function, the ramified theory restricts comprehension to only those propositional functions which possess both arguments and definitions which are acceptable in the appropriate sense.

The ramified theory is clearly more restrictive than the simple theory and, indeed, it has several drawbacks which have made it untenable for actual mathematical practice. To begin with, even Russell realized that ordinary proofs by induction would not be possible in the ramified theory and, as a consequence, was compelled to introduce the *axiom of reducibility* the addition of which renders the system impredicative (cf. section V of [30]). As becomes clear from Weyl's [36] study of predicative analysis, one faces many additional difficulties when trying to do mathematics in a predicative system. Yet, the impractical nature of the syntax alone has rendered the system sufficiently repugnant that it has won few (if any) true disciples.[6]

The myth, which is due in part to the shortcomings of the ramified theory of types, that predicative systems are either too weak or too complicated for one to carry out much of mathematics within them has repeatedly hindered the development of constructive mathematics and it was not until the publication of Bishop's [10] in 1967 that this myth began to be dispelled among specialists. Shortly after the publication of [10] there were attempts to develop both set theoretic and type theoretic systems of predicative mathematics in which the work of Bishop could be formalized.

On the side of type theory, Martin-Löf [24] has, on the basis of a constructive philosophy of mathematics, developed a sophisticated predicative type theory which is now taken to be the paradigm of constructivity. An alternative to the type theoretic framework of Martin-Löf has been offered in the form of constructive set theories (such set theories are predicative in the sense that they do not admit the powerset axiom). Such set theories were introduced and studied by Myhill [27], Friedman [16] and Aczel [1]. In [1], [2] and [3] Aczel showed that a particular form of constructive set theory known as **CZF** can be given an interpretation in a form of Martin-Löf type theory, thereby relating the two theories.

[6] One point of history which is glossed over in this introduction is the connection between intuitionistic logic and predicativity. This may appear to be a considerable omission since both the ramified and simple theories of types employ classical logic whereas all other theories considered in this thesis are intuitionistically acceptable. We do not, however, have the space here to address this historical issue fully and the reader is referred to [9] and [13] for the relevant historical details.

Summary of the Thesis

The approach adopted in this thesis is that of *algebraic set theory* as developed by Joyal and Moerdijk [21] and refined in the work of Simpson [34], Moerdijk and Palmgren [25, 26], Butz [12] and Awodey, Butz, Simpson and Streicher [6]. Awodey et al [6] have formulated an impredicative set theory **BIST** such that: (i) the category of sets in **BIST** form a topos, (ii) **BIST** is sound and complete with respect to models in (impredicative) categories of classes, (iii) every topos $\mathcal{E}$ may be embedded in a category of classes $\mathcal{E}'$ (the category of *(inclusion) ideals* of $\mathcal{E}$) and (iv) the set theory $\mathbf{BIST}_C$ obtained by adding the Strong Collection axiom to **BIST** is sound and complete with respect to models in (inclusion) ideal categories $\mathcal{E}'$ where $\mathcal{E}$ a topos. One way of understanding this result is as showing that **BIST** is the set theory 'generated' by Intuitionistic Higher Order Logic (the simple type theory associated with topoi). As mentioned earlier, it is the aim of this thesis to obtain similar results for the case of predicative type theories. While this may at first sound like a trivial task the reader should note that without the assumption that the underlying category possesses a type of propositions matters become sufficiently complicated that the task at hand requires a fair amount of work. In addition, the embedding results obtained in this thesis make use of the *ideal completion* of a category which is the completion of the original category under certain 'nice' colimits. This completion has several virtues over the inclusion ideals considered in [6]. For one, the ideal completion $\mathbf{Idl}(\mathcal{E})$ of a category $\mathcal{E}$ is a full subcategory of sheaves on $\mathcal{E}$ (the ideal completion is also considered in [28], [7] and [14]). As such, we may employ the traditional tools of sheaf theory (and those associated with functor categories in general).

This is not the first attempt to study predicative set theories and type theories using the methods developed in [21]. Specifically, Moerdijk and Palmgren [25, 26] have investigated such theories using these methods. However, just as there is a difference in aim and perspective between the early work of Joyal and Moerdijk in [21] and that of Awodey, Butz, Simpson and Streicher in [6] there is also a difference between the work of Moerdijk and Palmgren and that contained in this thesis. To begin with, one of the main goals of [25] and [26] was to arrive, through a category theoretic investigation of type theoretic constructions such as W-types, at an appropriate notion of 'predicative topos' which will relate to predicative type theories in the same way that topoi relate to intuitionistic higher-order logic (IHOL). In this context, Moerdijk and Palmgren are led to consider locally cartesian closed pretopoi with W-types. In this thesis lcc pretopoi (which are here dubbed Π-*pretopoi*) are also studied, but from a slightly different perspective; for whereas Moerdijk and Palmgren model **CZF** (using the Mostowski collapse of universal small maps) in such categories, we are concerned in this thesis with modeling weaker set theories in weaker categories. However, the set theory **CST** of which we provide models in this thesis is, in an appropriate sense to be made precise below, the set theory corresponding to Π-pretopoi. Nonetheless the results obtained by Moerdijk and Palmgren are certainly related to this thesis and, in some cases, suggest directions which future research in the area might take (e.g., working out an internal version of the ideal construction discussed in this thesis inside a Π-pretopos, issues to do with W-types, et cetera). As such, there is hope that future investigations will reveal more concrete connections between this research and that of Moerdijk and Palmgren.

Recently Gambino [17] has studied presheaf models of constructive set theories. One interesting aspect of this work is that it serves to relate the approach of Joyal and Moerdijk with that of Scott [31]. There is, unfortunately, insufficient space in this thesis to discuss this exciting research.

Finally, the reader should note that most of the results (and all of the major results) contained in this thesis were previously presented in the technical report [8] written jointly by the author of this thesis and Steve Awodey.

We now turn to a summary of the thesis and its main results.

Chapter 1: Π-Pretopoi and Constructive Set Theories

The purpose of the first chapter is to introduce the reader to the particular type theory and set theories which we intend to study. This task begins with the definition of Π-*pretopoi* which are, in some sense, 'predicative topoi'. A Π-pretopos is a locally cartesian closed pretopos and, from a logical perspective, is a sort of predicative type theory. Indeed, upon introducing Π-pretopoi we turn to a brief introduction to dependent type theory, after which it is shown that dependent type theory is sound and complete with respect to models in Π-pretopoi.

The focus then shifts from type theory to set theory and the two constructive set theories **BCST** (basic constructive set theory) and **CST** (constructive set theory) are introduced. Several basic properties of these set theories are derived.

The main results of this chapter are the following:

Corollary (1.1.4) *Dependent type theory is sound and complete with respect to models in* Π-*pretopoi.*

Theorem (1.2.16) *The category of sets of* **CST** *form a* Π-*pretopos.*

Chapter 2: Predicative Categories of Classes

In the second chapter the categories suitable for modeling **BCST** and **CST** are introduced axiomatically and their properties are developed. First, *categories with basic class structure* and *categories with class structure* are defined. Such categories are capable of modeling certain typed versions of the set theories under consideration. In short, a category $\mathcal{C}$ has basic class structure if and only if it is a positive Heyting category (and so is capable of modeling first-order logic), it is equipped with a system $\mathcal{S}$ of *small maps* required to satisfy certain stability conditions, and for each object C there exists a powerobject $\mathcal{P}_s(C)$ which has a universal property similar to that for powerobjects in topos theory. A category also has class structure (simpliciter) if, roughly, exponentials of small objects by small objects are small (one defines small objects in terms of the system $\mathcal{S}$). It is shown that the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects and maps forms a Heyting pretopos if $\mathcal{C}$ has basic class structure and a Π-pretopos if $\mathcal{C}$ has class structure.

In order to model the untyped set theories **BCST** and **CST** it is necessary to augment categories with (basic) class structure with a *universal object* U. That is, an object U such that for every object D there exists a monomorphism $D \rightarrowtail U$. A category with class structure and a universal object is called a *category of classes* (similarly for *basic categories of classes*). Once such categories are introduced it is shown that **BCST** is sound and complete with respect to models in basic categories of classes and **CST** is sound and complete with respect to models in categories of classes.

In particular, the main results are:

Theorem (2.1.25) *If $\mathcal{C}$ has basic class structure, then the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects and maps is a Heyting pretopos. If $\mathcal{C}$ has class structure, then $\mathcal{S}_{\mathcal{C}}$ is a Π-pretopos.*

Theorem (2.2.8, 2.2.13, 2.2.25) BCST *is sound and complete with respect to models in basic categories of classes and* **CST** *is sound and complete with respect to models in categories of classes.*

Chapter 3: The Ideal Completion of a Π-Pretopos

'Completions' arise in a wide range of different mathematical contexts. However, they are all instances of the general situation in which one is given some 'mathematical-object' X (an object of some category) which is lacking some important property and one wishes to somehow obtain a new object X' such that X' is 'just like' X in relevant respects but also has the new 'special' property. Examples of completions include: (i) the completion of a metric space (here one is given a metric space X in which Cauchy sequences need not converge and one obtains a new metric space X' in which they do along with an isometric embedding of X into X'), (ii) the ideal completion of a lattice L to a complete lattice L', obtained by taking order ideals in L, and (iii) the cocompletion of a category $\mathcal{C}$ into a cocomplete category $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$. Of course, all such completions possess descriptions in terms of universal mapping properties since they are instances of adjunctions and, as such, are uniquely determined up to isomorphism, independently of the particular construction used. In our case we are interested in completing a Π-pretopos (or Heyting pretopos) $\mathcal{E}$ to a category with class structure and containing a universe (an object U such that $\mathcal{P}_s U \rightarrowtail U$). It turns out that the completion we employ to this end has much in common with the ideal completion of a lattice.

Recall that an *ideal* in a lattice L is a non-empty subset $I \subseteq L$ satisfying: (i) if $a, b \in I$, then $a \vee b \in I$, and (ii) if $b \in I$ and $a \leq b$, then $a \in I$. Then, for any lattice L, there exists a complete lattice $\mathbf{Idl}(L)$ in which L is embedded via a lattice homomorphism $\downarrow : L \hookrightarrow \mathbf{Idl}(L)$. In particular, $\mathbf{Idl}(L)$ is given by the collection $\{I | I \text{ is an ideal in } L\}$ and the inclusion ordering, and the embedding $\downarrow$ takes an element $a \in L$ to the *principal ideal* $\downarrow (a) := \{x \in L | x \leq a\}$ *generated by* a. Moreover, $\mathbf{Idl}(L)$ is the *universal* way of mapping L to a complete lattice.

Similarly, where $\mathcal{C}$ is a category, we shall define an *ideal* in $\widehat{\mathcal{C}}$ is a presheaf which is a colimit of an *ideal diagram* of representables (a special kind of diagram with a filtered domain). Intuitively, the ideals are those presheaves that may be 'patched together' or approximated by their representable components (this is, of course, all made precise in the thesis). The most useful tool for studying ideals is a characterization of the ideals as certain sheaves. Roughly, a map $\varphi : F \longrightarrow G$ of sheaves is said to be *small* provided that it has representable fibers. Then it turns out that the ideals are exactly those sheaves with small diagonals (cf. [7]). This characterization, which was suggested by André Joyal and is called the *Joyal Condition* in this thesis, provides a useful tool for investigating the category $\mathbf{Idl}(\mathcal{C})$ of ideals on $\mathcal{C}$. We should note here that the base category $\mathcal{C}$ must be a pretopos in order for this characterization of ideals to hold.

Once the basic properties of $\mathbf{Idl}(\mathcal{E})$, for $\mathcal{E}$ a pretopos, have been developed we turn to the definition of powerobjects in $\mathbf{Idl}(\mathcal{E})$. This point is somewhat tricky in that, given an ideal X, once we have defined the powerobject $\mathcal{P}_s X$ of X as a presheaf, we must then show that $\mathcal{P}_s X$ is also an ideal. In order to show that this is the case, and thereby justify our definition, the

Joyal Condition is employed (i.e., it is shown that $\mathcal{P}_s X$ has a small diagonal when X is an ideal). The reader should note that in order for $\mathcal{P}_s X$ to be an ideal the base category must be Heyting in addition to being a pretopos. In particular, the proof that $\mathcal{P}_s X$ is an ideal makes crucial use of the existence of dual-images in the base category (i.e., the existence of universal quantification in the internal language of the base category). Once this key result has been established we turn to showing that $\mathbf{Idl}(\mathcal{R})$ has basic class structure and a universe if $\mathcal{R}$ is a Heyting pretopos and that it has class structure and a universe if $\mathcal{R}$ is a Π-pretopos.

Since $\mathbf{Idl}(\mathcal{R})$ will possess a universe U for those $\mathcal{R}$ with which we are concerned we may then restrict to the subcategory $\downarrow(U)$ of $\mathbf{Idl}(\mathcal{R})$ consisting of those ideals below U. This subcategory is then a basic category of classes if $\mathcal{R}$ is a Heyting pretopos, and a category of classes if $\mathcal{R}$ is a Π-pretopos, in which $\mathcal{R}$ occurs as the category of 'sets' – which is our main result.

The ideal categories $\mathbf{Idl}(\mathcal{R})$ have additional nice properties. For instance, the Strong Collection axiom will be satisfied by models of set theories in ideal categories. This fact and additional special properties are considered next. Finally, a sketch of the proof that $\mathbf{BCST}_C$ and $\mathbf{CST}_C$ are sound and complete with respect to ideal models is given.

The main results of this chapter are:

Proposition (3.0.39, 3.0.41) *If $\mathcal{R}$ is a Heyting pretopos and X is an ideal, then the small powerobject functor $\mathcal{P}_s X$ is also an ideal.*

Theorem (3.0.46, 3.0.49) *If $\mathcal{R}$ is a Heyting pretopos, then $\mathbf{Idl}(\mathcal{R})$ has basic class structure. Moreover, if $\mathcal{R}$ is a Π-pretopos, then $\mathbf{Idl}(\mathcal{R})$ has class structure.*

Theorem (3.0.50) *If $\mathcal{R}$ is a Heyting pretopos, then $\downarrow(U)$ is a basic category of classes and if $\mathcal{R}$ is a Π-pretopos then $\downarrow(U)$ is a category of classes.*

Theorem (The Main Result) *Every Heyting pretopos occurs as the category of sets in a basic category of classes. Every Π-pretopos occurs as the category of sets in a category of classes.*

Theorem (3.0.62) $\mathbf{BCST}_C$ *sound and complete with respect to ideal models over Heyting pretopoi and similarly for* $\mathbf{CST}_C$ *over Π-pretopoi.*

Appendix: Some Basic Category Theory

Finally, the appendix contains some basic information on category theory and categorical logic which may be of interest to the reader. However, this information is not at all comprehensive and is merely included as a handy reference to remind the reader of several facts.

Notation and Conventions

Throughout this thesis 'we' is employed to mean the reader and the author. However, all mistakes which 'we' might make are the responsibility of the author alone. Nearly all category theoretic terminology and notation follows Johnstone's usage (cf. [19] and [20]). In particular, a *cover* is a regular epimorphism (written $\longrightarrow\!\!\!\!\triangleright$) and a *cartesian category* is a left exact category. The names of the set theories under consideration are somewhat arbitrary (as such things usually are) and the name **CST**, which stands for 'Constructive Set Theory', was employed by Myhill [27] to name a slightly different theory. Hopefully this nomenclature will not result in any confusion; for the best alternative, which was an abbreviation of the rather long-winded 'Predicative Intuitionistic Set Theory', seemed a bit inappropriate for English speakers (although good for jokes). Other notation and conventions are mentioned in the appropriate chapters (e.g., set theoretic notation is introduced at the beginning of the section on set theories).

Chapter 1

Π-Pretopoi and Constructive Set Theories

In this chapter we will introduce both dependent type theory and constructive set theory. Specifically, we begin by introducing a certain class of categories (the locally cartesian closed categories and the Π-pretopoi) which will provide models of dependent type theory and which will, later on, be completed to models of constructive set theories. Some basic facts about the type theories and set theories under consideration are also derived.

1.1 Π-Pretopoi and Dependent Type Theory

In this section we define Π-pretopoi and prove that dependent type theory is sound and complete with respect to such categories.

1.1.1 Π-Pretopoi

A locally cartesian closed category $\mathcal{C}$ is a cartesian category such that each slice $\mathcal{C}/D$ is cartesian closed. That is, each slice has finite products (including the terminal object) and exponentials (therefore each slice is a model of the simply typed lambda calculus). We will be interested in those locally cartesian categories which possess additional structure; namely, those which are also pretopoi (the reader who is unfamiliar with lcccs or pretopoi should refer to the appendix on basic category theory). As such we adopt the following definition.

Definition 1.1.1 A Π-*pretopos* is a locally cartesian closed pretopos.

Although we will not be particularly concerned with studying the properties of Π-pretopoi (as the properties of pretopoi and lcccs are well known and can be found, for instance, in [19]) the following fact should be noted.

Proposition 1.1.2 *Every Π-pretopos $\mathcal{R}$ is Heyting.*

PROOF Straightforward. $\square$

1.1.2 Dependent Type Theory

Simple types (not to be confused with Russell's theory of simple types mentioned in the introduction) such as product and exponential types are useful for many purposes; but they do not exhaust the expressive capabilities needed to formulate more sophisticated mathematics. For example, the type of n by m matrices $\mathrm{Mat}(n : \mathbb{N}, m : \mathbb{N})$ depends on the terms n and m of type $\mathbb{N}$ (cf. [18]). The appropriate type theory for implementing such types as the aforementioned is *dependent type theory*, the syntax of which we will introduce.

First, though, some comments are in order regarding the notation and other conventions surrounding the syntax of dependent type theory. Uppercase Roman letters with our without indices are used to denote types: A_1, B, C_4, et cetera. Lowercase Roman letters are employed similarly for terms: a_n, x, m_5, et cetera. We reserve uppercase Greek letters for contexts (to be explained below): Γ, Δ, et cetera. Finally, we denote judgments by lowercase Greek letters: φ, ψ, et cetera.

A context is a finite sequence of variable declarations $x_1 : A_1, \ldots, x_n : A_n$ such that $\cdot|A_1 : \mathsf{type}$, $x_1 : A_1, \ldots, x_i : A_{i-1}|A_i : \mathsf{type}$ for each $i \leq n$ and the free variables of each A_j must have been previously declared (i.e., $\mathrm{FV}(A_j) \subseteq \{x_1, \ldots, x_{j-1}\}$ and $x_j \notin \{x_1, \ldots, x_{j-1}\}$). We denote the empty context by $\cdot$, but will often omit mention of it altogether.

Because types are allowed to depend on terms it will be useful to introduce some new terminology which should help to make precise the concept of type dependence. A type A is *closed* if $\cdot|A : \mathsf{type}$. Alternatively, A is *open* if, for some non-empty context Γ, $\Gamma|A : \mathsf{type}$ and it is not the case that $\cdot|A : \mathsf{type}$. For example, $\mathbb{N}$ is a closed type, whereas $\mathrm{Mat}(n, m)$ is an open type. In the context of dependent type theory judgments in non-empty contexts are often said to be *hypothetical.* The notion of dependence is capture by the distinction between open and closed types and hypothetical and non-hypothetical (*categorical*) judgments.

The type forming operations are, *term equality* $\mathsf{eq}_A(x : A, y : A)$, *dependent product* $\Pi x : A.B$ and *dependent sum* $\Sigma x : A.B$. As mentioned in the introduction, the latter two new type forming operations correspond under the Curry-Howard 'isomorphism' to universal and existential quantification, respectively. The type formation rules are as follows:

$$\frac{\Gamma, x : A | B : \mathsf{type}}{\Gamma | \Pi x : A.B : \mathsf{type}}\ \Pi\text{-form}, \qquad \frac{\Gamma, x : A | B : \mathsf{type}}{\Gamma | \Sigma x : A.B : \mathsf{type}}\ \Sigma\text{-form},$$

and:

$$\frac{\Gamma | A : \mathsf{type}}{\Gamma, x : A, y : A | \mathsf{eq}_A(x, y) : \mathsf{type}}\ \mathsf{eq}_A\text{-form}.$$

The introduction and elimination rules for dependent product and sum are:

$$\frac{\Gamma, x : A | y : B}{\Gamma | \lambda x : A.y : (\Pi x : A.B)}\ \Pi I, \qquad \frac{\Gamma | y : (\Pi x : A.B) \qquad \Gamma | z : A}{\Gamma | \mathsf{app}(y, z) : B[z/x]}\ \Pi E,$$

$$\frac{\Gamma, x : A | y : B[x/z]}{\Gamma | \mathsf{pair}(x, y) : (\Sigma z : A.B)}\ \Sigma I, \qquad \frac{\Gamma | y : (\Sigma x : A.B)}{\Gamma | \pi_1(y) : A}\ \Sigma E_1,$$

and:

$$\frac{\Gamma | y : (\Sigma x : A.B)}{\pi_2(y) : B[\pi_1(y)/x]}\ \Sigma E_2.$$

The introduction and elimination rules for type equality are as follows:

$$\frac{\Gamma | x : A}{\Gamma | \mathsf{r}_A(x) : \mathsf{eq}_A(x, x)}\ \mathsf{eq}_A I, \qquad \frac{\Gamma | y : \mathsf{eq}_A(x, x')}{\Gamma | x = x' : A}\ \mathsf{eq}_A E_1,$$

and:

$$\frac{\Gamma | y : \mathsf{eq}_A(x, x')}{\Gamma | y = \mathsf{r}_A(x) : \mathsf{eq}_A(x, x')}\ \mathsf{eq}_A E_2.$$

We also also adopt the following conversions:

$$\begin{aligned} \pi_1(\mathsf{pair}(x, y)) &= x, \\ \pi_2(\mathsf{pair}(x, y)) &= y, \text{ and} \\ \mathsf{pair}(\pi_1(x), \pi_2(x)) &= x. \end{aligned}$$

Note that simple types are subsumed by defining $A \times B := \Sigma_A B$ and $B^A := \Pi_A B$.

Some comments are now in order regarding the intended meaning of the foregoing introduction and elimination rules. First, notice that we have adopted the *strong* elimination rules for Σ and eq_A. As discussed in [18], in the presence of the aforementioned conversions these rules are equivalent to the following:

$$\frac{\Gamma, z : (\Sigma x : A.B)|C : \mathsf{type} \qquad \Gamma, x : A, y : B|w : C[\mathsf{pair}(x,y)/z]}{\Gamma, z : (\Sigma x : A.B)|(\mathsf{unpack}\ z\ \mathsf{as}\ \mathsf{pair}(x,y)\ \mathsf{in}\ w) : C}\ \Sigma E^*,$$

and:

$$\frac{\Gamma, x : A, x' : A, z : \mathsf{eq}_A(x,x')|B : \mathsf{type} \qquad \Gamma, x : A|w : B[x/x', \mathsf{r}_A(x)/z]}{\Gamma, x : A, x' : A, z : \mathsf{eq}_A(x,x')|(w\ \mathsf{with}\ x' = x\ \mathsf{via}\ z) : B}\ \mathsf{eq}_A E^*,$$

where $\mathsf{unpack}\ z\ \mathsf{as}\ \mathsf{pair}(x,y)\ \mathsf{in}\ w$ is the new term forming operation for ΣE^* and $w\ \mathsf{with}\ x' = x\ \mathsf{via}\ z$ is, similarly, the new term forming operation for $\mathsf{eq}_A E^*$.

