Relating topos theory and set theory via categories of classes

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Abstract

We investigate a certain system of intuitionistic set theory from three points of view: an elementary set theory with bounded separation, a topos with distinguished inclusions, and a category of classes with a system of small maps. The three presentations are shown to be equivalent in a strong sense.

Contents

1 Introduction

One way to understand the notion of a topos is as a category of generalized sets, and thus as a generalization of the category of sets. As such, some features of sets as classically conceived are retained, but others are abstracted away. The global relations of membership and inclusion, in particular, are discarded. But at what loss?

An essential aspect of the classical notion of set is associated with these relations, namely the generation of the universe by iterated subset collection, or — more generally — its closure under such collection. Thus the universe U of sets satisfies $\mathcal{P}(U) \subseteq U$ for a suitably determined powerset functor $\mathcal{P}$, and this fact is what gives rise to the global membership and inclusion relations of classical set theory. Indeed, it implies many of the other conventional facts about sets as well.

The purpose of this work is, first of all, to analyze algebraically the theory of sets according to this "iterative" conception, with the universe occurring as a "sub-fixed-point" of the powerset functor, and the elementary-logical structure of sets resulting from this. The second, less superficial, purpose is to compare the resulting theory of sets with topos theory, and to determine in particular just how much is lost in passing from the "iterative," elementary-logical conception of set to the abstract, algebraic-categorical one.

Our basic tool in this investigation is the notion of a *category of classes*, which derives from the algebraic set theory of Joyal and Moerdijk. Roughly speaking, this notion is to the Gödel-Bernays-von Neumann theory of classes, what topos theory is to elementary set theory: the objects of the respective categories are the (first-order) objects of the respective elementary theories. We begin by showing how to interpret set theory in such a category, using the universe U. The elementary set theory of such universes can be completely axiomatized; we call the resulting theory bIST, for basic Intuitionistic Set Theory. It is noteworthy for including the unrestricted Axiom of Replacement in the absence of the full Axiom of Separation.

The universe U is an internal subcategory of sets in the category of classes, and is easily shown to be a topos. Our first logical completeness theorem for the set theory bIST with respect to toposes follows from this fact. Whether a topos satisfies an elementary logical condition depends in general on the ambient category of classes; thus some care is required in formulating the notions of soundness and completeness. Indeed, not only is it the case that every topos of sets in a category of classes satisfies BIST; but in fact, every topos whatsoever satisfies bIST, with respect to some category of classes.

This strong form of soundness follows from the fact that, as we show, every topos occurs as the category of sets in a category of classes. The proof of this fact is of independent interest; for it shows that in a sense, every topos has its own class structure, consisting of ideals of objects under a selected ordering by a system of distinguished inclusions. There is an analogy here to the ideal completion of a semilattice into a Heyting algebra. Indeed the analogy extends to the following representation theorem, which is our main mathematical result:

Theorem. *Every category of classes embeds into one consisting of ideals in a topos.*

Since the embedding is a faithful functor preserving all class category structure, it follows from the completeness of bIST with respect to categories of classes that bIST is also logically complete with respect to toposes, equipped with their ideal class structure. This latter interpretation can be reformulated in a direct "forcing" semantics that makes no mention of classes, but only bIST and toposes. The corresponding completeness theorem provides the precise statement of our comparison between the elementary-logical and algebraic-categorical approaches to set theory.

Acknowledgements. There are some to be made. References must be added as well.

2 Categories of classes

We intend to describe categories of "classes", among which certain of the objects (and arrows) have the special property of being "small" — these will be the sets. A distinctive aspect of our approach is the great degree of flexibility, with respect to the classes, in determining the sets. Many different categories of classes may result in equivalent theories of sets. We let our choice be guided by convenience in exposition rather than, say, philosophy, parsimony or strength. In selecting axioms for classes, for instance, one could draw on conceptual considerations about forming collections and selecting or mapping elements for motivation; instead, we take a simple formulation that is sufficient for interpreting first-order logic.

By a *category of classes* we shall mean a (locally small) category $\mathcal{C}$ satisfying the following conditions:

(C1) $\mathcal{C}$ has *finite limits*, i.e. terminal object 1, binary products $C \times D$, as well as equalizers, pullbacks, etc.

(C2) $\mathcal{C}$ has *finite coproducts*, i.e. initial object 0 and binary coproducts $C+D$. Moreover, these coproducts are required to be disjoint and stable under pullbacks.