In terms of the Curry-Howard 'Isomorphism' ΣE^* is the usual elimination rule for the existential quantifier (with the caveat that the quantification is bounded). Secondly, we may understand $\mathsf{eq}_A E_1$ as stating that if the type $\mathsf{eq}_A(x,x')$ is inhabited, then $x = x' : A$ and $\mathsf{eq}_A E_2$ as stating that if y witnesses the equality of $x : A$ and $x' : A$, then y is identical to the canonical witness $\mathsf{r}_A(x)$ of $x = x : A$. The reason for adopting these particular forms of the strong elimination rules are their ease of interpretation in locally cartesian closed categories. For more details regarding (the categorical approach to) dependent type theory the reader may consult [18], D4.4 of [20] and [32].

1.1.3 Soundness and Completeness

The reader should recall the following theorem which affirms a tight connection between locally cartesian closed categories and dependent type theory (cf. [20] or [18]):

Theorem 1.1.3 (LCCC Soundness and Completeness) *For any judgment in context $\Gamma|\varphi$ of dependent type theory (DTT),*

$$\mathrm{DTT} \vdash \ \Gamma|\varphi \text{ iff, for every lccc } \mathcal{C}, \mathcal{C} \vDash \ \Gamma|\varphi.$$

However, before extending this result to Π-pretopoi it will be instructive to make the interpretation under consideration explicit so that the connection between locally cartesian closed categories and dependent type theory may be better appreciated.

The interpretation $[\![-]\!]$ of dependent type theory in a locally cartesian closed category $\mathcal{C}$ is defined as follows:

- The empty context is interpreted as the terminal object 1.
- A closed type is interpreted as an object $[\![A]\!]$ of $\mathcal{C}$.
- Given a context $\Gamma := (x_1 : A_1, \ldots, x_n : A_n)$ we will informally regard the interpretation $[\![\Gamma]\!]$ as a composite:

$$[\![A_n]\!] \longrightarrow [\![A_{n-1}]\!] \longrightarrow \ldots \longrightarrow [\![A_1]\!].$$

  Formally we define $[\![A_j]\!]$ to be an object of a category $(\mathcal{C}/\Gamma)_{j-1}$ where the categories $(\mathcal{C}/\Gamma)_j$ are defined inductively by:

$$\begin{array}{rcl} (\mathcal{C}/\Gamma)_0 & := & \mathcal{C}/[\![\cdot]\!] \cong \mathcal{C}, \\ (\mathcal{C}/\Gamma)_1 & := & (\mathcal{C}/\Gamma)_0/[\![A_1]\!], \\ & \ldots & \\ (\mathcal{C}/\Gamma)_i & := & (\mathcal{C}/\Gamma)_{i-1}/[\![A_i]\!], \end{array}$$

  for $1 \leq i \leq n$. The fact that, for $f : A \longrightarrow B$, $(\mathcal{C}/B)/f \cong \mathcal{C}/A$ justifies the informal understanding of the interpretation given above; specifically, we define $[\![\Gamma]\!]$ to be an object of $(\mathcal{C}/\Gamma)_{n-1}$.
- If $\Gamma|B : \mathsf{type}$, then $[\![B]\!] : \mathsf{d}[\![B]\!] \longrightarrow [\![\Gamma]\!]$, where $\mathsf{d}[\![B]\!]$ is the domain of $[\![B]\!]$. I.e., $[\![B]\!]$ is an object of $(\mathcal{C}/\Gamma)_n$.
- A term in context $\Gamma|b : B$ is interpreted as a global section of $[\![B]\!] \longrightarrow \mathsf{d}[\![\Gamma]\!]$ in $\mathcal{C}/\mathsf{d}[\![\Gamma]\!]$. In particular, $[\![\cdot|a : A]\!]$ is a map $1 \longrightarrow [\![A]\!]$ in $\mathcal{C}$.
- For dependent products and sums we let:

$$\begin{array}{rcl} [\![\Sigma x : A.B]\!] & := & \Sigma_{[\![A]\!]}([\![B]\!]), \text{ and} \\ [\![\Pi x : A.B]\!] & := & \Pi_{[\![A]\!]}([\![B]\!]), \end{array}$$

  where $[\![A]\!]$ is the canonical map $[\![A]\!] \longrightarrow 1$.

As an example, an open type B depending on $x : A$ is interpreted as an object $[\![x : A|B]\!]$ of $\mathcal{C}/[\![A]\!]$. We write $\mathsf{d}[\![x : A|B]\!]$ for the domain of $[\![x : A|B]\!]$. However, we will often shorten this to $\mathsf{d}[\![B]\!]$ when the codomain is apparent.

Since every Π-pretopos is locally cartesian closed we obtain the following:

Corollary 1.1.4 (Π-Pretopos Soundness and Completeness) *For any judgment in context* $\Gamma|\varphi$ *of dependent type theory,*

$$\text{DTT} \vdash \Gamma|\varphi \text{ iff, for every } \Pi\text{-pretopos } \mathcal{R},\ \mathcal{R} \vDash \Gamma|\varphi.$$

PROOF Soundness is trivial since every Π-pretopos is locally cartesian closed. For completeness notice that if $\mathcal{C}$ is locally cartesian closed, then the Yoneda embedding $y : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$ preserves all of the locally cartesian closed structure and $\widehat{\mathcal{C}}$ is a Π-pretopos. Suppose that, for all Π-pretopoi $\mathcal{R}$, $\mathcal{R} \models \Gamma|\varphi$. Then, in particular, $\widehat{\mathcal{C}} \models \Gamma|\varphi$ for every LCCC $\mathcal{C}$. But since y is conservative (i.e., reflects isomorphisms) and preserves lcc structure it follows that $\mathcal{C} \models \Gamma|\varphi$. By the foregoing theorem 1.1.3 we therefore have $\vdash \Gamma|\varphi$. □

Remark 1.1.5 Π-pretopoi can be regarded as categories of 'setoids' in the sense of Martin-Löf type theory, in the way that lcccs are categories of types in such theories.

1.2 Constructive Set Theories

All of the set theories under consideration are first-order intuitionistic theories in the language $\mathcal{L} := \{\mathsf{S}, \in\}$ where S ('sethood') and $\in$ ('membership') are, respectively, unary and binary predicates. We include S in the language because we intend to allow urelements or non-sets. The majority of the results of this section are to be found, either explicitly or implicitly, in [4] or [6].

1.2.1 Notation and Axioms

Where φ is a formula, $\mathrm{FV}(\varphi)$ denotes the set of free variables of φ. We will freely employ the class notation $\{x|\varphi\}$ as in common set theoretical practice. Frequently it will be efficacious to employ bounded quantification which is defined as usual:

$$\forall x \in y.\varphi(x) := \forall x.x \in y \Rightarrow \varphi(x) \quad \text{and} \quad \exists x \in y.\varphi(x) := \exists x.x \in y \wedge \varphi(x).$$

A formula φ is called Δ_0 if all of its quantifiers are bounded.

Another notational convenience is the introduction of the 'set-many' quantifier S defined as:

$$\mathsf{S}x.\varphi \quad := \quad \exists y.(\mathsf{S}(y) \wedge \forall x.(x \in y \Leftrightarrow \varphi)),$$

where $y \notin \mathrm{FV}(\varphi)$. We also write:

$$x \subseteq y \quad := \quad \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \forall z \in x.z \in y.$$

We write $\mathrm{func}(f, a, b)$ to indicate that f is a functional relation on $a \times b$ (which will exist in any of the set theories we consider):

$$\mathrm{func}(f, a, b) \quad := \quad f \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.(x, y) \in f$$

Finally, for any formula φ, we define:

$$\mathrm{coll}(x \in a, y \in b, \varphi) \quad := \quad (\forall x \in a.\exists y \in b.\varphi) \wedge (\forall y \in b.\exists x \in a.\varphi).$$

For the sake of brevity we omit the obvious universal quantifiers in the following axioms and schemata for set theories:

Membership: $x \in a \Rightarrow \mathsf{S}(a)$.

Universal Sethood: $\mathsf{S}(x)$.

Extensionality: $(a \subseteq b \wedge b \subseteq a) \Rightarrow a = b$.

Emptyset: $Ƨz.\bot$.

Pairing: $Ƨz.z = x \vee z = y$.

Binary Intersection: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow Ƨz.z \in a \wedge z \in b$.

Union: $\mathsf{S}(a) \wedge (\forall x \in a.\mathsf{S}(x)) \Rightarrow Ƨz.\exists x \in a.z \in x$.

Infinity: $\exists a.\mathsf{S}(a) \wedge (\exists x.x \in a \wedge (\forall x \in a)(\mathsf{S}(x) \wedge \exists y \in a.\mathsf{S}(y) \wedge x \in y))$.

$\in$-Induction: $[\forall a.(\mathsf{S}(a) \wedge \forall x \in a.\varphi(x) \Rightarrow \varphi(a))] \Rightarrow \forall a.(\mathsf{S}(a) \Rightarrow \varphi(a))$.

Replacement: $\mathsf{S}(a) \wedge \forall x \in a.\exists! y.\varphi \Rightarrow Ƨy.\exists x \in a.\varphi$

Strong Collection: $\mathsf{S}(a) \wedge (\forall x \in a.\exists y.\varphi) \Rightarrow \exists b.(\mathsf{S}(b) \wedge \mathrm{coll}(x \in a, y \in b, \varphi)$.

Exponentiation: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow Ƨz.\,\mathrm{func}(z, a, b)$.

Subset Collection: $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow$

$$\exists c.\mathsf{S}(c) \wedge [\forall v.\forall x \in a.\exists y \in b.\varphi \Rightarrow \exists d \in c.\mathsf{S}(d) \wedge \mathrm{coll}(x \in a, y \in d)\varphi].$$

Δ_0-Separation: $\mathsf{S}(a) \Rightarrow Ƨz.z \in a \wedge \varphi$, if φ is a Δ_0 formula.

The particular set theories with which we will be primarily concerned are given in Table 1.1. In Table 1.1 we employ a solid bullet $\bullet$ to indicate that the axiom in question is one of the axioms of the theory and a hollow bullet $\circ$ to indicate a consequence of the axioms. There are several points worth mentioning in connection with Table 1.1. First, **CZF** is conventionally formulated in the language $\{\in\}$ with all of the axioms suitably reformulated. In the present setting this amounts to the addition of Universal Sethood. Secondly, the form of Δ_0-Separation which holds in **BCST** and **CST** is

Axioms	**BCST**	**CST**	**CZF**
Membership	•	•	∘
Extensionality, Pairing, Union	•	•	•
Emptyset	•	•	∘
Binary Intersection	•	•	∘
Replacement	•	•	∘
Δ_0-Separation	∘	∘	•
Exponentiation		•	∘
Infinity			•
$\in$-Induction			•
Strong Collection			•
Subset Collection			•
Universal Sethood			•

Table 1.1: Several Constructive Set Theories

subject to the stipulation that φ is also well-typed in a sense which will be made precise shortly. Finally, the reader should note that although the theories we consider do not include an axiom of infinity the results of this paper are easily extended to theories augmented with (an appropriate version of) Infinity (cf. [33]).

1.2.2 Basic Properties

We begin by showing that a particularly useful axiom schema holds in **BCST**; namely, *Indexed Union*:

$$\mathsf{S}(a) \wedge (\forall x \in a.\text{Ƨ}y.\varphi) \Rightarrow \text{Ƨ}y.\exists x \in a.\varphi.$$

Lemma 1.2.1 **BCST** $\vdash$ Indexed Union.

Proof Suppose $\mathsf{S}(a)$ and $\forall x \in a.\text{Ƨ}y.\varphi(x,y)$, then for any $x \in a$ there is a unique b such that $\mathsf{S}(b)$ and $(\forall y)(y \in b \Leftrightarrow \varphi(x,y))$. By Replacement there exists a c such that $\mathsf{S}(c)$ and:

$$c \quad = \quad \{z | \exists x \in a.\mathsf{S}(z) \wedge (\forall y)(y \in z \Leftrightarrow \varphi(x,y))\}.$$

Clearly $\mathsf{S}(y')$ for any $y' \in c$. By Union $\mathsf{S}(\bigcup c)$. Intuitively, we want to show that the class $w := \{z | \exists x \in a.\varphi(x,z)\}$ is a set. The claim then is that $w = \bigcup c$.

To see that this is so suppose that $y \in \bigcup c$. Then there exists a $d \in c$ such that $y \in d$. By the definition of c there exists an $e \in a$ with $(\forall z)(z \in d \Leftrightarrow \varphi(e,z))$. So, since $y \in d$ it follows that $\varphi(e,y)$ and $y \in w$.

Next, suppose that $y \in w$. Then there exists an $e \in a$ such that $\varphi(e, y)$. By the original assumption there exists a set d such that:

$$d \quad := \quad \{z | \varphi(e, z)\}.$$

Also $d \in c$ and since $\varphi(e, y)$ it follows that $y \in d$ and $y \in \bigcup c$. Thus, $\text{Ƨ}y.\exists x \in a.\varphi$, as required. $\square$

We now show that, although **BCST** lacks a separation axiom, it is possible to recover some degree of separation. To this end we define:

$$\varphi[a, x]\text{-Sep} := \mathsf{S}(a) \Rightarrow \text{Ƨ}x.(x \in a \wedge \varphi).$$

Here the free variables a and x need not occur in φ. Additionally we say that a formula φ is *simple* when the following, written $!\varphi$, is provable:

$$\text{Ƨ}z.(z = \emptyset \wedge \varphi)$$

and $z \notin \mathrm{FV}(\varphi)$. The intuition behind simplicity is that certain formulas are sufficiently lacking in logical complexity that their truth values are indeed sets. In particular, we will write t_φ for $\{z | z = \emptyset \wedge \varphi\}$ which we call the *truth value* of φ (and, if necessary, we will exhibit the free variable of φ: $t_{\varphi(x)}$). Separation holds for such simple formulae:

Lemma 1.2.2 (Simple Separation) **BCST** $\vdash (\forall x \in a.!\varphi(x)) \Rightarrow \varphi[a, x]\text{-Sep}$.

Proof We will show that, given the assumptions, $\{z | z = x \wedge \varphi(x)\}$ is a set for each $x \in a$. The conclusion then is an easy consequence of Union-Rep. By assumption $\mathsf{S}(a)$ and for every $x \in a$ the truth value:

$$t_{\varphi(x)} := \{z | z = \emptyset \wedge \varphi(x)\}$$

of $\varphi(x)$ is a set. Suppose $y \in t_{\varphi(x)}$, then $y = \emptyset \wedge \varphi(x)$. But then $\exists! z.z = x \wedge y = \emptyset \wedge \varphi(x)$. By Replacement:

$$q := \{z | \exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)\}$$

is a set. But $\exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)$ is equivalent to $z = x \wedge \varphi(x)$ so that $\{z | z = x \wedge \varphi(x)\}$ is a set, as required. $\square$

Lemma 1.2.3 (The Equality Axiom) **BCST** *proves the* Equality Axiom *(cf. [33]):*

$$\forall x, y.\text{Ƨ}z.z = x \wedge z = y.$$

Proof Let x and y be given. Then $\{x\}$ and $\{y\}$ are sets and, by Binary Intersection, their intersection $\{x\} \cap \{y\}$ is also a set which has the required property. □

Henceforth, given x and y, we write δ_{xy} for the set $\{z | z = x \wedge z = y\}$.

Lemma 1.2.4 (cf. [6]) *In* **BCST***:*

1. $!(a = b)$.
2. *If* $\mathsf{S}(a)$ *and* $\forall x \in a.!\varphi(x)$*, then* $!(\exists x \in a.\varphi(x))$ *and* $!(\forall x \in a.\varphi(x))$.
3. $!(x \in a)$*, when* $\mathsf{S}(a)$.
4. *If* $!\varphi$ *and* $!\psi$*, then* $!(\varphi \wedge \psi)$, $!(\varphi \vee \psi)$, $!(\varphi \Rightarrow \psi)$*, and* $!(\neg\varphi)$.
5. *If* $\varphi \vee \neg\varphi$*, then* $!\varphi$.

Proof In turn:

1. By Equality δ_{ab} is a set. There exists, for each $x \in \delta_{ab}$, a unique y such that $y = \emptyset$. By Replacement $\text{Ƨ}z.\exists x \in \delta_{ab}.z = \emptyset$. But this is equivalent to saying that $\text{Ƨ}z.z = \emptyset \wedge a = b$.

2. Suppose $\mathsf{S}(a)$ and for all $x \in a$ $!\varphi(x)$. Then for each $x \in a$ there exists a unique set $t_{\varphi(x)} := \{z | z = \emptyset \wedge \varphi(x)\}$ and, by Replacement, there is a set $w := \{t_{\varphi(x)} | x \in a\}$. By Union $\bigcup w = \{z | z = \emptyset \wedge \exists x \in a.\varphi(x)\}$ is a set.

   For the second part note that, by Simple Separation, $b := \{x | x \in a \wedge \varphi(x)\}$ is a set and, as such, δ_{ab} is also a set. Moreover, **BCST** proves $!(z = z)$ for all $z \in \delta_{ab}$ by (1). By Simple Separation **BCST** also proves that $!(\exists z \in \delta_{ab}.z = z)$. Since:

   $$(\exists z \in \delta_{ab}.z = z) \Leftrightarrow (\forall x \in a.\varphi(x))$$

   we're done.

3. By (1) and (2).

4. Suppose $!\varphi$ and $!\psi$. Then $!(\varphi \wedge \psi)$, by Binary Intersection, and $!(\varphi \vee \psi)$, by Union. For $!(\varphi \Rightarrow \psi)$ notice that $\varphi \Rightarrow \psi$ if and only if $\forall z \in t_{\varphi}.\psi$. By assumption t_{φ} is a set and $!\psi$, so by (2), $!(\forall z \in t_{\varphi}.\psi)$. $!(\neg\varphi)$ is an easy consequence of Emptyset.

5. Suppose $\varphi \vee \neg\varphi$. For all $x \in \{\emptyset\}$ there is a unique set y such that:

$$(y = \emptyset \wedge \varphi) \vee (y = \{\emptyset\} \wedge \neg\varphi),$$

and by Replacement $v := \{z|(z = \emptyset \wedge \varphi) \vee (z = \{\emptyset\} \wedge \neg\varphi)\}$ is a set. It follows from (1) and Simple Separation that $\{z|z \in v \wedge z = \emptyset\} = \{z|z = \emptyset \wedge \varphi\}$ is a set. $\square$

Corollary 1.2.5 *Given the other axioms of* **BCST** *the following are equivalent:*

1. *Binary Intersection,*
2. *Equality, and*
3. *Intersection.*

PROOF (1)⇒(2) we have already proved. For (2)⇒(1), note that neither the proof of Simple Separation nor that of lemma 1.2.4.(1)-(3) employs Binary Intersection. As such, let a set a and a set b be given. Therefore Binary Intersection is an easy consequence of 1.2.4.(3) and Simple Separation.

Clearly (3)⇒(1), and by Simple Separation and lemma 1.2.4 (1)⇒(3).$\square$

Definition 1.2.6 Let a Δ_0 formula φ and a variable x occurring in φ be fixed. We say that x is an *orphan* if $x \in \mathrm{FV}(\varphi)$. If $x \notin \mathrm{FV}(\varphi)$, then we define the *parent of* x *in* φ to be the variable y such that x occurs as a bound variable of one of the following forms in φ: $\forall x \in y$ or $\exists x \in y$ (note that every x which is not an orphan has a unique parent in φ). The *family tree of* x *in* φ, denoted by $\Phi(\varphi, x)$, is the singleton $\{x\}$ if x is an orphan and otherwise it is the tuple $\langle x, y_1, y_2, \ldots, y_n\rangle$ such that the following conditions are satisfied: (i) y_1 is the parent of x in φ, (ii) each y_{m+1} is the parent of y_m for $1 \leq m \leq n-1$, and (iii) y_n is an orphan. The reader may easily verify that, for each variable x occurring in φ, $\Phi(\varphi, x)$ is unique.

Definition 1.2.7 Given a Δ_0 formula φ and a variable x occurring in φ we adopt the following abbreviation:

$$\begin{aligned} \mathsf{S}(\Phi(\varphi, x)) \;:=\; & \mathsf{S}(y_n) \wedge \forall y_{n-1} \in y_n. \\ & \mathsf{S}(y_{n-1}) \wedge \forall y_{n-2} \in y_{n-1}.\mathsf{S}(y_{n-2}) \wedge \ldots \forall x \in y_1.\mathsf{S}(x), \end{aligned}$$

where $\Phi(\varphi, x) = \langle x, y_1, \ldots, y_{n-1}, y_n\rangle$.

Definition 1.2.8 If φ is a Δ_0 formula of **BCST** such that there are no occurrences of the S predicate in φ and $x_1, \ldots, x_n$ are all of those variables of φ either bound or free which occur on the right hand side of the $\in$ predicate in φ, then we define a formula $\tau(\varphi, m)$ for each $1 \leq m \leq n$ by induction on n:

$$\begin{array}{rcl} \tau(\varphi, 0) & := & \top. \\ \tau(\varphi, m+1) & := & \tau(\varphi, m) \wedge \mathsf{S}(\Phi(\varphi, x_{m+1})). \end{array}$$

Then $\tau(\varphi) := \tau(\varphi, n)$.

Corollary 1.2.9 (Δ_0-Separation) *If φ is a Δ_0 formula in which there are no occurrences of S and $x_1, \ldots, x_n$ are all of those free variables of φ that occur on the right hand side of occurrences of $\in$, then:*

$$\mathbf{BCST} \vdash \tau(\varphi) \wedge \mathsf{S}(y) \Rightarrow \text{Ƨ} z \in y.\varphi.$$

Remark 1.2.10 If the Simple Sethood axiom, which states that the sethood predicate S is simple, is satisfied then full Δ_0-separation holds.

1.2.3 The Category of Sets

We will now show that the category of sets of **BCST** form a Heyting pretopos and that the sets of **CST** form a Π-pretopos (what we mean by 'the category of sets' will be made precise shortly). First, we consider quotients of equivalence relations.

Lemma 1.2.11 *If $\mathsf{S}(a)$ and $r \subseteq a \times a$ is an equivalence relation, then for each $x \in a$ the equivalence class:*

$$[x]_r := \{z | z \in a \wedge (x, z) \in r\}$$

is a set.

Proof Let $x \in a$ be given to show that Ƨ$z.z \in a \wedge (x, z) \in r$. In order to apply Simple Separation let an arbitrary $y \in a$ be given. It is an obvious consequence of part (2) of Lemma 1.2.4 that $\forall z \in r.!(z = (x, y))$. By part (1) of the lemma $!(\exists z \in r.z = (x, y))$. Since we shown that, for all $y \in a$, $!(\exists z \in r.z = (x, y))$ it follows from Simple Separation that Ƨ$y.(y \in a \wedge (\exists z \in r.z = (x, y))$. I.e., $[x]_r$ is a set, as required. $\square$

Lemma 1.2.12 *If $\mathsf{S}(a)$ and $r \subseteq a \times a$ is an equivalence relation, then the quotient*

$$a/r := \{[x]_r | x \in a\}$$

of the set a modulo r is a set.

PROOF This is an easy application of Replacement. $\square$

Let "**Sets**" be the category consisting of sets and functions between them in **BCST**. More precisely, objects are those x of **BCST** such that $\mathsf{S}(x)$ and arrows $f : x \longrightarrow y$ are those f of **BCST** such that $\mathrm{func}(f, x, y)$. By the foregoing lemmas and some obvious facts that we omit, we have the following:

Theorem 1.2.13 **BCST** *proves that* "**Sets**" *is a Heyting pretopos.*

Now we regard "**Sets**" as the category of sets in **CST**:

Lemma 1.2.14 *For any object I of* "**Sets**"*, the category* "**Sets**"$/I$ *is equivalent to* "**Sets**"^{I} *where I is regarded as a discrete category.*

PROOF The usual proof goes through in **BCST**: Define F : "**Sets**"$/I \longrightarrow$ **"Sets"**^{I} by:

$$\begin{array}{rcl} X \xrightarrow{f} I & \longmapsto & (X_i)_{i\in I}, \text{ and} \\ h : f \longrightarrow g & \longmapsto & (h_i)_{i\in I}, \end{array}$$

where X_i is the fiber $f^{-1}(i)$ of f over i and:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

commutes in **"Sets"**. Notice that each X_i is a set by Simple Separation.

Let G : "**Sets**"$^I \longrightarrow$ "**Sets**"$/I$ by:

$$(X_i)_{i\in I} \quad \longmapsto \quad f : X \longrightarrow I,$$

where $X := \coprod X_i$ and, for any $x \in X$, $f(x)$ is the $i \in I$ such that $x \in X_i$. Here $\coprod X_i := \{(x, i)|x \in X_i\}$ is a set by Simple Separation.

It is easily verified that F and G constitute an equivalence of categories just as in classical set theory. $\square$

Given $f : X \longrightarrow Y$ the pullback functor Δ_f : "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ serves to reindex a family of sets $(C_y)_{y\in Y}$ as $(C_{f(x)})_{x\in X}$. Note also that given a set I and a family of sets X_i for each $i \in I$, the class $\{X_i|i \in I\}$ is a set by Replacement.

Lemma 1.2.15 *For any map $f : X \longrightarrow Y$ in "**Sets**", the pullback functor Δ_f : "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ has both a left adjoint Σ_f and a right adjoint Π_f.*

Proof We may employ the usual definitions of the adjoints:

$$\begin{array}{ccc} \text{“}\mathbf{Sets}\text{”}^X & \xrightarrow{\Sigma_f} & \text{“}\mathbf{Sets}\text{”}^Y \\ (C_x)_{x\in X} & \longmapsto & (S_y)_{y\in Y}, \end{array}$$

where $S_y := \coprod_{f(x)=y} C_x$, and Π_f:

$$(C_x)_{x\in X} \longmapsto (P_y)_{y\in Y},$$

where $P_y := \prod_{f(x)=y} C_x$. Here the arbitrary product:

$$\prod_{i\in I} X_i := \{f : I \longrightarrow \bigcup_{i\in I} X_i | \forall i \in I.f(i) \in X_i\}$$

is a set. In particular, $\bigcup X_i$ is a set by Union and $(\bigcup X_i)^I$ is a set by Exponentiation. The result follows directly from Lemma 1.2.4 and Simple Separation. □

By the foregoing lemmas we have proved:

Theorem 1.2.16 CST *proves that* "**Sets**" *is a Π-pretopos.*

1.2.4 Axioms of Infinity

In this section we will consider the theories obtained by augmenting **BCST** and **CST** with axioms of infinity and induction. In what follows we will consider an extension of the language $\mathcal{L}$ of set theory obtained by adding three new constants o, s, N. We write $\mathcal{L}^+$ for the language so obtained. Let ψ be the conjunction of the following three formulae:

$$\forall x \in N.o \neq s(x),$$

$$\forall x, y \in N.s(x) = s(y) \Rightarrow x = y, \text{ and}$$

$$\forall a.[a \subseteq N \wedge o \in a \wedge \forall x \in a.(s(x) \in a)] \Rightarrow N = a,$$

then we may state the infinity and induction axioms under consideration (as well, for the sake of comparison, as the analogous axioms for **CZF**):

Infinity: $\exists a.\mathsf{S}(a) \wedge (\exists x.x \in a \wedge (\forall x \in a)(\mathsf{S}(x) \wedge \exists y \in a.x \in y))$.

Infinity$*$: $\mathsf{S}(N) \wedge o \in N \wedge \mathrm{func}(s, N, N) \wedge \psi$.

Induction: $[\varphi(o) \wedge \forall x \in N.(\varphi(x) \Rightarrow \varphi(s(x)))] \Rightarrow \forall x \in N.\varphi(x)$.

$\in$-Induction: $[\forall a.(\mathsf{S}(a) \wedge \forall x \in a.\varphi(x) \Rightarrow \varphi(a))] \Rightarrow \forall a.(\mathsf{S}(a) \Rightarrow \varphi(a))$.

Henceforth we denote the theory obtained by adding Infinity$*$ to **BCST** by **BCST**$^+$ and similarly for **CST**. The immediate point to note about the particular axiom of infinity adopted is that it places no unnecessary constraints on the actual elements of N. In particular, we do not know that the elements of N are built up using Pairing and the emptyset; indeed, the elements of N need not even be sets. The benefits of adopting such an axiomatization are both practical and ideological; for such an axiomatization allows us greater ease in the later interpretation of the set theory and permits an 'implementation-invariance' not otherwise possible.

For any formula φ, we will henceforth write $\varphi[x]$-Ind to denote the formula which states that Induction holds for the formula φ; namely:

$$[\varphi(o) \wedge \forall x \in N.(\varphi(x) \Rightarrow \varphi(s(x)))] \Rightarrow \forall x \in N.\varphi(x).$$

Then we have the following useful fact:

Proposition 1.2.17 (Simple Induction) **BCST**$^+ \vdash \forall x \in N.!\varphi \Rightarrow \varphi[x]$-Ind.

Proof Suppose that for any $x \in N$ we have that $t_{\varphi(x)}$ is a set. Then by Simple Separation $a := \{x | x \in N \wedge \varphi(x)\}$ is also a set. Suppose that the antecedent of $\varphi[x]$-Ind holds. Then $o \in a$ and if $x \in a$ then also $s(x) \in a$. By Infinity$*$, $a = N$. $\square$

Chapter 2

Predicative Categories of Classes

In this chapter we introduce the axiomatic theory of categories of classes (as well as several variants of this notion) and derive soundness and completeness results for **BCST** and **CST**. Our approach is related to those developed in [21], [34], [12], [6], and [28]. Throughout this chapter we make extensive use of both the internal languages of the categories under consideration and the Kripke-Joyal semantics. For the internal language the reader is referred to section D1.3 of [20] and [23] (here the internal language is called the 'Mitchell-Bénabou language'). There is a brief discussion of the Kripke-Joyal semantics in the appendix of this paper, but for more detailed information the reader should see [23].

2.1 Categories with Class Structure

In this section the axiomatic theories of *categories with basic class structure* and *categories with class structure* are introduced. The internal languages of such categories are typed set theories analogous to **BCST** and **CST**, respectively. As such, after introducing the axioms we proceed to show that appropriate typed versions of the set theoretic axioms are validated in these categories.

2.1.1 Axioms for Categories with Basic Class Structure

Definition 2.1.1 A *system of small maps* in a positive Heyting category $\mathcal{C}$ is a collection $\mathcal{S}$ of maps of $\mathcal{C}$ satisfying the following axioms:

(S1) $\mathcal{S}$ is closed under composition and all identity arrows are in $\mathcal{S}$.

(S2) If the following is a pullback diagram:

$$\begin{tikzcd} C' \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "g"'] & D \end{tikzcd}$$

and f is in $\mathcal{S}$, then f' is in $\mathcal{S}$.

(S3) All diagonals $\Delta : C \longrightarrow C \times C$ are contained in $\mathcal{S}$.

(S4) If e is a cover, g is in $\mathcal{S}$ and the diagram:

$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, "g"'] & & D \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

commutes, then f is in $\mathcal{S}$.

(S5) If $f : C \longrightarrow A$ and $g : D \longrightarrow A$ are in $\mathcal{S}$, then so is the copair $[f, g] : C + D \longrightarrow A$.

A map f is *small* if it is a member of $\mathcal{S}$ and an object C is *small* if the canonical map $!_C : C \longrightarrow 1$ is small. Similarly, a relation $R \rightarrowtail C \times D$ is a *small relation* if the composite:

$$R \rightarrowtail C \times D \longrightarrow D$$

with the projection is a small map. Finally, a subobject $A \rightarrowtail C$ is a *small subobject* if $A \rightarrowtail C \times 1$ is a small relation; i.e., provided that A is a small object.

Definition 2.1.2 A *category with basic (predicative) class structure* is a positive Heyting category $\mathcal{C}$ with a system of small maps satisfying:

(P1) For each object C of $\mathcal{C}$ there exists a *(predicative) power object* $\mathcal{P}_s(C)$ and a small *membership relation* $\epsilon_C \rightarrowtail C \times \mathcal{P}_s(C)$ such that, for any

D and small relation $R \rightarrowtail C \times D$, there exists a unique map $\rho : D \longrightarrow \mathcal{P}_s\, C$ such that the square:

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \epsilon_C \arrow[d, tail] \\ C \times D \arrow[r, "1_C \times \rho"'] & C \times \mathcal{P}_s\, C \end{tikzcd}$$

is a pullback.

As in topos theory we call the unique map ρ in **(P1)** the *classifying map* of R and R the *relation classified by* ρ.

By the definition of small subobjects and small relations there are covariant functors $\mathrm{SSub}_{\mathcal{C}}(-)$ and $\mathrm{SRel}_B(-)$ induced by restricting, for any objects A and B, the covariant 'direct image' functors $\mathrm{Sub}_{\mathcal{C}}(B)$ and $\mathrm{Sub}_{\mathcal{C}}(B \times A)$ to the subposets of small subobjects of B and small relations on $B \times A$, respectively (this fact requires images of small subobjects to be small which follows by **(S4)**). The content of the small powerobject axiom **(P1)** is then that these functors are representable in the sense that:

$$\begin{aligned} \mathrm{Hom}(A, \mathcal{P}_s\, B) &\cong \mathrm{SRel}_B(A), \text{ and} \\ \mathrm{Hom}(1, \mathcal{P}_s\, B) &\cong \mathrm{SSub}_{\mathcal{C}}(B). \end{aligned}$$

These facts are proved below in proposition 2.1.7.

2.1.2 The Internal Language of Categories with Basic Class Structure

We will now develop some of the properties of the internal language of categories with basic class structure. This approach is influenced by the work of Rummelhoff [28] and will provide a useful stepping stone for deriving further results. In particular, our aim in developing the internal logic explicitly is twofold:

1. By deriving typed versions of the set theoretic axioms with which we are concerned we are able to provide more elegant soundness proofs; for the validity of the untyped axioms ultimately rests on the validity of their typed analogues.

2. Furthermore, we will make some use of the internal language to show that the subcategories of small things have certain category theoretic properties. E.g., if $\mathcal{C}$ is a category with basic class structure, then the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects is a Heyting pretopos.

More generally, the development of the theory via the internal language allows us to emphasize the contribution of the categorical structure already present in categories with basic class structure and to compare it with the additional structure provided by the move to categories of classes (cf. subsection 2.2.1 below).

Henceforth we will assume that the ambient category $\mathcal{C}$ is a category with basic class structure. We will denote by π_A the composite:

$$\pi_A : \epsilon_A \rightarrowtail A \times \mathcal{P}_s \longrightarrow \mathcal{P}_s\, A.$$

Throughout we employ infix notation for certain distinguished relations and maps as in the use of $x\ \epsilon_C\ y$ for the more cumbersome $\epsilon_C\ (x, y)$. We abbreviate $\forall x_1 : X_1.\forall x_2 : X_2.\forall \ldots .\forall x_n.X_n.\varphi$ by $\forall x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n.\varphi$ and similarly for existential quantifiers. Finally, we write $\forall x\ \epsilon_C\ y$ in place of $\forall x : C.x\ \epsilon_C\ y$.

Proposition 2.1.3 *1. A relation $R \rightarrowtail C \times D$ is small iff, for some $\rho : D \longrightarrow \mathcal{P}_s\, C$:*

$$\mathcal{C} \models \forall x : C, y : D.R(x, y) \Leftrightarrow x\ \epsilon_C\ \rho(y).$$

2. A map $f : C \longrightarrow D$ is small iff, for some $f^{-1} : D \longrightarrow \mathcal{P}_s\, C$:

$$\mathcal{C} \models \forall x : C, y : D.f(x) = y \Leftrightarrow x\ \epsilon_C\ f^{-1}(y).$$

PROOF Immediate from the definitions of small maps and relations. In particular, the map f^{-1}, which we call the *fiber map*, classifies the graph $\Gamma(f) \rightarrowtail C \times D$ of f. □

The following proposition will be one of the most useful tools at our disposal in the study of categories with basic class structure. Indeed, this proposition serves to establish the importance of axiom **(S3)** (which will become all the more obvious with the introduction of the category of ideals in chapter 3).

Proposition 2.1.4 *The following are equivalent given **(S1)**, **(S2)** and **(P1)** (cf. [6] and [28]):*

1. ***(S3)**.*
2. *Regular monomorphisms are small.*
3. *If $g \circ f$ is small then f is small.*
4. *$\in_C$: $\epsilon_C \rightarrowtail C \times \mathcal{P}_s\, C$ is a small map.*

5. $[\![x : C, u : \mathcal{P}_s\, C, v : \mathcal{P}_s\, C | x \,\epsilon_C\, u \wedge x \,\epsilon_C\, v]\!]$ *is a small relation*

6. *Sections are small.*

Proof For (1)⇒(2) notice that Δ is a regular mono and suppose that $m : A \rightarrowtail B$ is the equalizer of $h, k : B \rightrightarrows C$. Then:

$$\begin{tikzcd} A \arrow[r, "h\circ m = k\circ m"] \arrow[d, tail, "m"'] & C \arrow[d, tail, "\Delta"] \\ B \arrow[r, "\langle h,k\rangle"'] & C \times C \end{tikzcd}$$

is a pullback and m is small by **(S2)**.

To show that (2)⇒(3) suppose regular monos are small and $g \circ f$ is small where:

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

and consider the pullback:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "g\circ f"'] & C. \end{tikzcd}$$

There is a canonical map $\zeta : A \longrightarrow P$ such that $p_1 \circ \zeta = 1_A$. By **(S1)** f is a small map.

(3)⇒(1) is trivial. Also (3)⇒(4) is trivial. (4)⇒(1) is by **(S2)**. Both (3)⇒(6) and (6)⇒(1) are trivial.

For (4)⇒(5) notice that if $R \rightarrowtail C \times D$ is a small relation and the map $S \rightarrowtail C \times D$ is small, then $R \wedge S$ is a small relation. (5)⇒(1) is by the fact that:

$$\mathcal{C} \vDash \forall x : C, y : C. x = y \Leftrightarrow \forall z : C. z \,\epsilon_C\, \{x\}_C \wedge z \,\epsilon_C\, \{y\}_C.$$

□

Corollary 2.1.5 *All of the canonical maps* $!_A : 0 \longrightarrow A$ *are small and if* $f : A \longrightarrow B$ *and* $g : C \longrightarrow D$ *are small, then* $f + g : A + C \longrightarrow B + D$ *is also small.*

Proof First, note that each $!_A : 0 \longrightarrow A$ is a regular monomorphism since, by disjointness of coproducts, the pullback of each inclusion $i_1, i_2 : A \rightrightarrows A + A$ along the other is 0. As such, $!_A$ is an equalizer of i_1 and i_2.

Next, notice that if we have $f : A \longrightarrow B$ and $g : C \longrightarrow D$ both regular monomorphisms, then f equalizes some pair $x, x' : B \rightrightarrows X$ and g equalizes some pair $y, y' : D \rightrightarrows Y$. But then $f + g : A + C \longrightarrow B + D$ equalizes $x + y$ and $x' + y'$. Thus, $f + g$ is also a regular mono.

Now consider a coproduct $A + B$ with inclusions i_1 and i_2. Since A is isomorphic to $A + 0$ we have that $i_1 = 1_A + !_B$. But both $!_B$ and 1_A are regular monos. So i_1 and i_2 are also regular monos.

Finally, where $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are small it follows from **(S5)** and the previous remarks that $f + g$ is also small. $\square$

The reader should be alerted at this point that use of proposition 2.1.4 and its corollary will often be made without explicit mention.

Proposition 2.1.6 (Typed Axioms) *The following are true in any category $\mathcal{C}$ with basic class structure:*

Extensionality: *For any object C:*

$$\mathcal{C} \models \forall a, b : \mathcal{P}_s\, C.(\forall x : C.x \;\epsilon_C\; a \Leftrightarrow x \;\epsilon_C\; b) \Rightarrow a = b.$$

Emptyset: *For each object C there exists a map $\emptyset_C : 1 \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x : C.x \;\epsilon_C\; \emptyset_C \Leftrightarrow \bot.$$

Singleton: *For each object C the singleton map $\{-\}_C$, which is the classifying map for the diagonal $\Delta : C \rightarrowtail C \times C$, is a small monomorphism.*

Binary Union: *For each C there exists a map $\cup_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x : C, a, b : \mathcal{P}_s\, C.x \;\epsilon_C\; (a \cup_C b) \Leftrightarrow x \;\epsilon_C\; a \vee x \;\epsilon_C\; b.$$

Product: *For all C and D there exists a map $\times_{C,D} : \mathcal{P}_s\, C \times \mathcal{P}_s\, D \longrightarrow \mathcal{P}_s\,(C \times D)$ such that:*

$$\mathcal{C} \models \forall x : C, y : D, a : \mathcal{P}_s C, b : \mathcal{P}_s D.(x, y) \;\epsilon_{C \times D}\; (a \times_{C,D} b) \Leftrightarrow x \;\epsilon_C\; a \wedge y \;\epsilon_D\; b.$$

Pairing: *For any C there exists a map $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ such that:*

$$\mathcal{C} \models \forall x, y, z : C.x \;\epsilon_C\; \{y, z\}_C \Leftrightarrow x = y \vee x = z.$$

PROOF For Extensionality, let the subobject r be given by the following:

$$[\![a, b : \mathcal{P}_s\, C | (\forall x : C)(x \epsilon_C a \Leftrightarrow x \epsilon_C b)]\!] \overset{r}{\rightarrowtail} \mathcal{P}_s\, C \times \mathcal{P}_s\, C.$$

By **(P1)** there exist subobjects S, S' of $C \times R$ classified by $\pi_1 \circ r$ and $\pi_2 \circ r$, respectively. But by assumption $S = S'$. Notice that r factors through the diagonal Δ iff $\pi_1 \circ r = \pi_2 \circ r$ (recall that Δ is the equalizer of π_2 and π_2). Thus, by **(P1)**, R factors through Δ, as required.

For Emptyset it suffices to notice that $[\![x : C|\bot]\!]$ is small.

For Singleton note that by Proposition 2.1.3 we have that:

$$[\![x, y : C | x \epsilon_C \{y\}]\!] = \Delta,$$

so that if $\mathcal{C} \models \{x\}_C = \{y\}_C$, then $\mathcal{C} \models x = y$. To see that $\{-\}_C$ is small notice that where:

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, "\Delta"'] & \epsilon_C \arrow[d, "\in_C"] \\ C \times C \arrow[r, "1_C \times \{-\}_C"'] & C \times \mathcal{P}_s\, C \end{tikzcd}$$

we have $\{-\}_C = \pi_C \circ p$. But p is small since it has a retraction.

Binary Union follows from the fact that, by **(S4)** and **(S5)**, the join of two small subobjects is a small subobject. Product is by **(S2)**. Finally, for Pairing, the map $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ is the composite $\cup_C \circ (\{-\}_C \times \{-\}_C)$. $\square$

The foregoing is a good start, but before we are able to verify that more sophisticated principles (e.g., Replacement) we must first develop several additional properties of the categories in question.

Proposition 2.1.7 *$\mathcal{P}_s\,(-)$ is the object part of a covariant endofunctor $\mathcal{P}_s$ on $\mathcal{C}$.*

PROOF The proof follows [21] (and, indeed, is standard in topos theory). For any map $f : C \longrightarrow D$, notice that if $R \rightarrowtail C \times E$ is a member of $\mathrm{SRel}_C(E)$, then the image $\exists_{f\times 1}(R)$ is a member of $\mathrm{SRel}_D(E)$. Moreover, this is natural in E by the Beck-Chevalley condition and therefore determines a natural transformation:

$$\mathrm{SRel}_C(-) \cong \mathrm{Hom}(-, \mathcal{P}_s\, C) \longrightarrow \mathrm{Hom}(-, \mathcal{P}_s\, D) \cong \mathrm{SRel}_D(-).$$

By the Yoneda lemma there is a corresponding map $\mathcal{P}_s\,(f) : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$. Functoriality is easily verified. $\square$

Henceforth we write $f_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$ instead of $\mathcal{P}_s\,(f)$, where $f : C \longrightarrow D$.

Proposition 2.1.8 *Where $f : C \longrightarrow D$:*

$$\mathcal{C} \models \forall x : D, a : \mathcal{P}_s\, C.x \in_D f_!(a) \Leftrightarrow \exists y \in_C a.f(y) = x.$$

PROOF By Proposition 2.1.7 and the proof of the Yoneda lemma we have:

$$[\![x : D, a : \mathcal{P}_s\, C | x \in_D f_!(a)]\!] \quad = \quad \exists_{f \times 1_{\mathcal{P}_s\, C}}(\in_C).$$

Also, in a regular category, given any $\alpha \rightarrowtail X \times Y$ and $f : X \longrightarrow Z$ we have:

$$\exists_{f \times 1_Y}(\alpha) \quad = \quad [\![z : Z, y : Y | \exists x : X.\alpha(x, y) \wedge f(x) = z]\!],$$

as required. □

Corollary 2.1.9 *If $m : C \rightarrowtail D$ is monic, then so is $m_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$. I.e.,*

$$\mathcal{C} \models \forall x, x' : \mathcal{P}_s\, C.m_!(x) = m_!(x') \Rightarrow x = x'.$$

PROOF By Typed Extensionality and the internal language. □

Corollary 2.1.10 *If $m : C \rightarrowtail D$ is monic, then:*

$$\begin{tikzcd} \in_C \arrow[r, tail] \arrow[d, tail] & \in_D \arrow[d, tail] \\ C \times \mathcal{P}_s\, C \arrow[r, tail, "m \times m_!"'] & D \times \mathcal{P}_s\, D \end{tikzcd}$$

is a pullback.