(C3) $\mathcal{C}$ has *kernel quotients*, i.e. for every arrow $f : C \to D$, the kernel pair k_1, k_2 (the pullback of f against itself) has a coequalizer $q : C \to Q$.

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & C \arrow[r, two heads, "q"] \arrow[d, "f"'] & Q \\ & D & \end{tikzcd}$$

Morever, regular epimorphisms are required to be stable under pullbacks.

(C4) $\mathcal{C}$ has *dual images*, i.e. for every arrow $f : C \to D$, the pullback functor,

$$f^* : \mathrm{Sub}(D) \to \mathrm{Sub}(C) ,$$

has a right adjoint,

$$f_* : \mathrm{Sub}(C) \to \mathrm{Sub}(D) .$$

Conditions (C1) and (C3) imply that $\mathcal{C}$ has (stable) images, i.e. for every arrow $f : C \to D$, the pullback functor,

$$f^* : \mathrm{Sub}(D) \to \mathrm{Sub}(C) ,$$

also has a left adjoint,

$$f_! : \mathrm{Sub}(C) \to \mathrm{Sub}(D) .$$

Moreover, it follows that such categories have the following nice logical property, the proof of which is routine.

Proposition 1. *Every category of classes $\mathcal{C}$ is a "Heyting category", i.e. a regular category in which each subobject poset* $\mathrm{Sub}(C)$ *is a Heyting algebra, and the pullback functor* $f^* : \mathrm{Sub}(D) \to \mathrm{Sub}(C)$ *for every arrow* $f : C \to D$ *has both right and left adjoints satisfying the "Beck-Chevally condition" of stability under pullbacks. In particular, $\mathcal{C}$ models intuitionistic, first-order logic with equality.*

2.1 Small maps

Let $\mathcal{C}$ be a category of classes. We now want to axiomatize a notion of "smallness" based on the idea that a class map $f : A \to B$ is small when all of its fibers $f^{-1}(b) \subseteq A$ are "sets".

By a *system of small maps* on $\mathcal{C}$ we mean a collection of arrows $\mathcal{S}$ of $\mathcal{C}$ satisfying the following conditions:

(S1) $\mathcal{S} \hookrightarrow \mathcal{C}$ is a subcategory with the same objects as $\mathcal{C}$. Thus every identity map $1_C : C \to C$ is small, and the composite $g \circ f : A \to C$ of any two small maps $f : A \to B$ and $g : B \to C$ is again small.

(S2) The pullback of a small map along any map is small. Thus in an arbitrary pullback diagram,

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r] & D \end{tikzcd}$$

f' is small if f is small.

(S3) Every diagonal $\Delta : C \to C \times C$ is small.

(S4) If $f \circ e$ is small and e is regular epic, then f is small, as indicated in the diagram:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[dr, "f \circ e"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

(S5) Copairs of small maps are small. Thus if $f : A \to C$ and $g : B \to C$ are small, then so is $(f, g) : A + B \to C$.

Proposition 2. *Given (S1) and (S2), condition (S3) is equivalent to each of the following:*

1. *Every regular monomorphism is small.*

2. *if $g \circ f$ is small, then so is f, as indicated in the diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Proof. Suppose regular monos are small, and consider the following pullback diagram, with $g \circ f$ small:

$$\begin{tikzcd} P \arrow[rr, "p_2"] \arrow[d, "p_1"'] & & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

The arrow p_1 is a split epi, as can be seen by considering the pair $1 : A \to A$ and $f : A \to B$. Call the section $s : A \to P$, it is a regular mono and hence small. But p_2 is also small, so $f = p_2 \circ s$ is small.

The other entailments are even more direct. □

Proposition 3. *Given (S1)–(S5), the following also hold:*

1. *The canonical maps $0 \to C$ are all small.*

2. *If $f : C \to D$ and $f' : C' \to D'$ are small, then so is $f + f' : C + C' \to D + D'$.*

Proof. This follows easily from disjointness and stability of coproducts. □

2.2 Small powerobjects

We shall use the following terminology:

- an object A is called *small* if $A \to 1$ is a small map,
- a relation $R \rightarrowtail C \times D$ is called *small* if its second projection

  $$R \rightarrowtail C \times D \to D$$

  is a small map,
- a subobject $A \rightarrowtail C$ is called *small* if the relation $A \rightarrowtail C \times 1$ is small.