PROOF Easy. □

Proposition 2.1.11 *Every small map $f : C \longrightarrow D$ gives rise to an (internal) inverse image map $f^* : \mathcal{P}_s\, D \longrightarrow \mathcal{P}_s\, C$.*

PROOF The proof follows [21] and is also familiar from topos theory. However, note that the smallness of f is a necessary assumption in this case whereas no such assumption is required for topoi.

Let a small relation $R \rightarrowtail D \times E$ be given and consider the pullback:

$$\begin{tikzcd} R' \arrow[r, "p"] \arrow[d, tail, "i'"'] & R \arrow[d, tail, "i"] \\ C \times E \arrow[r, "f \times 1"'] & D \times E \end{tikzcd}$$

of R along $f \times 1$. Since f is small it follows that both $f \times 1$ and p are small. So the composite $R' \longrightarrow R \longrightarrow X$ of p and the projection from R to X is small and R' is a small relation. □

Proposition 2.1.12 *If $f : C \longrightarrow D$ is a small map, then:*

$$\mathcal{C} \models \forall x : C, a : \mathcal{P}_s\, D. x \; \epsilon_C \; f^*(a) \Leftrightarrow f(x) \; \epsilon_D \; a,$$

where f^* is as above.

Proof By Proposition 2.1.11 and the proof of the Yoneda lemma f^* corresponds to the small subobject $(f \times 1_{\mathcal{P}_s\, D})^*(\; \epsilon_D \;)$ of $C \times \mathcal{P}_s\, D$, where here the pullback $(f \times 1_{\mathcal{P}_s\, D})^*$ is external. □

In the following we write $\subseteq_C$ for the subobject of $\mathcal{P}_s\, C \times \mathcal{P}_s\, C$ given by:

$$\subseteq_C \quad := \quad [\![x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, C | \forall z \; \epsilon_C \; x. z \; \epsilon_C \; y]\!].$$

From this description of $\subseteq_C$ it easily follows that $\subseteq_C \rightarrowtail \mathcal{P}_s\, C \times \mathcal{P}_s\, C$ is the equalizer of $\pi_1, \cap_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \rightrightarrows \mathcal{P}_s\, C$ and that:

$$\mathcal{C} \models \forall x, y : \mathcal{P}_s\, C. x \subseteq_C y \Leftrightarrow x \cap_C y = x.$$

Lemma 2.1.13 *If $f : C \longrightarrow D$ is a small map, then $f_! \dashv f^*$ internally. That is:*

$$\mathcal{C} \models \forall x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, D. f_!(x) \subseteq_D y \Leftrightarrow x \subseteq_C f^*(y).$$

Proof Easy using the internal language. □

Proposition 2.1.14 (Internal Beck-Chevalley Condition) *If $f : C \longrightarrow D$ is a small map and the following diagram is a pullback:*

$$\begin{tikzcd} C' \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "g"'] & D \end{tikzcd}$$

then $f^* \circ g_! = g'_! \circ (f')^*$.

Proof By the external Beck-Chevalley condition. □

2.1.3 Slicing, Exponentiation and the Subcategory of Small Objects

In this subsection we first show that the structure of categories with basic class structure is preserved under slicing. Next, we show that small objects are exponentiable and introduce the (categorical) *exponentiation* axiom. Finally, we show that the category $\mathcal{S}_\mathcal{C}$ of small objects in a category $\mathcal{C}$ with basic class structure is a Heyting pretopos and, moreover, if $\mathcal{C}$ also satisfies the categorical exponentiation axiom, then $\mathcal{S}_\mathcal{C}$ is a Π-pretopos.

Theorem 2.1.15 *If $\mathcal{C}$ is a category with basic class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ is also a category with basic class structure.*

Proof The Heyting category structure of $\mathcal{C}$ is easily seen to be preserved under slicing. Also, the collection $\mathcal{S}_D$ of all maps in $\mathcal{C}/D$ that are small in $\mathcal{C}$ is a system of small maps in $\mathcal{C}/D$.

Where $f : C \longrightarrow D$ is an object in $\mathcal{C}/D$ we define the powerobject $\mathcal{P}_s\,(f : C \longrightarrow D)$ as the composite $p_f : V_f \rightarrowtail \mathcal{P}_s\, C \times D \longrightarrow D$ where V_f is defined as follows:

$$V_f \quad := \quad [\![x : \mathcal{P}_s\, C, y : D | f_!(x) \subseteq_D \{y\}_D]\!].$$

Notice that by previous results $V_f = [\![x, y | \forall z \; \epsilon_C \; x.f(z) = y]\!]$. Similarly, we define the membership relation ϵ_f as the composite $M_f \rightarrowtail D \times C \times \mathcal{P}_s\, C \longrightarrow D$ where:

$$M_f \quad := \quad [\![x : D, y : C, z : \mathcal{P}_s\, C | y \; \epsilon_C \; z \wedge \forall x' \; \epsilon_C \; z.f(x') = x]\!].$$

In order to show that **(P1)** holds let objects $a : A \longrightarrow D$ and $b : B \longrightarrow D$ of $\mathcal{C}/D$ be given and suppose $r : R \rightarrowtail A \times_D B$ is a small relation on $a \times b$. Then the composite $a' \circ r : R \longrightarrow B$ is small, where a' is as in the following diagram:

$$\begin{tikzcd} A \times_D B \arrow[r, "a'"] \arrow[d, "b'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & D. \end{tikzcd}$$

Then there exists a map $\rho : B \longrightarrow \mathcal{P}_s\, A$ in $\mathcal{C}$ classifying $i \circ r : R \rightarrowtail A \times B$ where i is the inclusion $A \times_D B \rightarrowtail A \times B$. We now need to show that there exists a map $\hat{\rho} : B \longrightarrow V_a$ such that $\mathcal{P}_s\,(a) \circ \hat{\rho} = b$. To this end note that by an easy application of the internal logic of $\mathcal{C}$ one can see that:

$$\pi_1 \circ (a_! \times \{-\}_D) \circ \langle \rho, b \rangle \quad = \quad \cap_D \circ (a_! \times \{-\}_D) \circ \langle \rho, b \rangle,$$

and, since $j : \subseteq_D \rightarrowtail \mathcal{P}_s\, D \times \mathcal{P}_s\, D$ equalizes π_1 and $\cap_D$, there exists a unique map $\lambda : B \longrightarrow \subseteq_D$ with $j \circ \lambda = (a_! \times \{-\}_D) \circ \langle \rho, b \rangle$. But, since $j' : V_a \rightarrowtail \mathcal{P}_s\, a \times D$ is the pullback of j along $(a_1 \times \{-\}_D)$ there exists a unique map $\hat{\rho} : B \longrightarrow V_a$ making the appropriate triangles commute. As such, $\mathcal{P}_s\,(a) \circ \hat{\rho} = b$.

Similarly, one shows that there exists a map $\sigma : R \longrightarrow M_a$ such that $\hat{\rho}$ is the classifying map for $r \rightarrowtail a \times b$ in $\mathcal{C}/D$. □

Lemma 2.1.16 *Given $f : B \longrightarrow A$ in $\mathcal{C}$ the pullback functor $\Delta_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ preserves all basic class structure.*

Proof It follows from **(S2)** that small maps are preserved by Δ_f. Next, to show that powerobjects are preserved by pullback an object $g : D \longrightarrow A$ of $\mathcal{C}/A$ be given and suppose $r : R \rightarrowtail \Delta_f(D) \times_B E$ is a small relation on $\Delta_f(g) \times e$ in $\mathcal{C}/B$ where $e : E \longrightarrow B$ is an arbitrary object and $\Delta_f(D)$ is the domain of $\Delta_f(g)$. As such, the following diagram is a pullback:

$$\begin{tikzcd} \Delta_f D \times_B E \arrow[r, "l"] \arrow[d, "e'"'] & E \arrow[d, "e"] \\ \Delta_f D \arrow[r, "\Delta_f(g)"'] & B. \end{tikzcd}$$

As above, there exists a map $\rho : E \longrightarrow \mathcal{P}_s(\Delta_f D)$ classifying the small relation $i \circ r : R \rightarrowtail \Delta_f D \times E$ where i is the inclusion $\Delta_f D \times_B E \rightarrowtail \Delta_f D \times E$. Moreover, there exists a map $\hat{\rho} : e \longrightarrow \mathcal{P}_s\ (\Delta_f(g))$ in $\mathcal{C}/B$ such that the following is a pullback in $\mathcal{C}/B$:

$$\begin{tikzcd} R \arrow[r, "\hat{\rho}'"] \arrow[d] & \epsilon_{\Delta_f g} \arrow[d, "\pi_{\Delta_f g}"] \\ E \arrow[r, "\hat{\rho}"'] & \mathcal{P}_s\ (\Delta_f g). \end{tikzcd}$$

Also note that, since small maps are preserved by Δ_f, there exists a map $\psi : \Delta_f(\mathcal{P}_s\ g) \longrightarrow \mathcal{P}_s\ (\Delta_f g)$ which classifies the small relation $\Delta_f(\ \epsilon_g\ ) \rightarrowtail \Delta_f g \times \Delta f(\mathcal{P}_s\, g)$ in $\mathcal{C}/B$. By the two pullbacks lemma it suffices to show that there exists a map $\varphi : E \longrightarrow \Delta_f(\mathcal{P}_s g)$ in $\mathcal{C}/B$ such that $\psi \circ \varphi = \hat{\rho}$. To see that this is so the reader need only verify that $E \Vdash g_! \circ f'_! \circ \rho \subseteq_A \{f \circ e\}_A$ yielding a map $q : E \longrightarrow \subseteq_A$. By definition the following square is a pullback:

$$\begin{tikzcd} V_g \arrow[r, "s"] \arrow[d, tail, "i"'] & \subseteq_A \arrow[d, tail, "j"] \\ \mathcal{P}_s\ D \times A \arrow[r, "g_! \times \{-\}_A"'] & \mathcal{P}_s\ A \times \mathcal{P}_s\ A \end{tikzcd}$$

there exists a unique map $\xi : E \longrightarrow V_g$ with $i \circ \xi = \langle f'_! \circ \rho, f \circ e \rangle$ and $s \circ \xi = q$. In particular, $\mathcal{P}_s\ (g) \circ \xi = f \circ e$. By the definition of $\Delta_f(\mathcal{P}_s\, g) : \Delta_f(V_g) \longrightarrow B$ as given by the following pullback:

$$\begin{tikzcd} \Delta_f(V_g) \arrow[r, "p'"] \arrow[d, "\Delta_f(\mathcal{P}_s\ g)"'] & V_g \arrow[d, "\mathcal{P}_s\ (g)"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

there exists a unique map $\varphi : E \longrightarrow \Delta_f(V_g)$ with $\Delta_f(\mathcal{P}_s\, g) \circ \varphi = e$ and $p' \circ \varphi = \xi$. We may now apply the two pullbacks lemma (and the uniqueness of $\hat{\rho}$ as a classifying map) to conclude that φ classifies $r \rightarrowtail \Delta_f(g) \times e$. Since powerobjects are unique up to isomorphism it follows that Δ_f preserves powerobjects. $\Box$

We will now show that exponentials D^C exist when C is a small object. We define the exponential in question as a subobject of $\mathcal{P}_s\,(C \times D)$ as follows:

$$D^C \quad := \quad [\![R : \mathcal{P}_s\,(C \times D) | \forall x : C.\exists! y : D.(x, y) \,\epsilon_{C\times D}\, R]\!].$$

Lemma 2.1.17 *If C is small, then the following special case of the adjunction $- \times C \dashv -^C$ holds:*

$$\frac{C \xrightarrow{f} D}{\overline{\overline{1 \xrightarrow{\ulcorner f \urcorner} D^C}}}\,. \tag{2.1}$$

That is to say, there exists a natural isomorphism $\mathrm{Hom}(C, D) \cong \mathrm{Hom}(1, D^C)$.

PROOF First, let a global element $f : 1 \longrightarrow D^C$ be given. Then:

$$1 \Vdash \forall x : C.\exists! y : D.(x, y) \,\epsilon_{C\times D}\, \tilde{f},$$

where $\tilde{f}$ is the composite $1 \xrightarrow{f} D^C \rightarrowtail \mathcal{P}_s\,(C \times D)$. So there exists a map $\bar{f} : C \longrightarrow D$ such that:

$$[\![x : C, y : D | \bar{f}(x) = y]\!] \quad = \quad [\![x : C, y : D | (x, y) \,\epsilon_{C\times D}\, \tilde{f}]\!].$$

For the other direction we have, using the lemma, that $\Gamma(f)$ is a small subobject of $C \times D$. Moreover, $1 \Vdash \forall x : C.\exists! y : D.(x, y) \,\epsilon_{C\times D}\, \tilde{f}$, where $\tilde{f}$ is the classifying map of $\Gamma(f)$. That is, $\tilde{f}$ factors through D^C via some map $\ulcorner f \urcorner$, as required.

These operations are easily seen to be mutually inverse. $\Box$

Now, using the fact that $\mathcal{C}/E$ has basic class structure and the pullback functor $\Delta_{!_E} : \mathcal{C} \longrightarrow \mathcal{C}/E$ preserves this structure we arrive at the more general lemma:

Lemma 2.1.18 *Where C is a small object we have the following natural isomorphisms:*

$$\frac{E \times C \xrightarrow{f} D}{\overline{\overline{E \xrightarrow{\ulcorner f \urcorner} D^C}}}\,. \tag{2.2}$$

Corollary 2.1.19 *Small objects are exponentiable.*

Proposition 2.1.20 *If $f : C \longrightarrow D$ is a small map, then the pullback functor $\Delta_f : \mathcal{C}/D \longrightarrow \mathcal{C}/C$ has a right adjoint Π_f.*

Proof Clearly $(f : C \longrightarrow D)$ is a small object in $\mathcal{C}/D$ and, hence, exponentiable there. The existence of the adjoint Π_f then follows as usual. $\Box$

Definition 2.1.21 A *category with (predicative) class structure* is a category $\mathcal{C}$ with basic class structure which also satisfies the following *exponentiation axiom*:

(E) If $f : C \longrightarrow D$ is a small map, then the functor $\Pi_f : \mathcal{C}/C \longrightarrow \mathcal{C}/D$ (which exists by the foregoing proposition) preserves small maps.

Proposition 2.1.22 *In a category with class structure if C and D are both small, then so is D^C.*

Proof Notice that D^C is $\Pi_C \circ \Delta_C(D)$. Moreover, since D is small so is $\Delta_C(D)$. By **(E)** it follows that $D^C \longrightarrow 1$ is also small. $\Box$

Proposition 2.1.23 *If $\mathcal{C}$ is a category with class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ also has class structure.*

Proof Use the fact that $(\mathcal{C}/D)/f \cong \mathcal{C}/\operatorname{dom}(f)$. $\Box$

In the following proposition and theorem we will be concerned with the properties of the full subcategory $\mathcal{S}_{\mathcal{C}} := \mathcal{S}/1$ of $\mathcal{C}$ consisting of small objects and small maps between them.

Proposition 2.1.24 *Let $\mathcal{C}$ be a category with basic class structure. If $\partial_0, \partial_1 : R \rightrightarrows C \times C$ is an equivalence relation in $\mathcal{S}_{\mathcal{C}}$, then the coequalizer of ∂_0 and ∂_1 exists in $\mathcal{S}_{\mathcal{C}}$ and ∂_0, ∂_1 is its kernel pair.*

Proof We define the coequalizer C/R by:

$$C/R \quad := \quad [\![z : \mathcal{P}_s\, C \mid \exists x : C. \forall y : C. y \,\epsilon_C\, z \Leftrightarrow R(x, y)]\!].$$

Notice that since ∂_0 and ∂_1 are small maps so is $\langle \partial_0, \partial_1 \rangle : R \rightarrowtail C \times C$. As such, $\langle \partial_0, \partial_1 \rangle$ is also a small relation and there exists a unique $\alpha : C \longrightarrow \mathcal{P}_s C$ such that:

$$\begin{tikzcd}
R \arrow[r, "p"] \arrow[d, tail] & \epsilon_C \arrow[d, tail] \\
C \times C \arrow[r, "1 \times \alpha"'] & C \times \mathcal{P}_s\, C
\end{tikzcd}$$

is a pullback. That is:

$$\mathcal{C} \models \forall x, y : C.R(x, y) \Leftrightarrow x \epsilon_C \alpha(y). \tag{2.3}$$

By (2.3) and Typed Extensionality it follows that C/R is the image of α:

$$\mathrm{im}(\alpha) \quad = \quad [\![z : \mathcal{P}_s\, C | \exists x : C.\alpha(x) = z]\!],$$

and, as such, that α factors through $i : C/R \rightarrowtail \mathcal{P}_s\, C$ via a cover $\bar{\alpha}$. Moreover, by **(P1)**, $\bar{\alpha} \circ \partial_0 = \bar{\alpha} \circ \partial_1$ since $\langle \partial_0, \partial_1 \rangle$ is an equivalence relation. Notice that since C is small it follows that $\bar{\alpha}$ is a small map and, by **(S4)**, that C/R is a small object.

Finally, we will show that ∂_0, ∂_1 is the kernel pair of $\bar{\alpha}$; i.e., that:

$$\begin{tikzcd} R \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] & C \arrow[d, "\bar{\alpha}"] \\ C \arrow[r, "\bar{\alpha}"'] & C/R \end{tikzcd}$$

is a pullback. Let an object Z and maps $z_0, z_1 : Z \rightrightarrows C$ be given such that $\bar{\alpha} \circ z_0 = \bar{\alpha} \circ z_1$. Then we also have that $\alpha \circ z_0 = \alpha \circ z_1$. Define a map $\eta : Z \longrightarrow \epsilon_C$ by $\eta := p \circ r \circ z_0$, where r is the 'reflexivity' map. Then we have:

$$\begin{aligned} \in \circ \eta &= \langle \partial_0, \alpha \circ \partial_1 \rangle \circ r \circ z_0 \\ &= \langle z_0, \alpha \circ z_0 \rangle \\ &= (1_C \times \alpha) \circ \langle \partial_0, \partial_1 \rangle. \end{aligned}$$

By the universal property of pullbacks there exists a unique map $\bar{\eta} : Z \longrightarrow R$ with $p \circ \bar{\eta} = \eta$ and $\langle \partial_0, \partial_1 \rangle \circ \bar{\eta} = \langle z_0, z_1 \rangle$. Moreover $\bar{\eta}$ is the unique map from Z to R such that $\partial_0 \circ \bar{\eta} = z_0$ and $\partial_1 \circ \bar{\eta} = z_1$. It follows from the fact that covers coequalize their kernel pairs that $\bar{\alpha}$ is a coequalizer of ∂_0 and ∂_1. It is easily seen that if Z together with z_0 and z_1 are in $\mathcal{S}_\mathcal{C}$, then so is $\hat{\eta}$. $\square$

Theorem 2.1.25 *If $\mathcal{C}$ has basic class structure, then $\mathcal{S}_\mathcal{C}$ is a Heyting pretopos. Moreover, if $\mathcal{C}$ has class structure, then $\mathcal{S}_\mathcal{C}$ is a Π-pretopos.*

Proof By Proposition 2.1.24 $\mathcal{S}_\mathcal{C}$ has coequalizers of equivalence relations. It suffices to show that $\mathcal{S}_\mathcal{C}$ is a positive Heyting category. But, this structure is easily seen exist since $\mathcal{C}$ is a positive Heyting category. For instance, to show that $\mathcal{S}_\mathcal{C}$ has disjoint finite coproducts note that if C and D are small objects then so is $C + D$ together with the maps $C \longrightarrow C + D$ and $D \longrightarrow C + D$ by **(S5)**. Disjointness and stability are consequences of **(S3)**.

Similarly, by the description of $C \times D$ as the pullback of $!_C$ along $!_D$, it follows that $C \times D$ is a small object when C and D are. $\mathcal{S}_\mathcal{C}$ is seen to be regular by **(S3)**. Finally, for dual images, let a map $f : C \longrightarrow D$ and a subobject $m : S \rightarrowtail C$ be given in $\mathcal{S}_\mathcal{C}$. Consider the subobject $i : \forall_f(m) \rightarrowtail D$. Notice that, in general, if a monomorphism $C \rightarrowtail D$ in a category $\mathcal{C}$ with basic class structure is small, then it is also regular since it is a pullback of the section $\top : 1 \longrightarrow \mathcal{P}_s\, 1$. Moreover since, by Proposition 2.1.20, Π_f exists and is a right adjoint, it follows that i is a small map.

The further result is a consequence of Proposition 2.1.22. $\square$

2.1.4 Typed Union and Replacement

We now show that typed versions of Union and Replacement are valid in categories with basic class structure. To this end, we introduce a typed version of the '$\text{Ƨ}z.\varphi$' notation from above as follows:

$$\text{Ƨ}x : C.\varphi \quad := \quad \exists y : \mathcal{P}_s\, C.\forall x : C.(x \in_C y \Leftrightarrow \varphi),$$

where $y \notin \mathrm{FV}(\varphi)$.

Proposition 2.1.26 *A relation $R \rightarrowtail C \times D$ is small if and only if $\mathcal{C} \models \forall y : D.\text{Ƨ}x : C.R(x, y)$.*

Proof Suppose $R \rightarrowtail C \times D$ is a small relation and $\rho : D \longrightarrow \mathcal{P}_s\, C$ is the classifying map. Then by Proposition 2.1.3 we have $\mathcal{C} \models \forall y : D.\forall x : C.R(x, y) \Leftrightarrow x \in_C \rho(y)$. The conclusion may be seen to follow from this (use ρ to witness the existential).