A small subobject $A \rightarrowtail C$ is evidently just one that is represented by a monomorphism $m : A \to C$ with a small domain A. Note that such a mono is necessarily small. We write:

$$\mathrm{SSub}(C) \;\subseteq\; \mathrm{Sub}(C)$$

for the subposet of small subobjects. We write:

$$\mathrm{SRel}_C(X) \;\subseteq\; \mathrm{Sub}(C \times X)$$

for the subposet of small relations on $C \times X$.

Via their graphs, the small maps determine and are determined by the small relations. We make the latter *representable*, in the following sense:

(P1) Every object C has a *small powerobject*: an object $\mathcal{P}C$ with a small relation $\in_C \rightarrowtail C \times \mathcal{P}C$ such that, for any object X and any small relation $R \rightarrowtail C \times X$, there is a unique arrow $\rho : X \to \mathcal{P}C$ such that the following is a pullback diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \in_C \arrow[d, tail] \\ C \times X \arrow[r, "1_C \times \rho"'] & C \times \mathcal{P}C \end{tikzcd}$$

(P2) The internal *subset* relation $\subseteq_C \rightarrowtail \mathcal{P}C \times \mathcal{P}C$ is small.

Intuitively, axiom (P1) requires every class to have a powerclass of all small subobjects or "subsets", and axiom (P2) requires the powerclass of a small object to be small, i.e. the powerclass of a set to be a set. (P1) is of course much like the universal mapping property of powerobjects familiar from topos theory, only adjusted for small relations. It says that the object $\mathcal{P}C$ represents the (contravariant) functor of small relations on $C \times -$,

$$\mathrm{Hom}(X, \mathcal{P}C) \;\cong\; \mathrm{SRel}_C(X)\ .$$

In particular there is then an isomorphism,

$$\mathrm{Hom}(1, \mathcal{P}C) \;\cong\; \mathrm{SSub}(C)\ .$$

The subset relation $\subseteq_C \rightarrowtail \mathcal{P}C \times \mathcal{P}C$ mentioned in (P2) can be constructed logically in the expected way as:

$$\subseteq_C = [\![(y,z) : \mathcal{P}C \times \mathcal{P}C | \ \forall x : C.\ x \in y \Rightarrow x \in z]\!]$$

Here we use the canonical interpretation $[\![-]\!]$ of first-order logic in the internal logic of $\mathcal{C}$, interpreting the atomic formula $x \in y$ as:

$$[\![x \in y]\!] = \in_C \rightarrowtail C \times \mathcal{P}C\ .$$

Remark 4. The subset relation can in fact be specified using only finite limits and representability (P1). First observe that the intersection $P \cap Q \rightarrowtail C \times X$ of any small relations $P \rightarrowtail C \times X$ and $Q \rightarrowtail C \times X$ is again small, for consider the diagram below, in which the square is a pulback:

$$\begin{tikzcd}
P \cap Q \arrow[r, tail] \arrow[d, tail, "q'"'] & Q \arrow[d, tail, "q"'] \arrow[dr] & \\
P \arrow[r, tail, "p"'] & C \times X \arrow[r, "\pi_2"'] & X
\end{tikzcd}$$

Since Q is a small relation, q is a small map. Thus q' is also small, as is $\pi_2 \circ p \circ q'$. It follows that there is a unique internal *intersection* operation,

$$\cap : \mathcal{P}C \times \mathcal{P}C \to \mathcal{P}C$$

with the property that:

$$[f \cap g] = [f] \cap [g] \ ,$$

where we are writing $[\rho] \rightarrowtail C \times X$ for the small relation corresponding uniquely to $\rho : X \to \mathcal{P}C$ by

$$[\rho] = (1_C \times \rho)^*(\in_C)$$

under (P1). The internal *subset* relation is defined to be the equalizer:

$$\begin{tikzcd} \subseteq \arrow[r, tail] & \mathcal{P}C \times \mathcal{P}C \arrow[r, shift left, "\cap"] \arrow[r, shift right, "\pi_1"'] & \mathcal{P}C \end{tikzcd}$$

One sees easily that a pair of arrows,

$$\langle f, g \rangle : X \to \mathcal{P}C \times \mathcal{P}C$$

factors through the relation $\subseteq$ if and only if $[f] \leq [g]$ holds for the corresponding small relations. It follows that this specification is equivalent to the logical one given above.