For the other direction suppose $\mathcal{C} \models \forall y : D.\text{Ƨ}x : C.R(x, y)$. Then, by Typed Extensionality:

$$\mathcal{C} \models \forall y : D.\exists! z : \mathcal{P}_s\, C.\forall x : C(x \in_C z \Leftrightarrow R(x, y)),$$

and there is a map $\rho : D \longrightarrow \mathcal{P}_s\, C$ with the requisite property. $\square$

Proposition 2.1.27 (Typed Union) *For all C:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\, (\mathcal{P}_s\, C).\text{Ƨ}z : C.\exists x \in_{\mathcal{P}_s\, C} a.z \in_C x.$$

Proof Let H be defined as:

$$H \quad := \quad [\![x : C, y : \mathcal{P}_s\, C, z : \mathcal{P}_s\, (\mathcal{P}_s\, C) | y \in_{\mathcal{P}_s\, C} z \wedge x \in_C y]\!],$$

and note that the projection:

$$H \rightarrowtail C \times \mathcal{P}_s\, C \times \mathcal{P}_s\, (\mathcal{P}_s\, C) \longrightarrow \mathcal{P}_s\, (\mathcal{P}_s\, C)$$

is small. By **(S4)** it follows that $[\![x : C, z : \mathcal{P}_s\,(\mathcal{P}_s\,C) | \exists y \; \epsilon_{\mathcal{P}_s\,C} \; z \wedge x \; \epsilon_C \; y]\!]$ is a small relation. We write $\bigcup_C : \mathcal{P}_s\,(\mathcal{P}_s\,C) \longrightarrow \mathcal{P}_s\,C$ for the classifying map. □

Corollary 2.1.28 (The 'Powerset' Monad) *If $\mathcal{C}$ has basic class structure, then the covariant powerobject functor $\mathcal{P}_s : \mathcal{C} \longrightarrow \mathcal{C}$ together with natural transformations $\{-\}_- : 1_{\mathcal{C}} \longrightarrow \mathcal{P}_s$ and $\bigcup_- : {\mathcal{P}_s}^2 \longrightarrow \mathcal{P}_s$ induced by the singleton and union maps is a monad.*

PROOF Easy. □

Proposition 2.1.29 (Typed Replacement) *For all C and D:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\,C.(\forall x \; \epsilon_C \; a.\exists! y : D.\varphi) \Rightarrow (\text{Ƨ} y : D.\exists x \; \epsilon_C \; a.\varphi).$$

PROOF Let $a : 1 \longrightarrow \mathcal{P}_s\,C$ be given with $1 \Vdash \forall x \; \epsilon_C \; a.\exists! y : D.\varphi$. Let $\alpha \rightarrowtail C$ be the small subobject classified by a. Then the assumption yields a map $f : \alpha \longrightarrow C \longrightarrow D$ such that:

$$\Gamma(f) \quad = \quad [\![x : \alpha, y : D | \varphi(x, y)]\!].$$

Moreover, the image of f is the subobject:

$$I \quad := \quad [\![y : D | \exists x \; \epsilon_C \; a.\varphi(x, y)]\!].$$

Since α is a small subobject it follows by **(S4)** that I is also a small subobject. We may now pull the general problem back as usual. □

2.2 Soundness and Completeness

This section we consider the logical applications of categories with class structure. In particular, we prove soundness and completeness results for **BCST**, **CST** and their variants obtained by adding the Infinity* axiom.

2.2.1 Universes and Categories of Classes

All of the set theories introduced earlier are untyped (or, as we prefer to think of things, mono-typed) theories; yet the internal languages of the categories we have been considering are typed languages. As such, we will introduce a technical device which will allow us to model untyped theories. The use of universal objects for this purpose originated in [34] and has its roots in Dana Scott's earlier work on modeling the lambda calculus (cf. [??]).

Definition 2.2.1 A *universal object* in a category $\mathcal{C}$ is an object U of $\mathcal{C}$ such that for any object C there exists a monomorphism $m : C \rightarrowtail U$. Similarly, in a category $\mathcal{C}$ with basic class structure, a *universe* is an object U together with a monomorphism $\iota : \mathcal{P}_s(U) \rightarrowtail U$.

Notice that the monomorphisms m and ι in the definition need not be unique. Also, notice that if U is a universe in a category $\mathcal{C}$ then we may obtain a category $\mathcal{C}'$ containing a universal object, also U, by restricting to the full subcategory of $\mathcal{C}$ consisting of subobjects of U.

Definition 2.2.2 A *basic (predicative) category of classes* a category $\mathcal{C}$ with basic class structure satisfying the additional *universal object* axiom:

(U) There exists a universal object U.

Similarly, a *predicative category of classes* is a category with class structure satisfying **(U)**.

We will now turn to proving that **BCST** is sound and complete with respect to models in basic categories of classes and that **CST** is sound and complete with respect to models in predicative categories of classes.

2.2.2 Soundness

In order to interpret the theories in question in basic categories of classes (respectively, predicative categories of classes) we must choose a monomorphism $\iota : \mathcal{P}_s U \rightarrowtail U$ (this is because **(U)** is consistent with the existence of multiple monos $\mathcal{P}_s U \rightarrowtail U$). An *interpretation* of **BCST** in a basic category of classes $\mathcal{C}$ is a conventional interpretation $[\![-]\!]$ of the first-order structure $(\in, \mathsf{S})$ determined by the following conditions:

- $[\![\mathsf{S}(x)]\!]$ is defined to be:
$$\mathcal{P}_s U \overset{\iota}{\rightarrowtail} U.$$
- $[\![x \in y]\!]$ is interpreted as the subobject:
$$\epsilon_U \overset{\in}{\rightarrowtail} U \times \mathcal{P}_s U \overset{1\times\iota}{\rightarrowtail} U \times U.$$

Remark 2.2.3 We write $(\mathcal{C}, U) \vDash \varphi$ to indicate that φ is satisfied by the interpretation. As above $\mathcal{C} \vDash \varphi$ indicates that φ is true in the internal language and $Z \Vdash \varphi$ means that Z forces φ. See the appendix for more information on the Kripke-Joyal semantics.

We will now derive several useful results that will allow us to transfer results about the typed internal language to the untyped set theories in question.

Lemma 2.2.4 *Given $a : 1 \longrightarrow U$ such that a factors through $\iota : \mathcal{P}_s\, U \rightarrowtail U$ via some map $\bar{a}$:*

$$[\![x : U | x \in a]\!] = [\![x : U | x \ \epsilon_U \ \bar{a}]\!]$$

in $\mathrm{Sub}_{\mathcal{C}}(U)$.

Proof The proof is by the fact that $a = \iota \circ \bar{a}$ and the following diagram:

$$\begin{tikzcd}
{[\![x|x \in a]\!]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_U \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_U \arrow[d, tail, "\in"] \\
U \arrow[r, "1\times\bar{a}"] \arrow[d, tail, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times \mathcal{P}_s\, U \arrow[r, tail, "1"] \arrow[d, tail, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times \mathcal{P}_s\, U \arrow[d, tail, "1\times\iota"] \\
U \arrow[r, "1\times\bar{a}"'] & U \times \mathcal{P}_s\, U \arrow[r, tail, "1\times\iota"'] & U \times U.
\end{tikzcd}$$

Lemma 2.2.5 *If $m : C \rightarrowtail D$ is a small subobject with classifying map $c : 1 \longrightarrow \mathcal{P}_s\, D$ and $r : R \rightarrowtail D \times E$ is a small relation with classifying map $\rho : E \longrightarrow \mathcal{P}_s\, D$ such that $E \Vdash \rho \subseteq_D c$, then there exists a restriction $r' : R' \rightarrowtail C \times E$ of R to C which is a small relation with classifying map $\rho' : E \longrightarrow \mathcal{P}_s\, C$ such that $\rho = m_! \circ \rho'$.*

Proof Let $r' : R' \rightarrowtail C \times E$ be the pullback of $r : R \rightarrowtail D \times E$ along $m \times 1_E$, then, since $!_C$ is small so is $m \times 1_E$. As such, $r' : R' \rightarrowtail C \times E$ is a small relation and there exists a classifying map $\rho' : E \longrightarrow \mathcal{P}_s\, C$.

We use **(P1)** to show that $\rho = m_! \circ \rho'$. In particular, let $p : P \rightarrowtail D \times E$ be the small relation which results by pulling ϵ_D back along $1 \times (m_! \circ \rho')$, then it is a straightforward application of the internal language to show that the following holds:

$$\mathcal{C} \models \forall x : D, y : E. x \ \epsilon_D \ \rho(y) \Leftrightarrow x \ \epsilon_D \ m_! \circ \rho'(y).$$

By **(P1)** it follows that $\rho = m_! \circ \rho'$. $\square$

Proposition 2.2.6 *Suppose $i : \alpha \rightarrowtail C$ is a small subobject with classifying map $a : 1 \longrightarrow \mathcal{P}_s\, C$, then:*

$$\mathcal{P}_s\, \alpha \quad = \quad [\![x : \mathcal{P}_s\, C | x \subseteq_C a]\!],$$

where $\mathcal{P}_s\, \alpha$ is regarded as a subobject of $\mathcal{P}_s\, C$ via $i_! : \mathcal{P}_s\, \alpha \rightarrowtail \mathcal{P}_s\, C$.

Proof By Corollaries 2.1.9 and 2.1.10 we have $i_! : \mathcal{P}_s\,\alpha \rightarrowtail \mathcal{P}_s\,C$. As such, we need only find a map $\xi : \mathcal{P}_s\,\alpha \longrightarrow \subseteq_C$ such that:

$$\begin{tikzcd}
\mathcal{P}_s\,\alpha \arrow[r, "\xi"] \arrow[d, tail, "i_!"'] & \subseteq_C \arrow[d, tail, "j"] \\
\mathcal{P}_s\,C \arrow[r, "1\times a"'] & \mathcal{P}_s\,C \times \mathcal{P}_s\,C
\end{tikzcd}$$

is a pullback. We use the description of $\subseteq_C$ as the equalizer of π and $\cap_C$ to show that ξ exists. To this end, notice that, by **(P1)** it suffices to show that:

$$\mathcal{C} \vDash \forall x : C, y : \mathcal{P}_s\,\alpha.(x \;\epsilon_C\; i_!(y) \wedge x \;\epsilon_C\; a) \Leftrightarrow (x \;\epsilon_C\; i_!(y));$$

for then $\pi_1 \circ (1 \times a) \circ i_! = i_1 = \cap_C \circ (1 \times a) \circ i_!$. This is a straightforward application of the internal language and, by **(P1)** and the universal property of equalizers, there exists a map $\xi : \mathcal{P}_s\,\alpha \longrightarrow \subseteq_C$ such that $j \circ \xi = (1 \times a) \circ i_!$.

To show that the square in question is a pullback suppose given an object Z and maps $l : Z \longrightarrow \mathcal{P}_s\,C$ and $k : Z \longrightarrow \subseteq_C$ such that $j \circ k = (1 \times a) \circ l$. Then l is the classifying map of a small relation $L \rightarrowtail C \times Z$. By Lemma 2.2.5 the restriction of L to α exists with classifying map $\lambda : Z \longrightarrow \mathcal{P}_s\,\alpha$ such that $\lambda : Z \longrightarrow \mathcal{P}_s\,\alpha$. Moreover, λ is the unique map $Z \longrightarrow \mathcal{P}_s\,\alpha$ with this property since $i_!$ is monic. Finally, we have that $\pi_1 \circ (1\times a) \circ l = \cap_C \circ (1 \times a) \circ l$ so that there exists a unique map $\mu : Z \longrightarrow \subseteq_C$ with $j \circ \mu = (1 \times a) \circ l$. Hence $\mu = k$. Moreover $j \circ (\xi \circ \lambda) = (1 \times a) \circ l$. Therefore, $\xi \circ \lambda = k$, as required. $\square$

Lemma 2.2.7 *If $a : 1 \longrightarrow U$ and $1 \Vdash \mathsf{S}(a)$ via some map $\bar{a} : 1 \longrightarrow \mathcal{P}_s\,U$, then:*

$$[\![x | \mathsf{S}(x) \wedge (\forall y)(y \in x \Rightarrow y \in a)]\!] \quad = \quad \mathcal{P}_s\,\alpha,$$

where $i : \alpha \rightarrowtail U$ is the small subobject classified by $\bar{a}$ and $\mathcal{P}_s\,\alpha$ is regarded as a subobject of U via $\iota \circ i_!$.

Proof Note that $[\![x, z | (\forall y)(y \in x \Rightarrow y \in z)]\!]$ is the composite:

$$\subseteq_U \overset{j}{\rightarrowtail} \mathcal{P}_s\,U \times \mathcal{P}_s\,U \overset{\iota\times\iota}{\rightarrowtail} U \times U.$$

Using Proposition 2.2.6 the proof is by the following diagram:

$$\begin{tikzcd}
\mathcal{P}_s\,\alpha \arrow[r, tail, "1"] \arrow[d, tail, "i_!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_s\,\alpha \arrow[r] \arrow[d, tail, "i_!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \subseteq_U \arrow[d, tail, "j"] \\
\mathcal{P}_s\,U \arrow[r, tail, "1"] \arrow[d, tail, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_s\,U \arrow[r, "1\times\bar{a}"] \arrow[d, tail, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_s\,U \times \mathcal{P}_s\,U \arrow[d, tail, "\iota\times\iota"] \\
\mathcal{P}_s\,U \arrow[r, tail, "\iota"'] & U \arrow[r, "1\times a"'] & U \times U.
\end{tikzcd}$$

Theorem 2.2.8 (Soundness of BCST) BCST *is sound with respect to models in basic categories of classes.*

Proof The Membership axiom is trivial and all of the other axioms follow from the previous results contained in this subsection and the fact that their typed analogues are valid in the internal languages of categories with basic class structure (see 2.1.6, 2.1.27 and 2.1.29). $\square$

In order to prove the soundness of **CST** we will need a way to eliminate the defined terms such as $\mathrm{func}(f, a, b)$, $\{a, b\}$, et cetera which occur in Exponentiation. We now prove several lemmas which will provide us with the requisite methods (which will also be needed to prove the soundness of **BCST$^+$**).

Lemma 2.2.9 (Eliminating Defined Terms) *In any basic category of classes $\mathcal{C}$:*

1. *Given $a : 1 \longrightarrow U$ (such an a will usually occur for us as the interpretation of a constant) we have that $[\![\{a\}]\!] = \iota \circ \{-\}_U \circ a$.*
2. *If $a, b : 1 \longrightarrow U$, then $[\![\{a, b\}]\!] = \iota \circ \{-, -\}_U \circ \langle a, b\rangle$.*
3. *There exists a map $\mathrm{pair} : U \times U \longrightarrow U$ such that, given a, b as above, $\mathrm{pair}(\langle a, b\rangle) = [\![\langle a, b\rangle]\!]$ (in the latter the $\langle a, b\rangle$ is the set theoretic, defined, ordered pair).*

Proof (1) The proof is using Extensionality. I.e., where $b := \iota \circ \{-\}_U \circ a$ we will show that:

$$[\![x | x \in \{a\}]\!] = [\![x | x \in b]\!].$$

By soundness $[\![x|x \in \{a\}]\!] = [\![x|x = a]\!]$. But $[\![x|x \in b]\!] = [\![x|x = a]\!]$ by the following diagram:

$$
\begin{tikzcd}
E' \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_U \arrow[r, tail, "1"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon_U \arrow[d, tail, "\in"] \\
U \arrow[r, "1\times a"] \arrow[d, tail, "1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U \arrow[r, tail, "1\times\{-\}_U"] \arrow[d, tail, "1"] \arrow[dr, phantom, "\lrcorner", very near start] & U \times \mathcal{P}_s\, U \arrow[r, tail, "1"] \arrow[d, tail, "1"] \arrow[dr, phantom, "\lrcorner", very near start] & U \times \mathcal{P}_s\, U \arrow[d, tail, "1\times\iota"] \\
U \arrow[r, "1\times a"'] & U \times U \arrow[r, tail, "1\times\{-\}_U"'] & U \times \mathcal{P}_s\, U \arrow[r, tail, "1\times\iota"'] & U \times U.
\end{tikzcd}
$$

Next, the proof of (2) is similar to that of (1). In particular, by a similar application of soundness and a diagram chase one sees that $[\![\{a, b\}]\!] = \iota \circ \{-,-\}_U \circ \langle a, b\rangle$.

Finally, for (3) define pair $: U \times U \longrightarrow U$ as follows:

$$\text{pair} \;:=\; \iota \circ \{-,-\}_U \circ (\iota \times \iota) \circ (\{-\}_U \times \{-,-\}_U) \circ \sim \circ (\delta \times 1),$$

where $\sim$ is the 'twist' isomorphism $(U \times U) \times U \longrightarrow U \times (U \times U)$. At this point the reader should note that this definition depends on the encoding of ordered pairs in set theory as Kurotowski ordered pairs. However, similar maps exist for the other implementations of ordered pairs in the set theory.$\Box$

Lemma 2.2.10 *Let $a, b : 1 \rightrightarrows U$ each factoring through ι via maps $\bar{a}, \bar{b} : 1 \rightrightarrows \mathcal{P}_s\, U$ be given, then $[\![a \times b]\!] = \iota \circ (\text{pair})_! \circ \times_{U,U} \circ \langle \bar{a}, \bar{b}\rangle$.*

Proof First, let $i : \alpha \rightarrowtail U$ and $j : \beta \rightarrowtail U$ be the small subobjects classified by $\bar{a}$ and $\bar{b}$, respectively. Define $\overline{a \times b} := (\text{pair})_! \circ \times_{U,U} \circ \langle \bar{a}, \bar{b}\rangle$ and $k := \text{pair} \circ (i \times j)$. It is straightforward to verify that:

$$[\![x : U | x \,\epsilon_U\, \overline{a \times b}]\!] \;=\; [\![x : U | x \,\epsilon_U\, k]\!].$$

The required result then follows by soundness. $\Box$

Corollary 2.2.11 *If $a, b : 1 \rightrightarrows U$ factor through ι via $\bar{a}$ and $\bar{b}$, respectively, and $i : \alpha \rightarrowtail U$ and $j : \beta \rightarrowtail U$ are the subobjects classified by $\bar{a}$ and $\bar{b}$, respectively, then:*

$$[\![x|\mathsf{S}(x) \wedge (\forall y)(y \in x \Rightarrow y \in a \times b)]\!] \;=\; \mathcal{P}_s\,(\alpha \times \beta),$$

where $\mathcal{P}_s\,(\alpha\times\beta)$ is regarded as a subobject of U via the map $\iota \circ (\text{pair})_! \circ (i \times j)_!$.

Proof By lemmas 2.2.10 and 2.2.7. $\Box$

Corollary 2.2.12 *Given the same assumptions as in the foregoing corollary:*

$$\beta^\alpha \quad = \quad [\![z|z \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.\langle x, y\rangle \in z]\!],$$

where β^α is regarded as a subobject of U via the map $\iota \circ (\mathrm{pair})_! \circ (i \times j)_! \circ l$ and $l : \beta^\alpha \rightarrowtail \mathcal{P}_s\,(\alpha \times \beta)$.

PROOF Using the foregoing corollary as well as the internal language. □

Theorem 2.2.13 (Soundness of CST) CST *is sound with respect to models in predicative categories of classes.*

PROOF All that remains to be checked is that $(\mathcal{C}, U) \models$ Exponentiation where $\mathcal{C}$ is a predicative category of classes.

We will first show that for any $a, b : 1 \longrightarrow U$ factoring through $\iota : \mathcal{P}_s\,U \rightarrowtail U$ via maps $\bar{a}$ and $\bar{b}$, respectively, the subobject $[\![z|\,\mathrm{func}(z, a, b)]\!]$ is small. By definition there exist small subobjects α and β of U corresponding to $\bar{a}$ and $\bar{b}$.

Since these subobjects are small so is the exponential β^α by Proposition 2.1.22 and, by the foregoing lemma and Proposition 2.2.6, it follows that:

$$\beta^\alpha \quad = \quad [\![z|z \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.\langle x, y\rangle \in z]\!]. \tag{2.4}$$

The general result follows from the fact that, given $a, b : Z \rightrightarrows U$ such that $Z \Vdash \mathsf{S}(a) \wedge \mathsf{S}(b)$, we may pull the problem back to $\mathcal{C}/Z$ along $\Delta_{!_Z}$. □

2.2.3 Completeness

We will now prove the completeness of **BCST** with respect to models $(\mathcal{C}, U)$ in basic categories of classes and the completeness of **CST** with respect to models in predicative categories of classes. The proof of the completeness theorem for **BCST** is (essentially) already to be found in [6] and the proof which we now give is along the same lines (the approach to such completeness theorems followed both here and in [6] has its origins in [34]). For more information on the approach to completeness theorems adopted herein the reader is directed to consult section D1.4 of [20].

We begin by defining the *syntactic categories* (or 'classifying categories') $\mathcal{C}_{\mathbf{BCST}}$ and $\mathcal{C}_{\mathbf{CST}}$ for **BCST** and **CST**, respectively.

Definition 2.2.14 The *syntactic category* $\mathcal{C}_{\mathbf{BCST}}$ *of* **BCST** is given by the following data:

Objects: α-equivalence classes of formulae in context, which are written $\{\vec{x}|\varphi\}$.

Arrows: An arrow $\{\vec{x}|\varphi\} \longrightarrow \{\vec{y}|\psi\}$ is a provable equivalence class of formulae θ with $\mathrm{FV}(\theta) \subseteq \{\vec{x}, \vec{y}\}$ in context which are provably functional. That is, (the universal closures of) the following are provable in **BCST**:

$$\begin{array}{rcl} \theta & \vdash_{\vec{x},\vec{y}} & \varphi \wedge \psi, \\ \theta \wedge \theta[\vec{z}/\vec{y}] & \vdash_{\vec{x},\vec{y},\vec{z}} & \vec{y} = \vec{z}, \text{ and} \\ \varphi & \vdash_{\vec{x}} & \exists \vec{y}.\theta. \end{array}$$

Maps are written as $[\theta]$.

Identities: The identity map $1_{\{\vec{x}|\varphi\}}$ of an object $\{\vec{x}|\varphi\}$ is defined to be the equivalence class $[\vec{x} = \vec{x}' \wedge \varphi(\vec{x})]$.

Composition: The composite $[\theta(\vec{y}, \vec{z})] \circ [\xi(\vec{x}, \vec{y})]$ is defined to be $[\exists \vec{y}.\xi(\vec{x}, \vec{y}) \wedge \theta(\vec{y}, \vec{z})]$.

The syntactic category $\mathcal{C}_{\mathbf{CST}}$ for **CST** is defined similarly (with the obvious modifications). In the following we will often abbreviate $\{\vec{x}|\varphi\}$ to $\{\varphi\}$ where no confusion should arise.

Lemma 2.2.15 $\mathcal{C}_{\mathbf{BCST}}$ *is a positive Heyting category.*

Proof Although this proof is standard (cf. D1.4.10 of [20] and [6]) we will give a brief sketch in order to recall the definitions of various categorical structure in syntactic categories. Given objects $\{\vec{x}|\varphi\}$ and $\{\vec{y}|\psi\}$ we define:

Terminal Object: The terminal object of $\mathcal{C}_{\mathbf{BCST}}$ is $\{\cdot|\top\}$.

Products: $\{\varphi\} \times \{\psi\} := \{\vec{x}, \vec{y}|\varphi \wedge \psi\}$ with the projections:

$$\{\vec{w}|\varphi[\vec{w}/\vec{x}]\} \xleftarrow{[\varphi \wedge \psi \wedge (\vec{x} = \vec{w})]} \{\vec{x}, \vec{y}|\varphi \wedge \psi\} \xrightarrow{[\varphi \wedge \psi \wedge (\vec{y} = \vec{z})]} \{\vec{z}|\psi[\vec{z}/\vec{y}]\}.$$

Pullbacks: Given a third object $\{\vec{w}|\xi\}$ and maps $[\theta] : \{\varphi\} \longrightarrow \{\xi\}$ and $[\theta'] : \{\psi\} \longrightarrow \{\xi\}$ the pullback of $[\theta]$ along $[\theta']$ is defined to be $\{\vec{x}, \vec{y}|\exists \vec{w}.\theta(\vec{z}, \vec{w}) \wedge \theta'(\vec{v}, \vec{w})\}$ where $\vec{z}$ and $\vec{v}$ are those variables in $\vec{x}$ and $\vec{y}$, respectively, occurring as free variables of θ and θ'.