Small powerobjects represent the system of small maps in the following sense. For every object C, the second projection of the elementhood relation:

$$\pi : \in_C \rightarrowtail C \times \mathcal{P}C \to \mathcal{P}C$$

is a *small map classifier for* C, i.e. every small map $f : C \to D$ is a pullback of π along the *fiber map* $f^{-1} : D \to \mathcal{P}C$, as indicated in the diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & \in_C \arrow[d, "\pi"] \\ D \arrow[r, "f^{-1}"'] & \mathcal{P}C \end{tikzcd}$$

The domain $\in_C$ of π can be regarded as a disjoint sum:

$$\in_C = \sum_{A \in \mathcal{P}C} A$$

The fiber map $f^{-1} : D \to \mathcal{P}C$ of a small map $f : C \to D$ is used to establish the following important property of small maps. It essentially says that a map is small if it is so "locally on a cover".

Proposition 5. *(Descent) In a pullback diagram,*

$$\begin{tikzcd} A \arrow[r, "e'"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, two heads, "e"'] & D \end{tikzcd}$$

if e is regular epic and g is small, then f is small.

Proof. (Sketch) First consider the diagram:

$$\begin{tikzcd} C \arrow[r, two heads, "e"] \arrow[d, "g^{-1}"'] & D \arrow[d, dotted, "h"] \\ \mathcal{P}A \arrow[r, "e'_!"'] & \mathcal{P}B \end{tikzcd}$$

in which the map h results from the fact that $e'_! \circ g^{-1}$ coequalizes the kernel pair of e. Next we show that h is f^{-1}, so that f is indeed small. To that end, observe that the outer rectangle in the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & \in_A \arrow[r] \arrow[d, "\pi_A"] & \in_B \arrow[d, "\pi_B"] \\ C \arrow[r, "g^{-1}"'] & \mathcal{P}A \arrow[r, "e'_!"'] & \mathcal{P}B \end{tikzcd}$$

Therefore the outer square in the following is also a pullback:

$$\begin{tikzcd}
A \arrow[rr] \arrow[d, equal] & & \in_B \arrow[d, equal] \\
A \arrow[r, two heads, "e'"] \arrow[d, "g"'] & B \arrow[r, dotted, "j"] \arrow[d, "f"'] & \in_B \arrow[d, "\pi_B"] \\
C \arrow[r, two heads, "e"'] & D \arrow[r, "h"'] & \mathcal{P}B
\end{tikzcd}$$

It follows that the indicated map j exists making the lower righthand square a pullback, as required. $\square$

A consequence of the descent property is that a map is small just if it pulls all small subobjects back to small subobjects, provided this is taken in an appropriate internal sense:

Proposition 6. *The following conditions are equivalent:*

1. *$f : A \to C$ is a small map.*

2. *$f_! : \mathcal{P}A \to \mathcal{P}C$ has a right adjoint $f^* : \mathcal{P}C \to \mathcal{P}A$.*

3. *There exists a "covering family of (small) subobjects of C":*

$$\begin{tikzcd}
 & U \arrow[dl] \arrow[d] \arrow[dr, "q"] & \\
C & C \times I \arrow[l, "p_1"] \arrow[r, "p_2"'] & I
\end{tikzcd}$$

(with the indexing map q small) such that in the following pullback

diagram, the map f_U is small:

$$\begin{tikzcd}
P \arrow[r, "f_U"] \arrow[d] & U \arrow[d, tail] \\
A \times I \arrow[r, "f \times I"'] & C \times I
\end{tikzcd}$$

Proof. Suppose $f : A \to C$ is small. Then for any small relation $R \rightarrowtail C \times X$, the pullback $R' = (f \times X)^* R$ is clearly also small.

$$\begin{tikzcd}
R' \arrow[r] \arrow[d, tail] & R \arrow[d, tail] \\
A \times X \arrow[r, "f \times X"'] & C \times X
\end{tikzcd}$$

Thus we have a function:

$$\mathrm{Hom}(X, \mathcal{P}C) \;\cong\; \mathrm{SRel}_C(X) \longrightarrow \mathrm{SRel}_A(X) \;\cong\; \mathrm{Hom}(X, \mathcal{P}A)$$

which is evidently natural in X. The yoneda lemma therefore gives the desired map $f^* : \mathcal{P}C \to \mathcal{P}A$.

Given the (internal) right adjoint $f^* : \mathcal{P}C \to \mathcal{P}A$, consider the covering family of *all* small subobjects of C:

$$\begin{tikzcd}
& \in_C \arrow[dl, "e"'] \arrow[d, tail] \arrow[dr, "\pi"] & \\
C & C \times \mathcal{P}C \arrow[l] \arrow[r] & \mathcal{P}C
\end{tikzcd}$$

The map $e : \in_C \to C$ is a regular epi since it is split by (a factorization of) the singleton $\{-\} : C \to \mathcal{P}C$, which exists since the diagonal of C is small.

Consider the pullback diagram:

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d] & \in_C \arrow[d, tail] \\
A \times \mathcal{P}C \arrow[r, "f \times I"'] & C \times \mathcal{P}C
\end{tikzcd}$$

The realtion $P \rightarrowtail A \times \mathcal{P}C$ is $f^*(R)$, and so is small, whence the required map f' is small.

The remaining entailment (3) to (1) is a simple consequence of descent. $\square$

Finally, as a warning, we emphasize that not all monomorphisms are small. Thus despite some intuition to the contrary, it is not the case that every subobject of a small object is small. The reason for this choice is that we intend to capture a conception of "set" that is not only motivated by limitation of size, but also by definability. The following proposition indicates some of the consequences of this choice:

Proposition 7. *The following conditions are equivalent:*

1. *Every mono in $\mathcal{C}$ is small.*
2. *Every mono in $\mathcal{C}$ is regular.*
3. *$\mathcal{C}$ has a subobject classifier.*

Remark 8. Adding one of these assumptions to our system of axioms for small maps gives a system equivalent to that in [**?**], which we shall here call *full class structure*. The resulting system can be stated more simply, however, by requiring only the axioms (C1), (C3), (S1), (S2), and the condition that all monos are small. Axioms (C2), (C4), (S3), (S4), and (S5) then follow. This system captures the notion of "set" formalized by (I)ZF, and motivated by "limitation of size" alone.

2.3 Topos of sets

We now begin to investigate some of the consequences of our axioms. Let $\mathcal{C}$ be a category of classes with a system $\mathcal{S}$ of small maps. For any object X, the slice category $\mathcal{S}/X$ is the (full) subcategory of $\mathcal{C}/X$ with objects all small maps into X. Let us write:

$$\mathcal{S}_X \hookrightarrow \mathcal{C}/X$$

for the larger collection of *all maps* in $\mathcal{C}/X$ that are small as maps in $\mathcal{C}$.

Lemma 9. *For any object X in $\mathcal{C}$,*

1. *The collection $\mathcal{S}_X$ is a system of small maps in $\mathcal{C}/X$.*

2. *If $\mathcal{C}$ has small powerobjects, then so does $\mathcal{C}/X$.*

3. *These structures are preserved by pullbacks.*

Proof. The axioms in groups (C) and (S) are immediately seen to be preserved by slicing. To verify (P1) we construct the "powermap" $\mathcal{P}f$ of any map $f : C \to X$. Consider the following diagram, in which the rectangle is a pullback, and $\mathcal{P}f$ is the indicated composite:

$$\begin{tikzcd}
C \arrow[d, "f"'] & V \arrow[r] \arrow[d] \arrow[dl, dotted, "\mathcal{P}f"'] & \supseteq \arrow[d] \\
X & X \times \mathcal{P}C \arrow[l, "\pi_1"] \arrow[r, "\{-\} \times f_!"'] & \mathcal{P}X \times \mathcal{P}X
\end{tikzcd}$$

In the internal logic of $\mathcal{C}$, we thus have:

$$V = [\![x, y \mid \{x\} \supseteq f_! y]\!]$$

The idea is that the first projection$\mathcal{P}f : V \to X$ is the "fiberwise powerobject":

$$\sum_{x \in X} \mathcal{P}(f^{-1}(x))$$

since $f_! y \subseteq \{x\} \Leftrightarrow y \subseteq f^{-1}(x)$ when $f^{-1}(x)$ exists.