Equalizers: Given maps $[\theta], [\theta'] : \{\varphi\} \rightrightarrows \{\psi\}$ the equalizer is the object $\{\vec{z}|\exists \vec{y}.\theta[\vec{z}/\vec{x}] \wedge \theta'[\vec{z}/\vec{x}]\}$ and the map $[\exists \vec{y}.\theta \wedge \theta' \wedge (\vec{z} = \vec{x})]$.

Initial Object: Use $\{\cdot|\bot\}$.

Coproducts: Let $\{\varphi\}+\{\psi\} := \{w,\vec{z}|(w=0\wedge\varphi(\vec{z}))\vee(w=1\wedge\psi(\vec{z}))\}$ with the obvious injections.

Direct Images: The image of a map $[\theta] : \{\varphi\} \longrightarrow \{\psi\}$ is the object $\{\vec{y}|\exists\vec{x}.\theta\}$.

Dual Images: Given a subobject $\{\chi\} \rightarrowtail \{\varphi\}$ and a map $[\theta]$ as above the dual image is given by $\forall_{[\theta]}(\{\chi\}) := \{\vec{y}|\psi \wedge (\forall\vec{x})(\theta \Rightarrow \chi)\}$.

$\square$

Remark 2.2.16 The reader may have noted that, e.g., the construction of equalizers requires arbitrary choices of elements from the equivalence relations $[\theta]$ and $[\theta']$ such that different choices will yield distinct but isomorphic equalizers. For more information about this issue and how it may be resolved see remark D1.4.3 of [20].

As in [6] small maps are defined to be those maps $[\theta] : \{\vec{x}|\varphi\} \longrightarrow \{\vec{y}|\psi\}$ such that:

$$\psi \quad \vdash_{\vec{y}} \quad \text{Ƨ}\vec{x}.\theta(\vec{x},\vec{y})$$

is provable in **BCST**.

Lemma 2.2.17 $\mathcal{C}_{\mathbf{BCST}}$ *satisfies axioms **(S1)**-**(S5)**.*

Proof The proof of this fact is straightforward using the foregoing descriptions of the categorical structure of $\mathcal{C}_{\mathbf{BCST}}$. For more details see [6]. $\square$

Lemma 2.2.18 $\mathcal{C}_{\mathbf{BCST}}$ *satisfies axiom **(P1)**.*

Proof Given an object $\{\vec{x}|\varphi\}$ of **BCST** the powerobject is given by the following definition:

$$\mathcal{P}_s\left(\{\varphi\}\right) \quad := \quad \{\vec{y}|\mathsf{S}(\vec{y}) \wedge \forall\vec{x} \in \vec{y}.\varphi\}. \qquad \square$$

Then, given a small relation $\{\vec{x},\vec{y}|\rho\}$ on $\{\vec{x}|\varphi\} \times \{\vec{y}|\psi\}$ the classifying map is as follows (cf. [6]):

$$[\mathsf{S}(\vec{z}) \wedge \forall\vec{x}.\vec{x} \in \vec{z} \Leftrightarrow \rho(\vec{x},\vec{y})].$$

Lemma 2.2.19 $\mathcal{C}_{\mathbf{BCST}}$ *satisfies axiom **(U)**.*

Proof The universal object U is defined as follows:

$$U \quad := \quad \{u|u = u\}.$$

Then for each $\{\vec{x}|\varphi\}$ there exists a canonical monomorphism $[\varphi(\vec{x}) \wedge \vec{x} = \vec{u}] : \{\vec{x}|\varphi\} \rightarrowtail \{\vec{u}|\vec{u} = \vec{u}\}$ (note that $U \cong \{\vec{u}|\vec{u} = \vec{u}\}$ where the vectors $\vec{u}$ are of any length). □

Definition 2.2.20 Given an object $\{\vec{x}|\varphi\}$ of $\mathcal{C}_{\mathbf{CST}}$ we define a new language $\mathcal{L}(\{\vec{x}|\varphi\}) := \mathcal{L} \cup \{\vec{c}\}$ where the length of the vector $\vec{c}$ is the same as that of $\vec{x}$. Also, let $\mathbf{CST}(\{\vec{x}|\varphi\})$ be the theory $\mathbf{CST}+\varphi(\vec{c})$ in the language $\mathcal{L}(\{\vec{x}|\varphi\})$. Finally, we define a new category $\mathcal{C}_{\mathbf{CST}}[\{\vec{x}|\varphi\}]$ as the syntactic category of the theory $\mathbf{CST}(\{\vec{x}|\varphi\})$.

Lemma 2.2.21 *$\mathcal{C}_{\mathbf{CST}}[\{\vec{x}|\varphi\}]$ is equivalent to the slice category $\mathcal{C}_{\mathbf{CST}}/\{\vec{x}|\varphi\}$.*

Proof The functor $\Phi : \mathcal{C}_{\mathbf{CST}}/\{\vec{x}|\varphi\} \longrightarrow \mathcal{C}_{\mathbf{CST}}[\{\vec{x}|\varphi\}]$ sends an object $[\theta] : \{\vec{y}|\psi\} \longrightarrow \{\vec{x}|\varphi\}$ of the slice category to $\{\vec{y}|\theta(\vec{y},\vec{c})\}$. The pseudo-inverse $\Psi : \mathcal{C}_{\mathbf{CST}}[\{\varphi\}] \longrightarrow \mathcal{C}_{\mathbf{CST}}/\{\vec{x}|\varphi\}$ sends an object $\{\vec{y}|\psi\}$ to:

$$[\vec{z}, \vec{x'}|\exists\vec{x}.\psi \wedge \varphi \wedge \vec{x} = \vec{x}'] : \{\vec{z}|\exists\vec{x}.\psi \wedge \varphi(\vec{x})\} \longrightarrow \{\varphi\},$$

where (up to α-equivalence) $\vec{z} = \vec{y}\backslash\vec{c}$ and $\vec{x}$ is of the same length as $\vec{x}$. □

Lemma 2.2.22 *For any object $\{\vec{x}|\varphi\}$ of $\mathcal{C}_{\mathbf{CST}}$, the category $\mathcal{C}_{\mathbf{CST}}[\{\vec{x}|\varphi\}]$ has basic class structure and has the additional property that the exponential of a small object by another small object is itself small.*

Proof We have already showed that basic class structure is preserved under slicing. So, let small objects $\{\vec{y}|\psi\}$ and $\{\vec{z}|\xi\}$ be given. Then both $\text{Ƨ}\vec{y}.\psi$ and $\text{Ƨ}\vec{z}.\xi$ are provable in $\mathbf{CST}(\{\vec{x}|\varphi\})$. By Exponentiation, $\{\vec{y}|\psi\}^{\{\vec{z}|\xi\}}$ is also small. □

Lemma 2.2.23 *$\mathcal{C}_{\mathbf{CST}}$ satisfies axiom **(E)**.*

Proof We consider first the case where Π is taken along one of the canonical maps into the terminal object. Let a small object $\{\vec{x}|\varphi\}$ and a small map $[\alpha] : \vec{y}|\psi\} \longrightarrow \{\varphi\}$ be given. Then we conclude that $\{\vec{y}|\psi\}$ is also a small object so that both $\text{Ƨ}\vec{x}.\varphi(\vec{x})$ and $\text{Ƨ}\vec{y}.\psi(\vec{y})$ are provable. But then, by Exponentiation, $\{\psi\}^{\{\varphi\}}$ is a small object. So we have the following situation:

$$\begin{tikzcd}
\Pi_{!_{\{\varphi\}}}\alpha \arrow[r] \arrow[dd] & \{\psi\}^{\{\varphi\}} \arrow[dd, "\alpha^{\{\varphi\}}"] \arrow[dr] & \\
 & & 1 \\
1 \arrow[r] & \{\varphi\}^{\{\varphi\}} \arrow[ur] &
\end{tikzcd}$$

with the canonical map from $\{\psi\}^{\{\varphi\}}$ small. By the small map axioms it follows that $\Pi_{!_{\{\varphi\}}}[\alpha]$ is a small object, as required.

The general case, where we are considering $\Pi_{[\theta]}$ for an arbitrary small map $[\theta]$ is a consequence of the foregoing lemma. $\square$

As usual we may now consider the canonical interpretation of **BCST** or **CST** in its syntactic category. Specifically, given a formula φ with free variables $\vec{x}$ the interpretation of φ is easily seen to be given by the following:

$$[\![\vec{x}|\varphi]\!] \quad = \quad \{\vec{x}|\varphi\},$$

in $\mathrm{Sub}_{\mathcal{C}_{\mathbf{BCST}}}(U^n)$ (this is prove by induction as usual). Then we have the lemma:

Lemma 2.2.24 *A formula φ is valid in $(\mathcal{C}_{\mathbf{BCST}}, U)$ if and only if it is provable in **BCST** (resp. **CST**).*

Proof As usual (cf. D1.4.5 of [20]). $\square$

Theorem 2.2.25 (Completeness) *For any formula φ of $\mathcal{L}$, if $(\mathcal{C}, U) \models \varphi$ for all models $(\mathcal{C}, U)$ with $\mathcal{C}$ a category of classes, then* $\mathbf{BCST} \vdash \varphi$. *Similarly, if $(\mathcal{C}, U) \models \varphi$ for all models with $\mathcal{C}$ a predicative category of classes, then* $\mathbf{CST} \vdash \varphi$.

Proof Suppose φ is valid in all models $(\mathcal{C}, U)$. Then, in particular, it is valid in $(\mathcal{C}_{\mathbf{BCST}}, U)$ (resp. $\mathcal{C}_{\mathbf{CST}}$). But then it is provable in **BCST** (resp. **CST**). $\square$

We obtain analogous theorems for the theories $\mathbf{BCST}^+$ and $\mathbf{CST}^+$ if we restrict attention only to those basic categories of classes (resp. categories of classes) $\mathcal{C}$ such that there exists a natural number object in the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects and maps. Explicitly:

Theorem 2.2.26 *For any formula φ of $\mathcal{L}^+$, $(\mathcal{C}, U) \models \varphi$ for all models $(\mathcal{C}, U)$ with such that $\mathcal{C}$ is a basic category of classes and $\mathcal{S}_{\mathcal{C}}$ has a natural number object if and only if* $\mathbf{BCST}^+ \vdash \varphi$ *(and similarly for* $\mathbf{CST}^+$*).*

Proof Cf. [6]. $\square$

Chapter 3

The Ideal Completion of a Π-Pretopos

In this chapter we construct models of **CST** 'over' Π-pretopoi $\mathcal{R}$. Intuitively, the construction proceeds by freely adjoining certain 'nice' colimits to the base category $\mathcal{R}$. This is achieved explicitly by considering a certain subcategory of the category $\mathbf{Sh}(\mathcal{R})$ of sheaves over $\mathcal{R}$.

This approach is related to that developed in [6] and [28]. However, in [6] the 'ideals' are obtained by first requiring a system of inclusions on the base category, thereby supplying the order-theoretic structure required to consider order ideals. In this chapter, as in [28], [7] and [14], we will avoid employing such inclusions (although, to some extent, they are lurking implicitly in the ideal completion of this chapter).

3.0.4 Definitions and Basic Properties

In the theory of lattices and ordered sets one is often concerned with those lattices (or semi-lattices) which possess directed joins. The category theoretic analogue of directedness is *filteredness* and is equally useful.

Definition 3.0.27 A category $\mathcal{C}$ is *filtered* provided that it is non-empty and:

1. For any objects A, B of $\mathcal{C}$ there exists an object D and maps:
$$A \longrightarrow D \longleftarrow B.$$
2. For any two arrows $f, g : A \rightrightarrows B$ there exists a D and an arrow $h : B \longrightarrow D$ such that $h \circ f = h \circ g$.

For some useful facts about filtered categories and filtered colimits the reader is referred to section A.0.13 of the appendix. Given a category $\mathcal{C}$, the inductive completion $\mathbf{Ind}(\mathcal{C})$ studied by Grothendieck et al [5] for its uses in algebraic geometry is the subcategory of $\widehat{\mathcal{C}}$ consisting of filtered colimits of representables. We will be concerned only with those Ind-objects which are *ideals*, in the following sense:

Definition 3.0.28 A diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$ is an *ideal diagram* in $\mathcal{C}$ provided that $\mathcal{I}$ is a small filtered category such that for every map $\alpha : i \longrightarrow j$ in $\mathcal{I}$ the map $D(\alpha)$ is a monic. An *ideal* I on a category $\mathcal{C}$ is an object of $\widehat{\mathcal{C}}$ which is (up to isomorphism) a colimit of an ideal diagram of representables.

Using this definition, the *ideal completion* $\mathbf{Idl}(\mathcal{C})$ of a category $\mathcal{C}$ is the full subcategory of $\widehat{\mathcal{C}}$ consisting of ideals. Indeed, if $\mathcal{C}$ is a pretopos then since every ideal is a sheaf for the coherent coverage (cf. [7]), $\mathbf{Idl}(\mathcal{C})$ is also a subcategory of $\mathbf{Sh}(\mathcal{C})$.

In $\widehat{\mathcal{C}}$ the representable functors have many nice properties. One such property is that they satisfy a sort of compactness (or presentability) condition analogous to the definition of compact elements of a lattice (cf. the definition of finitely presentable in the appendix):

Proposition 3.0.29 (Representable Compactness) *In $\widehat{\mathcal{C}}$, where X is a colimit $\varinjlim_i yD_i$ of representables, any map $f : yC \longrightarrow X$ factors through at least one of the canonical maps $l_i : yD_i \longrightarrow X$.*

Proof Let X, yC and f be given as in the statement of the theorem and let $P := \coprod_i yD_i$ be the coproduct of the yD_i. Then, by the construction of $\varinjlim_i yD_i$ as a coequalizer of coproducts, there is a canonical map $\xi : P \longrightarrow X$ such that each $l_j : yD_j \longrightarrow X$ factors as $l_j = \xi \circ \iota_j$, where $\iota_j : yD_j \longrightarrow P$ is the coproduct inclusion. It is easily verified that ξ is a cover.

Since representables are projective it follows that there is a map $\zeta : yC \longrightarrow P$ such that $\xi \circ \zeta = f$. Therefore there exists a yD_j such that ζ factors through some $\iota_j : yD_j \longrightarrow P$ via some map η (cf. A.0.65 in the appendix). But then:

$$\begin{aligned} f &= \xi \circ \zeta \\ &= \xi \circ \iota_j \circ \eta \\ &= l_j \circ \eta, \end{aligned}$$

as required. □

Of course, when X is an ideal any such factorization will occur also in $\mathbf{Idl}(\mathcal{R})$ since $\mathbf{Idl}(\mathcal{R})$ is a full subcategory of $\widehat{\mathcal{R}}$.

Definition 3.0.30 Where $\mathcal{C}$ and $\mathcal{D}$ are categories with colimits of ideal diagrams, a functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ is said to be *continuous* provided that it preserves colimits of ideal diagrams.

Proposition 3.0.31 *If $\mathcal{C}$ is a category with colimits of ideal diagrams and $\mathcal{R}$ is any category, then any functor $F : \mathcal{R} \longrightarrow \mathcal{C}$ which preserves monomorphisms extends to a functor $\bar{F} : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathcal{C}$ which is continuous and unique up to natural isomorphism. In this sense $\mathbf{Idl}(\mathcal{R})$ is the free completion of $\mathcal{R}$ with colimits of ideal diagrams:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, hook, "y"] \arrow[dr, "F"'] & \mathbf{Idl}(\mathcal{R}) \arrow[d, dotted, "\bar{F}"] \\ & \mathcal{C}. \end{tikzcd}$$

Proof Let $\bar{F}(\varinjlim_{i \in \mathcal{I}} yC_i) := \varinjlim_{i \in \mathcal{I}} F(C_i)$. Notice that the assumption that F preserves monomorphisms is necessary so that the colimit $\varinjlim_i F(C_i)$ exists in $\mathcal{C}$. $\square$

3.0.5 Class Structure in $\mathbf{Idl}(\mathcal{C})$

Let $\mathcal{C}$ be a pretopos. Throughout the remainder of this chapter we will occasionally be concerned with the category $\mathbf{Sh}(\mathcal{C})$ of sheaves on $\mathcal{C}$. In all such cases we assume that sheaves are taken with respect to the *coherent coverage* (also called the 'finite-epi' in pre-Johnstonian terminology). I.e., the (Grothendieck) coverage J such that, for every object D of $\mathcal{C}$, $J(D)$ the set of all finite families of maps $(f_i : E_i \longrightarrow D)_{i \in I}$ such that the induced map to D from the coproduct of the E_i is a cover (cf. A.2.1.11 of [20]):

$$[f_i] : \coprod_i E_i \longrightarrow\!\!\triangleright D.$$

Sheaves with respect to the coherent coverage have a particularly nice description (the proof of which is a straightforward application of the definitions):

Proposition 3.0.32 $F : \mathcal{C}^{op} \longrightarrow \mathbf{Sets}$ *is a sheaf for the coherent coverage if and only if F sends finite coproducts to finite products and covers to monomorphisms.*

In explaining the basic class structure on $\mathbf{Idl}(\mathcal{C})$ we make use of the intuition that the representables should be the small objects and that the small maps should be those with small fibers. This intuition is made explicit in the following definition:

Definition 3.0.33 A map $f : X \longrightarrow Y$ in $\mathbf{Sh}(\mathcal{C})$ is *small* provided that it pulls representables back to representables. I.e., f is small provided that, for every $yC \longrightarrow Y$ the object P in the following pullback diagram is a representable:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & yC \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

As such, an object is *small* if and only if it is a representable since the terminal object of $\mathbf{Idl}(\mathcal{C})$ is $y1$.

Definition 3.0.34 A sheaf F is *separated* if and only if its diagonal $\Delta : F \rightarrowtail F \times F$ is a small map. Note though that being a separated sheaf is not the same as being a separated presheaf (all sheaves are trivially separated presheaves).

Using this definition of small maps between sheaves we are able to employ a characterization, which was proposed by André Joyal, of the ideals as precisely the separated sheaves. This is stated explicitly in the following theorem.

Theorem 3.0.35 (The Joyal Condition) *Let $\mathcal{C}$ be a pretopos, then, for any sheaf F in $\mathbf{Sh}(\mathcal{C})$, the following are equivalent:*

1. *F is an ideal.*
2. *F is separated.*
3. *For all arrows $f : yC \longrightarrow F$ with representable domain, the image of f is representable; i.e., $f : yC \longrightarrow\!\!\!\!\!\triangleright\, yD \rightarrowtail F$ for some yD.*

PROOF See [7]. $\square$

Using the Joyal condition one may easily show that $\mathbf{Idl}(\mathcal{C})$ has several nice properties.

Theorem 3.0.36 *If $\mathcal{C}$ is a pretopos, then:*

1. *$\mathbf{Idl}(\mathcal{C})$ is a positive Heyting category.*
2. *All of the positive Heyting structure of $\mathbf{Idl}(\mathcal{C})$ may be computed in $\mathbf{Sh}(\mathcal{C})$.*

3. *The (restricted) Yoneda embedding $y : \mathcal{C} \longrightarrow \mathbf{Idl}(\mathcal{C})$ preserves the pretopos structure, all limits existing in $\mathcal{C}$ and, moreover, if $\mathcal{C}$ is Heyting, then it is a Heyting functor.*

Proof See [7]. $\square$

Next we show that the small map axioms from 2.1.1 are satisfied in $\mathbf{Idl}(\mathcal{C})$:

Proposition 3.0.37 *Let $\mathcal{C}$ be a pretopos, then $\mathbf{Idl}(\mathcal{C})$ satisfies axioms **(S1)**-**(S5)**.*

Proof **(S1)** and **(S2)** are easy. **(S3)** is by the Joyal Condition.

For **(S4)** suppose we have:

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, "f\circ e"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

with e a cover and $f \circ e$ small. Let $i : yC \longrightarrow Z$ be given and consider the diagram:

$$\begin{tikzcd} yC' \arrow[r, "e'"] \arrow[d] & P \arrow[r, "f'"] \arrow[d] & yC \arrow[d] \\ X \arrow[r, "e"'] & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

where both squares are pullbacks (as is the outer rectangle). Then it follows that P is representable.

For **(S5)** notice that the pullback of $yC \longrightarrow Z$ along $[f, g] : X + Y \longrightarrow Z$ is the coproduct $f^*(yC) + g^*(yC)$ which is representable since both $f^*(yC)$ and $g^*(yC)$ are representable. $\square$

We will strengthen this result by showing that, where $\mathcal{R}$ is a Heyting pretopos, the category $\mathbf{Idl}(\mathcal{R})$ is a category with basic class structure. In order to motivate the definition of the (predicative) powerobjects $\mathcal{P}_s(X)$ in $\mathbf{Idl}(\mathcal{R})$ suppose that the indexing category is a topos $\mathcal{E}$ and consider a provisional definition of the powerobject of an object yC in $\widehat{\mathcal{E}}$ as follows:

$$\mathcal{P}_s(yC) \;:=\; y(\mathcal{P}(C)),$$

where $\mathcal{P}(C)$ is the usual (topos) powerobject of C in $\mathcal{E}$. Then we have $\mathcal{P}_s(yC) \cong y(\Omega^C)$ and at any object E in $\mathcal{E}$:

$$\begin{aligned} \mathcal{P}_s(yC)(E) &\cong y(\Omega^C)(E) \\ &= \mathrm{Hom}_{\mathcal{E}}(E, \Omega^C) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(E \times C, \Omega) \\ &\cong \mathrm{Sub}_{\mathcal{E}}(E \times C). \end{aligned}$$

Dropping both the assumption that the indexing category is a topos and that we are working in presheaves, we therefore adopt the following provisional definition of the small powerobject of yC in $\mathbf{Idl}(\mathcal{R})$:

$$\mathcal{P}_s\,(yC) \quad := \quad \mathrm{Sub}_{\mathcal{R}}(- \times C).$$

We then extend $\mathcal{P}_s\,(-)$ continuously to ideals $X = \varinjlim_i yC_i$ by:

$$\mathcal{P}_s\,(X) \quad := \quad \varinjlim_i \mathcal{P}_s\,(yC_i).$$

We will show that this definition of $\mathcal{P}_s\,(X)$ is justified by first showing that there is a functor $\mathrm{Sub}^r_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathbf{Idl}(\mathcal{R})$ which takes C to $\mathrm{Sub}_{\mathcal{R}}(- \times C)$ and which preserves monomorphisms. Then it will be possible to apply Proposition 3.0.31 to arrive at an extension $\mathcal{P}_s : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathbf{Idl}(\mathcal{R})$ which will be seen to be a powerobject functor in the sense of satisfying **(P1)**.