Indeed, a small relation in $\mathcal{C}/X$ looks like this:

$$
\begin{tikzcd}
R \arrow[r, tail] \arrow[dr] & C \times_X D \arrow[r] \arrow[d] & D \arrow[d] \\
 & C \arrow[r] & X
\end{tikzcd}
$$

with a small second projection $R \to D$. Composing

$$R \rightarrowtail C \times_X D \rightarrowtail C \times D$$

thus determines a small relation on $C \times D$ in $\mathcal{C}$. The classifying map $r^{-1} : D \to \mathcal{P}C$ in $\mathcal{C}$ gives rise to the required corresponding one ρ in $\mathcal{C}/X$:

$$
\begin{tikzcd}
D \arrow[r, "\rho"] \arrow[dr] & V \arrow[d, "\mathcal{P}f"] \\
 & X
\end{tikzcd}
$$

$\square$

Proposition 10. *The following conditions are equivalent.*

1. *Axiom (P2) holds in $\mathcal{C}$; i.e., for every object C, the subset relation $\subseteq \rightarrowtail \mathcal{P}C \times \mathcal{P}C$ is small.*

2. *In every slice category $\mathcal{C}/X$, if A is small, then $\mathcal{P}A$ is small.*

Proof. If $f : C \to X$ is small, the powermap $\mathcal{P}f$ has the equivalent description:

$$
\begin{tikzcd}
 & V \arrow[r] \arrow[dl, "\mathcal{P}f"'] \arrow[d] & \supseteq \arrow[d, tail] \\
X & X \times \mathcal{P}C \arrow[l] \arrow[r, "f^{-1} \times \mathcal{P}C"'] & \mathcal{P}C \times \mathcal{P}C
\end{tikzcd}
$$

with a pullback on the right. But then we also have a pullback:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\mathcal{P}f"'] & \supseteq \arrow[d] \\ X \arrow[r, "f^{-1}"'] & \mathcal{P}C \end{tikzcd}$$

So (P2) implies that $\mathcal{P}f$ is small.

Conversely, the second projection $\subseteq \rightarrowtail \mathcal{P}C \times \mathcal{P}C \to \mathcal{P}C$ is the powermap $\mathcal{P}\pi$ in $\mathcal{C}/\mathcal{P}C$ of the second projection $\pi : \in \to C \times \mathcal{P}C \to \mathcal{P}C$. □

Corollary 11. *If (P2) holds in $\mathcal{C}$, then it also does in every slice category $\mathcal{C}/X$. The axioms (C), (S), and (P) are thus all preserved by slicing.*

Proposition 12. *For every small map $f : A \to B$, the reindexing functor,*

$$f^* : \mathcal{C}/B \to \mathcal{C}/A$$

has a right adjoint,

$$\Pi_f : \mathcal{C}/A \to \mathcal{C}/B\ .$$

Thus in partiular, every small class A is exponentiable.

Moreover, Π_f preserves small maps.

Proof. It suffices to show that every small object A is exponentiable, since the same will then hold in every slice category $\mathcal{C}$/X. Given objects C and small A, we can construct C^A as a subobject of $\mathcal{P}(A \times C)$ as in a topos:

$$C^A = [\![R \subseteq A \times C \mid \forall a \exists ! c. R(a, c)]\!] \rightarrowtail \mathcal{P}(A \times C)\,,$$

which will exist because the domain R of such a functional relation is small if A is small.

Preservation of small maps follows from (P2). Briefly, it suffices to show that for small objects A and B, also B^A is small. One construction gives B^A as a pullback:

$$\begin{tikzcd} B^A \arrow[r] \arrow[d, tail] & 1 \arrow[d, "`A'"] \\ \mathcal{P}(A \times B) \arrow[r, "g"'] & \mathcal{P}A \end{tikzcd}$$

where 'A' classifies the maximal subobject of A and, intuitively, the map g takes a small subobject $R \rightarrowtail A \times B$ to the subobject $[\![a \mid \exists! b.\ R(a,b)]\!] \rightarrowtail A$, which can be shown to be small. □

Finally, we have the desired result:

Theorem 13. *The full subcategory $\mathcal{S}/1 \hookrightarrow \mathcal{C}$ of small objects and small maps between them is an elementary topos.*

Proof. It only remains to show that there is a subobject classifier for the category of small objects. Indeed, for any small object A we have natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}/1}(A, \mathcal{P}1) &\cong \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{P}1) \\ &\cong \mathrm{SRel}_1(A) \\ &\cong \mathrm{SSub}_{\mathcal{C}}(A) \\ &\cong \mathrm{Sub}_{\mathcal{S}/1}(A) \end{aligned}$$

And $\mathcal{P}1$ is small by proposition 10. □

It is possible to show more directly that $\mathcal{S}/1$ is a topos, by an argument for small powerobjects similar to the one just given for a subobject classifier. Indeed, one only requires some of the conditions on $\mathcal{C}$ and $\mathcal{S}$ for this to hold.