Similarly, we will define the membership relation $\epsilon_X \rightarrowtail X \times \mathcal{P}_s\,X$ as the restriction of the sheaf (hence also of the presheaf) membership relation to $\mathcal{P}_s\,X$. Explicitly, for a representable yC, an ideal $X := \varinjlim_i yC_i$ and an object D of the base category:

$$\begin{aligned}\epsilon_{yC}\,(D) \quad &:= \quad \{\langle f, S\rangle \in yC(D) \times \mathcal{P}_s\,(yC)(D) | \Gamma(f) \leq S\}, \text{ and}\\ \epsilon_X \quad &:= \quad \varinjlim_i \epsilon_{yC_i}\,.\end{aligned}$$

3.0.6 $\mathcal{P}_s$ is an Ideal

Lemma 3.0.38 *If $\mathcal{R}$ is a pretopos and C is an object of $\mathcal{R}$, then the purported powerobject presheaf $\mathcal{P}_s\,(yC) := \mathrm{Sub}_{\mathcal{R}}(- \times C)$ is a sheaf.*

Proof Notice that $\mathcal{P}_s\,(yC)(0) \cong \{*\}$ and, since coproducts in $\mathcal{R}$ are stable, $\mathcal{P}_s\,(yC)(A+B) \cong \mathcal{P}_s\,(yC)(A) \times \mathcal{P}_s\,(yC)(B)$. Suppose $f : A \longrightarrow B$ is a cover and let $h, k : Z \rightrightarrows \mathrm{Sub}_{\mathcal{R}}(B \times C)$ be given such that $\mathrm{Sub}_{\mathcal{R}}(f \times C) \circ h = \mathrm{Sub}_{\mathcal{R}}(f \times C) \circ k$. Then, for any $z \in Z$, $h(z), k(z) \in \mathrm{Sub}_{\mathcal{R}}(B \times C)$ and the pullback P of $h(z)$ along $f \times 1_C$ is also the pullback of $k(z)$ along $f \times 1_C$. But covers are preserved under pullback in $\mathcal{R}$ so that $h(z) = k(z)$ by the uniqueness of image factorizations. $\square$

Proposition 3.0.39 *If $\mathcal{R}$ is a Heyting pretopos and C is an object of $\mathcal{R}$, then the purported small powerobject $\mathcal{P}_s\,(yC)$ is an ideal.*

Proof Since $\mathcal{R}$ is effective it suffices to show that $\mathcal{P}_s\,(yC)$ is separated. To that end let $yD \longrightarrow \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC)$ be given and consider the following diagram:

$$\begin{tikzcd} & yD \arrow[d, "i"'] & \\ \mathcal{P}_s\,(yC) \arrow[r, tail, "\Delta"] & \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathcal{P}_s\,(yC) \end{tikzcd}$$

We will show that the equalizer of $\pi_1 \circ i$ and $\pi_2 \circ i$ is representable.

By the Yoneda lemma there are subobjects $\alpha, \beta \in \mathrm{Sub}_{\mathcal{R}}(D \times C)$ corresponding to $\pi_1 \circ i$ and $\pi_2 \circ i$, respectively. We want to find some H and $h : H \longrightarrow D$ in $\mathcal{R}$ such that the result of pulling α back along $h \times 1_C$ is the same as the result of pulling β back along $h \times 1_C$.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, tail] & \alpha \arrow[d, tail] \\ H \times C \arrow[r, "h\times 1_C"'] & D \times C \end{tikzcd}$$

Define subobjects G and H of $D \times C$ and D, respectively, as follows:

$$G \;:=\; [\![x, y|\alpha(x, y) \Leftrightarrow \beta(x, y)]\!],$$

and:

$$\begin{aligned} H \;&:=\; \forall_{\pi_D}(G) \\ &=\; [\![x|(\forall z)(\alpha(x, z) \Leftrightarrow \beta(x, z))]\!], \end{aligned}$$

where π_D is the projection $D \times C \longrightarrow D$. Finally, let $h : H \rightarrowtail D$.

To see that α and β both pull back to the same thing along $h \times 1_C$ notice that, where $\bar{\alpha}$ is the pullback of α along $h \times 1_C$ and $\bar{\beta}$ is similarly defined:

$$\begin{aligned} \bar{\alpha} \;&=\; [\![x, y|\alpha(x, y) \wedge (\forall z)(\alpha(x, z) \Leftrightarrow \beta(x, z))]\!] \\ &=\; [\![x, y|\beta(x, y) \wedge (\forall z)(\alpha(x, z) \Leftrightarrow \beta(x, z))]\!] \\ &=\; \bar{\beta}. \end{aligned}$$

So, $\pi_1 \circ i \circ yh = \pi_2 \circ i \circ yh$.

To see that yH is the equalizer suppose given some $\eta : X \longrightarrow yD$ with $\pi_1 \circ i \circ \eta = \pi_2 \circ i \circ \eta$. It suffices to assume that X is representable, so suppose $X \cong yE$. Consider the image factorization yE' of η:

$$\begin{tikzcd} yE \arrow[rr, "ye'"] \arrow[dr, "\eta"'] & & yE' \arrow[dl, tail, "ye"] \\ & yD & \end{tikzcd}$$

Notice that $\pi_1 \circ i \circ ye = \pi_2 \circ i \circ ye$ since ye' is a cover. That is, it suffices to consider monomorphisms m into yD with $\pi_1 \circ i \circ m = \pi_2 \circ i \circ m$. In particular, if α and β pull back to the same thing along $\eta \times 1_C$, then they already are the same when pulled back along $e \times 1_C$. Let ϵ denote the result of pulling α, β back along $e \times 1_C$.

We will now show that $E' \overset{e}{\rightarrowtail} D$ factors through $H \overset{h}{\rightarrowtail} D$ in $\mathcal{R}$. Note that:

$$\begin{array}{rl} E' \leq H \text{ in } \mathrm{Sub}_{\mathcal{R}}(D) & \text{iff} \;\; \pi_D^*(E') \leq G \text{ in } \mathrm{Sub}_{\mathcal{R}}(D \times C), \\ & \text{iff} \;\; \pi_D^*(E') \leq \alpha \Rightarrow \beta \text{ and } \leq \beta \Rightarrow \alpha, \\ & \text{iff} \;\; \alpha \wedge \pi_D^*(E') \leq \beta \text{ and } \beta \wedge \pi_D^*(E') \leq \alpha. \end{array}$$

But $\alpha \wedge \pi_D^*(E') = \epsilon = \beta \wedge \pi_D^*(E')$ is $\leq \alpha$ and $\leq \beta$ by definition.

So there exists a map $\bar{e} : E' \longrightarrow H$ such that $h \circ \bar{e} = e$. To show $\bar{e} \circ e'$ is the unique map from E making η factor through H suppose that $f : E \longrightarrow H$ and $h \circ f = \eta$. By the uniqueness of image factorizations it follows that $f = \bar{e} \circ e'$. □

Lemma 3.0.40 *The functor* $\mathrm{Sub}^r_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathbf{Idl}(\mathcal{R})$ *defined by* $\mathrm{Sub}^r_{\mathcal{R}}(C) := \mathrm{Sub}_{\mathcal{R}}(- \times C)$ *preserves monomorphisms.*

PROOF A map $f : D \longrightarrow C$ induces a natural transformation $\varphi : \mathrm{Sub}_{\mathcal{R}}(- \times D) \longrightarrow \mathrm{Sub}_{\mathcal{R}}(- \times C)$ given at an object E of $\mathcal{R}$ by:

$$\begin{array}{rcl} S \in \mathrm{Sub}_{\mathcal{R}}(E \times D) & \overset{\varphi_E}{\longmapsto} & S' \in \mathrm{Sub}_{\mathcal{R}}(E \times C), \textit{where} \\ S' & := & (1_E \times f)_!(S). \end{array}$$

As such, we define $\mathrm{Sub}^r_{\mathcal{R}}(f) := \varphi$. Notice that φ is natural since $\mathcal{R}$ satisfies the Beck-Chevalley condition.

If f is monic, then each component φ_E is monic and, by the Yoneda lemma, φ is monic (since the monomorphisms, like other limits, in $\mathbf{Idl}(\mathcal{R})$ agree with those in $\widehat{\mathcal{R}}$). □

Definition 3.0.41 For any object $X = \varinjlim_i yC_i$ of $\mathbf{Idl}(\mathcal{R})$, where $\mathcal{R}$ is a Heyting pretopos, we have by Proposition 3.0.31 and the foregoing lemma that there is a unique functor $\mathcal{P}_s : \mathbf{Idl}(\mathcal{R}) \longrightarrow \mathbf{Idl}(\mathcal{R})$ with:

$$\begin{array}{rcl} \mathcal{P}_s\,(X) & \cong & \mathcal{P}_s\,(\varinjlim\limits_i yC_i) \\ & \cong & \varinjlim\limits_i \mathrm{Sub}^r_{\mathcal{R}}(C_i) \\ & = & \varinjlim\limits_i \mathrm{Sub}_{\mathcal{R}}(- \times C_i). \end{array}$$

3.0.7 $\mathcal{P}_s(X)$ is a Powerobject

We will now show that the axiom **(P1)** holds in $\mathbf{Idl}(\mathcal{R})$ where $\mathcal{R}$ is a Heyting pretopos. It will be more efficient to break the proof into several steps. Also, notice that we write $\in_X$ for the membership relation in $\widehat{\mathcal{R}}$ and ϵ_X for the membership relation in $\mathbf{Idl}(\mathcal{R})$. Similarly, we write $\mathcal{P}X$ for the power object in $\widehat{\mathcal{R}}$ and $\mathcal{P}_s X$ for the small power object in $\mathbf{Idl}(\mathcal{R})$.

Lemma 3.0.42 *Given any small relation $R \overset{r}{\rightarrowtail} X \times Y$ in $\mathbf{Idl}(\mathcal{R})$ there exists a unique classifying map $\hat{r} : Y \longrightarrow \mathcal{P}_s X$.*

Proof First consider the case where $R \rightarrowtail yC \times yD$. Then in $\widehat{\mathcal{R}}$ both of the following squares (and the outer rectangle):

$$\begin{tikzcd}
\epsilon_{yC} \arrow[r, tail] \arrow[d, tail] & \in_{yC} \arrow[d, tail] \\
yC \times \mathcal{P}_s\, yC \arrow[r, tail, "1\times i"'] \arrow[d, tail] & yC \times \mathcal{P}yC \arrow[d, tail] \\
\mathcal{P}_s\, yC \arrow[r, tail, "i"'] & \mathcal{P}yC
\end{tikzcd}$$

are pullbacks where $\in_{yC}$ and $\mathcal{P}yC$ are the presheaf membership and powerobject relations and i is the inclusion of $\mathcal{P}_s\, yC$ into $\mathcal{P}yC$ ($\mathcal{P}_s\, yC$ is, by definition, a subfunctor of $\mathcal{P}_s\, yC$). Notice that R is representable since r is a small relation. In particular, $R = yE$ for some object E of $\mathcal{R}$ and $r = ye$. So, using the 'twist' isomorphism $\tilde{}: C \times D \cong D \times C$, we have a relation $\tilde{e} : E \rightarrowtail D \times C$. By the Yoneda lemma such an element corresponds to a map $\hat{r} : yD \longrightarrow \mathcal{P}_s\, yC$.

We will now show that the canonical classifying map $\rho : yD \longrightarrow \mathcal{P}yC$ in $\widehat{\mathcal{R}}$ factors through $\hat{r}$. I.e., we show that:

$$\begin{tikzcd}
yD \arrow[rr, "\hat{r}"] \arrow[dr, "\rho"'] & & \mathcal{P}_s\, yC \arrow[dl, tail, "i"] \\
& \mathcal{P}yC &
\end{tikzcd}$$

commutes. Notice that, by the two pullbacks lemma, this will suffice to show that $\hat{r}$ is a classifying map for R in $\mathbf{Idl}(\mathcal{R})$. By the proof of the Yoneda lemma the action of $\hat{r}$ on a given member f of $yD(F)$ is:

$$f \longmapsto \mathcal{P}_s\,(yC)(f)(\tilde{e}).$$

But, $\rho_F(f) = (yf \times 1_{yC})^*(y\tilde{e}) = i(\mathcal{P}_s\,(yC)(f)(\tilde{e}))$.

For uniqueness suppose that $q : yD \longrightarrow \mathcal{P}_s\, yC$ such that:

$$\begin{tikzcd} yE \arrow[r] \arrow[d, tail] & \epsilon_{yC} \arrow[d, tail] \\ yC \times yD \arrow[r, "1\times q"'] & yC \times \mathcal{P}_s\, yC \end{tikzcd}$$

is a pullback. Then, in $\widehat{\mathcal{R}}$, ye is the pullback of $\in_{yC}$ along $i \circ q$ and along $i \circ \hat{r} = \rho$. Since ρ is unique with this property it follows that $i \circ \hat{r} = i \circ q$ and, since i is monic, $q = \hat{r}$.

Now, for any ideal $X \cong \varinjlim_i yC_i$ and small relation $r : R \rightarrowtail X \times yD$, R must be representable since the projection:

$$R \rightarrowtail X \times yD \longrightarrow yD$$

is small. I.e., $R \cong yE$ for some E. By Representable Compactness 3.0.29 there exists then a factorization of r:

$$R \rightarrowtail yC_i \times yD \rightarrowtail X \times yD$$

for some i. Thus indeed $\mathrm{SRel}_X \cong \varinjlim_i \mathrm{SRel}_{yC_i}$. □

Lemma 3.0.43 *For any ideal X, $\epsilon_X \rightarrowtail X \times \mathcal{P}_s\, X$ is a small relation.*

Proof It clearly suffices to verify this for the case where X is a representable yC. Let $yD \rightarrowtail \mathcal{P}_s\, yC$ be given. Then there is a $r : R \rightarrowtail C \times D$ in $\mathcal{R}$ such that:

$$\begin{tikzcd} yR \arrow[r, "\pi\circ yr"] \arrow[d] & yD \arrow[d] \\ \epsilon_{yC} \arrow[r, "\pi_{yC}"'] & \mathcal{P}_s\, yC \end{tikzcd}$$

is a pullback, as required. □

Corollary 3.0.44 *Any relation $R \rightarrowtail X \times Y$ such that there exists a unique classifying map $\rho : Y \longrightarrow \mathcal{P}_s\, X$ is a small relation.*

Proof By **(S2)** and the fact that ϵ_X is a small relation. □

Putting the foregoing together we have the following proposition:

Proposition 3.0.45 *If $\mathcal{R}$ is a Heyting pretopos and $X \cong \varinjlim_i yC_i$ is an object of* $\mathbf{Idl}(\mathcal{R})$*, then $\mathcal{P}_s\,(X) = \varinjlim_i \mathrm{Sub}_{\mathcal{R}}(- \times C_i)$ is a small powerobject.*

Moreover, when combined with the fact that axioms **(S1)**-**(S5)** are satisfied in pretopoi we have shown the following:

Theorem 3.0.46 *If $\mathcal{R}$ is a Heyting pretopos, then* $\mathbf{Idl}(\mathcal{R})$ *is a category with basic class structure.*

Remark 3.0.47 It should be mentioned that Alex Simpson was the first to give a proof of Proposition 3.0.39 (the main difference between his proof and the one given in this paper is our use of the Joyal Condition).

3.0.8 Exponentiation

We now extend the results of the preceding subsection by showing that if $\mathcal{R}$ is a Π-pretopos, then $\mathbf{Idl}(\mathcal{R})$ satisfies **(E)**. First we need the following beautiful and useful fact:

Proposition 3.0.48 *If $\mathcal{C}$ is a small category and P is an object of* $\mathbf{Idl}(\mathcal{C})$*, then:*

$$\mathbf{Idl}(\mathcal{C})/P \quad \simeq \quad \mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P}).$$

Proof Here $\int_{\mathcal{C}} P$ denotes the category of elements of P as in [23], p. 41. It is well known (cf. exercise 8 on p. 157 of [23]) that $\widehat{\mathcal{C}}/P \simeq \widehat{\int_{\mathcal{C}} \mathcal{P}}$. In particular, there are two functors $R : \widehat{\mathcal{C}}/P \longrightarrow \widehat{\int_{\mathcal{C}} \mathcal{P}}$ and $L : \widehat{\int_{\mathcal{C}} \mathcal{P}} \longrightarrow \widehat{\mathcal{C}}/P$ such that $L \dashv R$ and the two maps are pseudo-inverse to one another. These functors are defined as follows:

- $R(\eta : F \longrightarrow P)$ is a functor given by:

  $$(c, C) \quad \longmapsto \quad \mathrm{Hom}_{\widehat{\mathcal{C}}/P}(\tilde{c} : yC \longrightarrow P, \eta : F \longrightarrow P),$$

  where $\tilde{c}$ is the map in $\widehat{\mathcal{C}}$ corresponding to the element $c \in P(C)$ by the Yoneda lemma.

- $L(F) := \varinjlim_{\mathcal{J}} \pi \circ i$ where $\mathcal{J} := \int_{\int_{\mathcal{C}} P}$, $i : \int_{\mathcal{C}} P \longrightarrow \widehat{\mathcal{C}}/P$ is the map taking an object (c, C) to the corresponding $\tilde{c} : yC \longrightarrow P$ as above and π is the projection from the category of elements.

We begin by showing that if $(\eta : F \longrightarrow P)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$, then $R(P)$ is isomorphic to an object of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$. Let η be given as mentioned. Then, since F is an ideal we have $F \cong \varinjlim_{\mathcal{I}} yD_i$ with maps $\mu_i : yD_i \longrightarrow F$ making up the cocone.

We define a functor $G : \mathcal{I} \longrightarrow \int_{\mathcal{C}} P$ such that $\varinjlim_{\mathcal{I}} yG_i \cong R(\eta)$ and $\varinjlim_{\mathcal{I}} yG_i$ is an object of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$. Let $G(i) := \widetilde{\eta \circ \mu_i}$ be the object corresponding via the Yoneda lemma to $\eta \circ \mu_i$. Given $f : i \longrightarrow j$ in $\mathcal{I}$, let $G(f) := D(f)$. G is easily seen to be functorial.

Next, let $T := \varinjlim_{\mathcal{I}} yG$. We now define an isomorphism $\varphi : R(\eta) \longrightarrow T$. If $f \in R(\eta)(c, C)$ then we have $f : yC \longrightarrow F$. But using Representable Compactness there exists an i together with a map $yl : yC \longrightarrow yD_i$ such that $\mu_i \circ yl = f$. Now, an element of $T(c, C)$ is an equivalence class $[g : C \longrightarrow D_i]_\sim$ where $g : C \longrightarrow D_i \sim g' : C \longrightarrow D_{i'}$ if and only if there exists an object i'' of $\mathcal{I}$ together with maps $h : i \longrightarrow i''$ and $h' : i' \longrightarrow i''$ such that $D(h) \circ g = D(h') \circ g'$. So we define $\varphi_{(c,C)}(f) := [l]_\sim$. The naturality of φ follows from the fact that $\mathcal{I}$ is filtered and the maps $\mu_k : yD_k \longrightarrow F$ are monic.

Now we need an inverse map $\psi : T \longrightarrow R(\eta)$. If $[g : C \longrightarrow D_i]_\sim \in T(c, C)$, then let $\psi_{(c,C)}([g]_\sim) := \mu_i \circ yg$. This definition is independent of choice of representative by the fact that $\mathcal{I}$ is filtered and naturality is straightforward.

Finally, it is straightforward to verify, using the fact that $\mathcal{I}$ is filtered, that $\varphi \circ \psi = 1_T$. Moreover, $\psi \circ \varphi = 1_{R(\eta)}$ is trivial. Furthermore, G is easily seen to preserve monomorphisms. As such, we have shown that $R(\eta)$ is an ideal in $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$.

Similarly, given an object F of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$ it follows from the fact that $\pi : \int_{\int_{\mathcal{C}} P} \longrightarrow \int_{\mathcal{C}} P$ and $i : \int_{\mathcal{C}} P \longrightarrow \widehat{\mathcal{C}}/P$ both preserve monomorphisms that $L(F)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$. $\square$

Proposition 3.0.49 *If $\mathcal{R}$ is a Π-pretopos, then $\mathbf{Idl}(\mathcal{R})$ satisfies **(E)**.*

PROOF First, we show that given $!_{yC} : yC \longrightarrow 1$ and $f : X \longrightarrow yC$ the map $\Pi_{!_{yC}}(f) \longrightarrow 1$ is small. By definition we have the following pullback square:

$$\begin{tikzcd}
\Pi_{!_{yC}}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{yC} \arrow[d, "f^{yC}"] \\
1 \arrow[r, "\widetilde{\pi_{yC}}"'] & yC^{yC}
\end{tikzcd}$$

where $\widetilde{\pi_{yC}}$ is the transpose of 1_{yC}. However, since f is small it follows that X is representable. I.e., $X \cong yE$ for some E. But since $\mathcal{R}$ is a Π-pretopos it follows that:

$$\begin{aligned} yC^{yC} &\cong y(C^C), \text{ and} \\ yE^{yC} &\cong y(E^C). \end{aligned}$$

Therefore f^{yC} is a small map and by **(S2)** so is the map $\pi_{!_{yC}}(f) \longrightarrow 1$.

The general case then follows from the foregoing proposition. $\square$

3.0.9 Universes and Infinity

If $\mathcal{R}$ is a Heyting pretopos, then we may construct universes U in $\mathbf{Idl}(\mathcal{R})$ as fixed points for endofunctors (cf. [28] or [7]). Given such a universe U in $\mathbf{Idl}(\mathcal{R})$, the full subcategory $\downarrow(U)$ of $\mathbf{Idl}(\mathcal{R})$ consisting of those objects X of $\mathbf{Idl}(\mathcal{R})$ which are subobjects of U is a category of classes with U as the universal object (cf. [34]). Putting this fact together with the results of the foregoing subsections we have our main theorem:

Theorem 3.0.50 *If $\mathcal{R}$ is a Heyting pretopos, then there exists a universe U in $\mathbf{Idl}(\mathcal{R})$ such that $\downarrow(U)$ is a basic category of classes in which $\mathcal{R}$ is equivalent to the category of small objects:*

$$\mathcal{R} \;\simeq\; \mathcal{S}_{\mathbf{Idl}(\mathcal{R})}.$$

Moreover, if $\mathcal{R}$ is a Π-pretopos, then $\downarrow(U)$ is a predicative category of classes.

Proof Let $A := \coprod_{C \in \mathcal{R}} yC$ and U a fixed point of $F(X) = A + \mathcal{P}_s\,(X)$:

$$U \cong A + \mathcal{P}_s\,(U).$$

Cf. [7]. $\square$

The reader should note that such intitial models will satisfy the Simple Sethood axiom and, as such, Δ_0-separation.

One may also be interested in providing ideal models of set theories satisfying Infinity$*$. Such models are obtained by adding a natural number object to the base category:

Corollary 3.0.51 *If $\mathcal{R}$ is a Heyting pretopos with a natural number object, then:*

$$(\downarrow(U), U) \models \text{Infinity*}.$$

However, there is an additional question as to how (and whether) one may obtain ideals models of stronger induction principles such as induction for classes as well as sets. This is an issue of considerable interest, but is one which we will not take up here.

3.0.10 Collection and Ideal Completeness

Ideal categories actually have some additional properties which are worth briefly mentioning.

Definition 3.0.52 A category with basic class structure is *saturated* if and only if it satisfies the following:

Small covers: Given a cover $e : E \longrightarrow\!\triangleright D$ such that D is a small object, there exists a small subobject $m : E' \rightarrowtail E$ such that $e \circ m$ is a cover.