Definition 14. We will henceforth write:

$$\mathcal{S}_{\mathcal{C}} \;=\; \mathcal{S}/1$$

for the full subcategory of small objects, and will often refer to these as "sets".

2.4 Universes and Infinity

A category of classes $\mathcal{C}$ is said to have an *infinite set* if it satisfies the conditions in the following:

Proposition 15. *The following are equivalent:*

1. *There is a small object I with a monomorphism $I + 1 \rightarrowtail I$.*

2. *The category $\mathcal{S}_{\mathcal{C}}$ of small objects has a natural numbers object.*

Proof. $\mathcal{S}_{\mathcal{C}}$ is a topos. □

We note that if the sets have an NNO, it does not follow that $\mathcal{C}$ itself has an NNO, since there may be many more classes than sets. Indeed, all of the axioms we have considered up to now are compatible with the assumption that all maps are small; then $\mathcal{C}$ is itself an elementary topos. By Cantor's theorem on the size of powersets, the following condition forces there to be classes that are not sets:

(U) There is a *universal object* U, i.e. one such that every object C has a monomorphism $C \rightarrowtail U$.

Such a universal object U is in particular a *universe*, in the following sense:

Definition 16. A *universe* is an object V together with a monomorphism,

$$\mathcal{P}V \rightarrowtail V\ .$$

Conversely, if $\mathcal{C}$ is a category of classes satisfying axioms (C, S, P) and having a universe V, then one shows easily that the full subcategory $\mathcal{C}_V \hookrightarrow \mathcal{C}$ of objects C having a mono $C \rightarrowtail V$ also satisfies axioms (C, S, P) as well as (U).

Observe that in the presence of a universal object there is a single (weakly) universal small map, namely $\pi_U : \in_U \to \mathcal{P}U$. Every small map $f : A \to B$ is a pullback of π_U along a (not necessarily unique) arrow $\varphi : B \to \mathcal{P}U$. One may think of π_U as the indexed family of *all* sets.

2.5 Class structure

Summarizing, we shall call a category of classes with a system of small maps, small powerobjects, and a universal object a *category with class structure*. Specifically, this consists of a (locally small) category $\mathcal{C}$ satisfying the following conditions:

(C) $\mathcal{C}$ is a regular category with disjoint and stable finite coproducts and dual images.

(S) There is a subcategory $\mathcal{S} \hookrightarrow \mathcal{C}$ of small maps.

(P) Every class C has a small powerclass $\mathcal{P}(C)$ with small subset relation.

(U) There is a universal object $\mathcal{U}$.

Theorem 17. *If $\mathcal{C}$ is a category with class structure, then so is the slice category $\mathcal{C}/X$ for every object X. Moreover, the class structure $\mathcal{S}, \mathcal{P}, \mathcal{U}$ is preserved by pullback functors.*

Proof. It remains only to show that the pullback X^*U of a universal object U is universal in $\mathcal{C}/X$, but this is obvious. □

3 The set theory bIST

We recall informally the elementary set theory bIST (basic Intuitionistic Set Theory), presented formally in [**?**]. In addition to the binary *membership relation* $x \in y$, there is a predicate of *sethood* $\mathsf{S}(x)$, which is required because we admit the possibility of atoms. The theory bIST$^-$ has the following axioms:

(sethood) $\quad a \in b \to \mathsf{S}(b)$

(extensionality) $\quad \mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \forall x(x \in a \leftrightarrow x \in b) \to a = b$

Moreover, the following are all asserted to be sets:

(empty set)	$\emptyset$	$=$	$\{x \mid \bot\}$	
(pairs)	$\{a, b\}$	$=$	$\{x \mid x = a \vee x = b\}$	
(powerset)	$P(a)$	$=$	$\{x \mid \mathsf{S}(x) \wedge \forall y.\ y \in x \to y \in a\}$	if $\mathsf{S}(a)$
(union)	$\bigcup a$	$=$	$\{x \mid \exists y \in a.\ x \in y\}$	if $\mathsf{S}(a)$, and $y \in a \to \mathsf{S}(y)$
(intersection)	$\bigcap a$	$=$	$\{x \mid \forall y \in a.\ x \in y\}$	if $\mathsf{S}(a)$, and $y \in a \to \mathsf{S}(y)$
(replacement)	$\{F(x) \mid x \in a\}$		if $\mathsf{S}(a)$ and F is any functional relation.	