Small generators: If every small subobject $m : D \rightarrowtail E$ factors through some $l : E' \rightarrowtail E$, then $E' \cong E$.

Saturated categories with (impredicative) class structure were considered by Awodey et al [6] in connection with their (inclusion) ideal models (see above) of the set theory **BIST**. We will employ them to prove an analogous result for predicative theories.

Lemma 3.0.53 *If $\mathcal{R}$ is a Heyting pretopos, then* $\mathbf{Idl}(\mathcal{R})$ *is saturated.*

Proof Since the representable functors generate $\mathbf{Idl}(\mathcal{R})$ the second condition is easily seen to hold. For small covers, let a cover $\varphi : X \longrightarrow\!\triangleright yC$ be given with $X \cong \varinjlim_i yD_i$. We employ the description of covers in $\mathbf{Idl}(\mathcal{R})$ as those maps which are locally epimorphic (cf. the appendix). Applying the criteria for locally epimorphic maps to φ, C, and 1_C we have that there exists a covering family $(f_k : E_k \longrightarrow C)_{k \in K}$ where K is finite such that, for all $k \in K$, $f_k \in \mathrm{im}(\varphi_{E_k})$. Let $E := \coprod_k E_k$ and $p : E \longrightarrow\!\triangleright C$.

So, $yp : yE \longrightarrow\!\triangleright yC$. Moreover, for all $k \in K$, $yf_k : yE_k \longrightarrow yC$ factors through φ via the map ξ_{f_k} corresponding to x_{f_k} under the Yoneda lemma. Since yE is a coproduct it follows that there exists a unique map $\mu : yE \longrightarrow X$ such that, for each coproduct injection $l_k : E_k \longrightarrow E$, $\mu \circ yl_k = \xi_{f_k}$. As such, $\varphi \circ \mu = yp$. By definition $X \cong \varinjlim_i yD_i$. Therefore, by representable compactness, μ factors through some $yh_i : yD_k \rightarrowtail X$ via a map $\bar{\mu} : yE \longrightarrow yD_i$. But then $\varphi \circ yh_i \circ \bar{\mu} = yp$ and $\varphi \circ yh_i$ is a cover, as required. □

Small covers implies that the ideal models will satisfy, in addition to the other axioms of **BCST** (or **CST** if $\mathcal{R}$ is a Π-pretopos), the Strong Collection axiom mentioned in section 1.2.1. First, another axiom in which we will be interested is the (categorical) *strong collection axiom* [21]:

Definition 3.0.54 A system $\mathcal{S}$ of small maps in a category $\mathcal{C}$ with pullbacks is said to have *collection* if and only if it satisfies the following axiom:

(S6) For any cover $p : D \longrightarrow\!\!\!\!\triangleright\, C$ and $f : C \longrightarrow A$ in $\mathcal{S}$ there exists a quasi-pullback square:

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "f'"'] & D \arrow[r, "p"] & C \arrow[d, "f"] \\ A' \arrow[rr, "h"'] & & A \end{tikzcd}$$

such that h is a cover and f' is in $\mathcal{S}$.

Proposition 3.0.55 (Typed Strong Collection) *If a category $\mathcal{C}$ with basic class structure satisfies* ***(S6)****, then:*

$\mathcal{C} \models \forall a : \mathcal{P}_s C.(\forall x \epsilon_C\, a.\exists y : D.\varphi(x, y) \Rightarrow \exists b : \mathcal{P}_s D.\, \mathrm{coll}(x \epsilon_C\, a, y \epsilon_D\, b, \varphi(x, y)),$

where φ is any relation on $C \times D$.

Proof A routine but fairly lengthy exercise in the internal language. $\square$

Proposition 3.0.56 *If $\mathcal{C}$ is a category with basic class structure that has small covers, then $\mathcal{C}$ satisfies* ***(S6)****.*

Proof By Theorem 2.1.15, it suffices to show consider the case where we are given a cover $e : E \longrightarrow\!\!\!\!\triangleright\, C$ with $!_C : C \longrightarrow 1$ a small map. By (2) there exists a small subobject $m : B \rightarrowtail E$ and the following is easily seen to be a quasi-pullback:

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d] & E \arrow[r, "e"] & C \arrow[d] \\ 1 \arrow[rr, "1"'] & & 1 \end{tikzcd}$$

Using the foregoing facts (and results of the previous sections) we have:

Proposition 3.0.57 *For any Heyting pretopos $\mathcal{R}$:*

$$(\downarrow(U), U) \models \text{Strong Collection}.$$

Awodey et al [6] have obtained, for the impredicative set theory $\mathbf{BIST}_C$ (**BIST** augmented with Strong Collection), a strengthening of the completeness result with respect to models in (impredicative) categories of classes with collection. This so-called ‘topos-completeness’ result may be replicated for impredicative set theories as well, and we will now summarize this construction. In the statement of the following theorems we will state everything for **BCST** exclusively. However, all of the results are obtained for **CST** in the exact same way.

Lemma 3.0.58 *$\boldsymbol{BCST}_C$ is complete with respect to models in basic categories of classes which have collection.*

PROOF Cf. [6]. $\square$

Lemma 3.0.59 *For any basic category of classes $\mathcal{C}$ with collection there exists a basic category of classes $\mathcal{C}'$ which has collection and is saturated and a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ which is conservative and logical.*

PROOF The proof from [6] does not use any impredicative features of the starting category $\mathcal{C}$. $\square$

Remark 3.0.60 The proof of 3.0.59 requires some form of the axiom of choice. However, it is not entirely clear to the author whether the full (nonconstructive) strength of choice is required or whether a similar proof may be given in a predicative meta-theory (as codified by, say, **CZF** augmented with the axiom of multiple choice).

Lemma 3.0.61 *If a basic category of classes $\mathcal{C}$ is saturated, then there is a conservative logical functor $\mathsf{d} : \mathcal{C} \longrightarrow \mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$, namely, the restricted Yoneda embedding:*

$$\mathsf{d}(C) \quad := \quad \mathrm{Hom}_{\mathcal{C}}(i-, C),$$

where $i : \mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ is the inclusion functor.

Assembling the pieces as in [6] give the following result which says that $\mathbf{BCST}_C$ is complete with respect to models over Heyting pretopoi.

Theorem 3.0.62 *For any formula φ of $\boldsymbol{BCST}_C$, if, for all Heyting pretopoi $\mathcal{R}$, $(\downarrow(U), U) \models \varphi$, then $\mathbf{BCST}_C \vdash \varphi$.*

PROOF Again, the proof contained in [6] requires no impredicative means. $\square$

Appendix A

Some Basic Category Theory

This appendix contains some of the basic results from category theory and categorical logic with which the reader should have some familiarity. However, this is by no means a comprehensive survey and the reader should refer to such textbooks as [22] or [11] for more detail.

A.0.11 Locally Cartesian Closed Categories and Pretopoi

Recall that a category is *cartesian* if it has all finite limits. Synonyms in the literature include *finitely complete* and *left-exact.* A category with finite products (including the empty product 1) is *cartesian closed* provided that for every pair of object A, B there exists an object B^A (the *exponential*) and a map $\epsilon_B : B^A \times A \longrightarrow B$ (the evaluation map) with the universal property that for every $f : C \times A \longrightarrow B$ there exists a unique map $\tilde{f} : C \longrightarrow B^A$ such that $\epsilon_B \circ (\tilde{f} \times 1) = f$. That is to say, there is an adjunction:

$$- \times A \quad \dashv \quad -^A,$$

where ϵ is the counit. A cartesian category $\mathcal{C}$ is *locally cartesian closed* if, for every object C of $\mathcal{C}$, the slice category $\mathcal{C}/C$ is cartesian closed.

Where $f : A \longrightarrow B$ is a map in a locally cartesian closed category $\mathcal{C}$ there exists a functor $\Delta_f : \mathcal{C}/B \longrightarrow \mathcal{C}/A$ which takes an object $f : C \longrightarrow B$ of $\mathcal{C}/B$ to its pullback along f (Δ_f is also written f^*). We state the following well-known result about locally cartesian closed categories for the record (cf., e.g., [19]):

Proposition A.0.63 *If $\mathcal{C}$ is a cartesian category, then $\mathcal{C}$ is locally cartesian closed if and only if, for every map $f : A \longrightarrow B$ in $\mathcal{C}$, $\Delta_f : \mathcal{C}/B \longrightarrow \mathcal{C}/A$ has a right adjoint Π_f.*

Notice that in a cartesian category $\mathcal{C}$ the functor Δ_f always has a left adjoint $\Sigma_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ which takes an object $g : C \longrightarrow A$ to its composite $f \circ g : C \longrightarrow B$ (Π_f is also written f_* and Σ_f is also written $f_!$). So, in a locally cartesian closed category Δ_f has both a left and a right adjoint.

A category $\mathcal{C}$ is *regular* if every map uniquely factors as a cover followed by a monomorphism and such factorizations are stable. $\mathcal{C}$ is *coherent* provided that it is regular, and each $\mathrm{Sub}_{\mathcal{C}}(C)$ has finite joins which are stable under pullback (we often call such joins *unions* and write them as $A \cup B$). Such a category is *positive* if it also possesses finite coproducts which are disjoint. I.e., such that for any A and B in $\mathcal{C}$ the following is a pullback:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[d, "\iota_A"] \\
B \arrow[r, "\iota_B"'] & A+B.
\end{tikzcd}$$

Definition A.0.64 In a positive coherent category $\mathcal{C}$ an object C is *indecomposable* provided that it is not a union of pairwise disjoint proper subobjects.

Notice that in a positive coherent category the union of disjoint subobjects is isomorphic to their coproduct. We make some use of the following lemma in the body of the thesis:

Lemma A.0.65 *In a positive coherent category $\mathcal{C}$ the following are equivalent:*

1. *C is indecomposible.*
2. *If $f : C \longrightarrow \coprod_{n=1}^{m} A_n$, then f factors through one of the $i_j : A_j \longrightarrow \coprod_{n=1}^{m} A_n$.*

Proof For (1) $\Rightarrow$ (2) notice that given the assumptions $\bigcup_{n=1}^{m} A'_n \cong C$, where $A'_j = f^*(A_j)$. Then since C is indecomposable f factors through some i_j (notice that the A'_j are all pairwise disjoint so the application of (1) here is legitimate). For (2) $\Rightarrow$ (1) suppose $C \cong \bigcup A_n$ with the A_j all pairwise disjoint. Then $\bigcup A_n \cong \coprod A_n$. Let $e : C \longrightarrow \coprod_{n=1}^{m} A_n$ be the isomorphism, then by assumption there exists an A_j such that e factors through i_j. Therefore $C \cong A_j$. $\square$

Definition A.0.66 A relation $(\partial_0, \partial_1) : R \rightarrowtail C \times C$ in a cartesian category $\mathcal{C}$ is an *equivalence relation* provided that it is:

Reflexive: There exists a map $r : C \longrightarrow R$ such that $\partial_0 \circ r = \partial_1 \circ r = 1_C$.

Symmetric: There exists a map $s : R \longrightarrow R$ such that $\partial_0 \circ s = \partial_1$ and $\partial_1 \circ s = \partial_0$.

Transitive: There exists a map $t : R \times_C R \longrightarrow R$ where:

$$\begin{tikzcd} R \times_C R \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\partial_0"] \\ R \arrow[r, "\partial_1"'] & C \end{tikzcd}$$

such that $\partial_0 \circ t = \partial_0 \circ p_1$ and $\partial_1 \circ t = \partial_1 \circ p_2$.

An equivalence relation is *effective* provided that it occurs as a kernel pair. A *pretopos* is a positive coherent category in which all equivalence relations are effective. In particular there are coequalizers for all equivalence relations.

Recall that a coherent category $\mathcal{C}$ is *Heyting* provided that, for each $f : C \longrightarrow D$ in $\mathcal{C}$, the pullback functor $f^* : \mathrm{Sub}_{\mathcal{C}}(D) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(C)$ has a right adjoint $\forall_f$.

A.0.12 Some Basic Sheaf Theory

We follow the discussion of sheaves in [23] and in chapter C2 of [20]. In particular, proofs of all propositions may be found in the former.

Definition A.0.67 A *coverage* on a category $\mathcal{C}$ is a function J which for each object C of $\mathcal{C}$ yields a collection $J(C)$ of families of maps $(f_i : U_i \longrightarrow C)_{i \in I}$ with codomain C satisfying:

(C) If $(f_i)_{i \in I}$ is in $J(C)$ and $g : D \longrightarrow C$, then there exists a family $(h_k : V_k \longrightarrow D)_{k \in K}$ in $J(D)$ such that each $g \circ h_k$ factors through some f_i.

A *site* is a category $\mathcal{C}$ together with a coverage J.

Given this general notion of coverage we may now define sheaves.

Definition A.0.68 Given a presheaf P in $\widehat{\mathcal{C}}$ and a family $(f_i : U_i \longrightarrow C)_{i \in I}$ of maps for some object C of $\mathcal{C}$, a family of elements $s_i \in P(U_i)$ is *compatible* (for P and $(f_i)_I$) if and only if, given any object D and maps $g : D \longrightarrow U_i$ and $h : D \longrightarrow U_j$, if the following diagram commutes:

$$\begin{tikzcd} & U_i \arrow[dr, "f_i"] & \\ D \arrow[ur, "g"] \arrow[dr, "h"'] & & C \\ & U_j \arrow[ur, "f_j"'] & \end{tikzcd} \qquad \text{in } \mathcal{C},$$

then $s_i \cdot g = s_j \cdot h$.

If $\mathcal{C}$ is a site with coverage J, then P is a *sheaf* (for J) if and only if for any object C of $\mathcal{C}$ and family of elements $s_i \in P(U_i)$ compatible for some $(f_i)_{i \in I} \in J(C)$, there exists a unique element $s \in P(C)$ (the *amalgamation*) such that $s \cdot f_i = s_i$ for each $i \in I$.

Recall that a sieve S on an object C of a category $\mathcal{C}$ is a collection of maps with codomain C which is a right ideal for composition. That is, if $f \in S$ and g is a map with $\operatorname{cod} g = \operatorname{dom} f$, then $f \circ g \in S$. Also recall that $h^*(S) := \{g | \operatorname{cod} g = D \text{ and } h \circ g \in S\}$. A coverage J is *sifted* if and only if $J(C)$ is a collection of sieves on C for each object C of $\mathcal{C}$. If J is sifted, then a matching family for $S \in J(C)$ and a presheaf P is an assignment:

$$D \xrightarrow{f} C \quad \in S \quad \longmapsto \quad x_f \in P(D)$$

such that $x_f \cdot g = x_{f \circ g}$ for all maps g in $\mathcal{C}$ such that $\operatorname{cod} g = D$.

Moreover, since a sieve on C 'is' just a subfunctor of yC it is clear from the definition that a matching family for S of elements of P is a natural transformation $\mu : S \longrightarrow P$ and every such natural transformation gives rise to a matching family. As such, it will often be convenient to write x_f for $\mu_D(f)$ where $\operatorname{dom} f = D$. Similarly, an *amalgamation* of a matching family $\mu : S \longrightarrow P$ is an element $x \in P(C)$ such that $x \cdot f = x_f$ for each $f \in S$.

Traditionally a certain class of coverages with nice closure properties have been of interest. We too will be concerned with such coverages and their sheaves.

Definition A.0.69 A *Grothendieck coverage* (cf. [23] or [20]) is a function assigning to each object C of a category $\mathcal{C}$ a collection $J(C)$ of sieves on C such that:

(M) For each object C of $\mathcal{C}$, the maximal sieve t_C on C is in $J(C)$.

(C) If $S \in J(C)$, then $h^*(S) \in J(D)$ for any $h : D \longrightarrow C$.

(L) If $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \longrightarrow C$ in S, then $R \in J(C)$.

We write $\mathbf{Sh}(\mathcal{C}, \mathrm{J})$ for the category of sheaves on $\mathcal{C}$ for a given Grothendieck coverage J. In general the particular Grothendieck topology in question will be obvious from context we will accordingly omit explicit mention of it.

A sieve S on C is *J-closed* if and only if, for any $f : D \longrightarrow C$, $f^*(S) \in J(D)$ implies that $f \in S$.

Proposition A.0.70 *For A' a subobject of a sheaf A in $\widehat{\mathcal{C}}$ the following are equivalent:*

1. *A' is a sheaf.*
2. *For any object C of $\mathcal{C}$ and $x \in A(C)$ the following sieve:*
$$B_x \quad := \quad \{g |\, \mathrm{cod}(g) = C \wedge A(g)(x) \in A'(\mathrm{dom}(g))\}$$
is J-closed.
3. *For any object C of $\mathcal{C}$, $x \in A(C)$, $S \in J(C)$ and $f \in S$, if $A(f)(x) \in A'(\mathrm{dom}(f))$, then $x \in A'(C)$.*

Epimorphisms in sheaves, although they do not agree with the 'pointwise-surjective' description of presheaf epimorphisms, do have a nice characterization as the 'locally surjective' maps.

Proposition A.0.71 *A map $\eta : F \longrightarrow G$ in $\mathbf{Sh}(\mathcal{C})$ is an epimorphism if and only if, for any object C of $\mathcal{C}$ and element $c \in G(C)$, there exists a J-cover $S \in J(C)$ such that, for all $f : D \longrightarrow C \in S$, $G(f)(c) \in \mathrm{im}(\eta_D)$.*

A.0.13 Filtered Colimits

Some of the basic facts about filtered colimits and the inductive completion are contained in this subsection. The first useful fact about filtered colimits which one encounters is that filtered colimits behave nicely in several ways (all proofs in this subsection can be found in either [22] or [11]).

Proposition A.0.72 *Finite limits commute with filtered colimits in* **Sets***.*

Filtered colimits taken in **Sets** have a particularly nice description (unlike arbitrary colimits) which is useful when considering filtered colimits in presheaf categories.

Proposition A.0.73 *Given a diagram $D : I \longrightarrow$ **Sets** with I a (small) filtered category:*
$$\varinjlim_i D_i \quad \cong \quad \coprod_i D_i / \sim,$$
where $a \in D_i \sim a' \in D_{i'}$ if and only if there exists an object i'' of $\mathcal{I}$ together with maps $f : i \longrightarrow i''$ and $f' : i' \longrightarrow i''$ such that $Df(a) = Df'(a')$. The injections then send an object of D_i to its equivalence class under $\sim$.

Definition A.0.74 A functor $F : \mathcal{I} \longrightarrow \mathcal{J}$ is *final* provided that, for each $j \in \mathcal{J}$, the comma category $(j \downarrow F)$ is inhabited and connected. Here for $(j \downarrow F)$ to be connected means that any maps $j \longrightarrow Fi$ and $j \longrightarrow Fi'$ can be 'joined' to give a finite commutative diagram:

$$\begin{tikzcd}
& & & j \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
Fi \arrow[rr, dashed] & & \bullet & \ldots & \bullet & & Fi' \arrow[ll, dashed]
\end{tikzcd}$$

Final functors are useful because they allow us to re-index filtered colimits.

Theorem A.0.75 *If $F : \mathcal{I} \longrightarrow \mathcal{J}$ is a final functor with filtered domain and codomain and $D : \mathcal{J} \longrightarrow \mathcal{C}$ is any functor, then $\varinjlim_{\mathcal{J}} yD_j \cong \varinjlim_{\mathcal{I}} yDF_i$ in $\widehat{\mathcal{C}}$.*

Another useful notion related to filteredness is that of a finitely presentable object:

Definition A.0.76 An object C of a category $\mathcal{C}$ is *finitely presentable* if and only if, for any filtered colimit $X := \varinjlim_{i \in \mathcal{I}} D_i$ in $\mathcal{C}$ and map $f : C \longrightarrow X$, there exists some i such that f factors through the map $D_i \longrightarrow D$.

A.0.14 The Kripke-Joyal Semantics

The reader should have some familiarity with the usual way of modeling first-order logic in Heyting categories. If not, then the reader should consult D1 of [20]. However, we will remind the reader of the Kripke-Joyal semantics for reasoning in the internal language of a positive Heyting category. As such, we assume for the duration of this section that the ambient category $\mathcal{C}$ is a positive Heyting category. The usual reference for the Kripke-Joyal semantics is [23] (however, the treatment given there is concerned with topoi).

Definition A.0.77 Given a subobject $i : [\![x : X|\varphi(x)]\!] \rightarrowtail X$ with φ a formula of the internal language with the free-variable $x : X$ and $a : A \longrightarrow X$ we write:

$$A \Vdash \varphi(a)$$

and say that 'A forces $\varphi(a)$' if and only if there exists a map $\zeta : A \longrightarrow [\![x|\varphi]\!]$ such that $i \circ \zeta = a$.

The definition of forcing may be extended in the obvious way to formulae φ of the internal language which contain more than one free-variable. The following useful facts about the forcing relation are easily verified:

Proposition A.0.78 *Given a subobject* $i : [\![x : X|\varphi(x)]\!] \rightarrowtail X$ *in* $\mathcal{C}$ *the following hold:*

Monotonicity: *If* $A \Vdash \varphi(a)$ *and* $b : B \longrightarrow A$*, then* $B \Vdash \varphi(a \circ b)$*.*

Local Character: *If* $e : E \longrightarrow\!\!\!\triangleright A$ *is a cover and* $E \Vdash \varphi(a \circ e)$*, then* $A \Vdash \varphi(a)$*.*

Using these facts one arrives at the 'fundamental theorem' of the Kripke-Joyal semantics (here as above the results easily generalize to the case of φ with more than one free-variable).

Theorem A.0.79 *If* $\langle a, b\rangle : A \longrightarrow X \times Y$*, then:*

1. $A \Vdash \varphi(a) \wedge \psi(b)$ *if and only if* $A \Vdash \varphi(a)$ *and* $A \Vdash \psi(b)$*.*

2. $A \Vdash \varphi(a) \vee \psi(b)$ *if and only if there exist arrows* $t : T \longrightarrow A$ *and* $s : S \longrightarrow A$ *such that* $t + s$ *is a cover,* $T \Vdash \varphi(a \circ t)$ *and* $S \Vdash \psi(b \circ s)$*.*

3. $A \Vdash \varphi(a) \Rightarrow \psi(b)$ *if and only if, for any arrow* $t : T \longrightarrow A$*, if* $T \Vdash \varphi(a \circ t)$*, then* $T \Vdash \psi(b \circ t)$*.*

4. $A \Vdash \neg\varphi(a)$ *if and only if, for* $t : T \longrightarrow A$*, if* $T \Vdash \varphi(a \circ t)$*, then* $T \cong 0$*.*

5. $A \Vdash \exists z : Z.\varphi(a, z)$ *if and only if there exists a cover* $t : T \longrightarrow\!\!\!\triangleright A$ *and a map* $s : T \longrightarrow Z$ *such that* $T \Vdash \varphi(a \circ t, s)$*.*

6. $A \Vdash \forall z : Z.\varphi(a, z)$ *if and only if, for all* $t : T \longrightarrow A$ *and* $s : T \longrightarrow Z$*,* $T \Vdash \varphi(a \circ t, s)$*.*

Of course, in certain categories (such as in sheaves) one may show that different clauses also hold. However, the reader is referred to the literature for more detail on these points.

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