Here "$\{x \mid \varphi\}$ is a set" is of course a circumlocution for the formula:

$$\exists y.\ \mathsf{S}(y) \wedge \forall x.\ x \in y \leftrightarrow \varphi$$

which we also sometimes abbreviate to:

$$\S x.\ \varphi$$

The full theory bIST also includes an axiom of *infinity* stating formally:

(infinity) there is a set I with an injection $I + 1 \rightarrowtail I$.

Finally, we recall from [**?**] the convenient fact that bIST$^-$ satisfies the following *bounded separation* scheme ("Δ_0-separation"), valid for all formulas φ without the predicate S and in which all quantified variables are of the form $\forall x \in b$ or $\exists x \in b$, i.e. bounded by a set b:

(bounded separation) if a is a set, then so is $\{x \in a \mid \varphi\}$ for bounded φ.

3.1 Satisfaction of bIST

Our objective in this subsection is to show that this elementary theory is modelled by the sets in any category of classes $\mathcal{C}$. Indeed, we shall see that this is true in the strong sense that the universal object U is a model of bIST in the internal logic of $\mathcal{C}$. Since the full subcategory $\mathcal{S}_{\mathcal{C}}$ of sets is a topos equivalent to the global elements $1 \to U$ of U, the following expected fact is a good sanity check:

Proposition 18. *The sets and functions in bIST form a topos.*

Proof. The usual (intuitionistic) set-theoretic constructions of ordered pairs, cartesian products, and function sets are all available in bIST. The subobject classifier is $P(\{\emptyset\})$, as expected. $\square$

To now show that the topos of sets in any category of classes $\mathcal{C}$ form an internal model of bIST, we need to interpret the basic relations $x \in y$ and $\mathsf{S}(x)$ over the universal object U, which will be the domain of the model. For this we set:

$$\begin{aligned} [\![x \mid \mathsf{S}(x)]\!] &= \mathcal{P}U \rightarrowtail U \\ [\![x, y \mid x \in y]\!] &= \in_U \rightarrowtail U \times \mathcal{P}U \rightarrowtail U \times U \end{aligned}$$

where the indicated monos are the evident canonical ones.

Proposition 19. *Under this interpretation, all of the axioms of bIST$^-$ are valid in any category $\mathcal{C}$ with class structure. If $\mathcal{C}$ has an infinite set, then (infinity) is also satisfied.*

Proof. (sethood) $a \in b \to \mathsf{S}(b)$.

We need to show:

$$1 \leq [\![\forall x, y.\ x \in y \to \mathsf{S}(y)]\!] \qquad \text{in Sub}(1)$$

But this is equivalent to:

$$[\![x, y \mid x \in y]\!] \leq [\![x, y \mid \mathsf{S}(y)]\!] \qquad \text{in Sub}(U \times U)$$

Now

$$[\![x, y \mid x \in y]\!] = \in_U \rightarrowtail U \times \mathcal{P}U \rightarrowtail U \times U$$

and

$$[\![x, y \mid \mathsf{S}(y)]\!] = \pi^*(i) \rightarrowtail U \times U$$

where $\pi : U \times U \to U$ is the second projection. Thus it suffices to observe that the following diagram commutes.

$$\begin{tikzcd}
\in_U \arrow[rr] \arrow[d, tail] & & \mathcal{P}U \arrow[d, tail, "i"] \\
U \times \mathcal{P}U \arrow[r, tail, "1 \times i"'] & U \times U \arrow[r, "\pi"'] & U
\end{tikzcd}$$

(extensionality) $\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \forall x(x \in a \leftrightarrow x \in b) \to a = b$.

Suppose given arbitrary $\langle a, b\rangle : Z \to U \times U$ factoring through the subobject

$$[\![u, v \mid \mathsf{S}(u) \wedge \mathsf{S}(v) \wedge \forall x.(x \in u \leftrightarrow x \in v)]\!] \rightarrowtail U \times U$$

then by the first two conjuncts there are small relations $[\![y, z \mid y \in a(z)]\!]$ and $[\![y, z \mid y \in b(z)]\!]$ on $U \times Z$, and by the third one these satisfy

$$[\![y, z \mid y \in a(z)]\!] \leq [\![y, z \mid y \in b(z)]\!]$$

and

$$[\![y, z \mid y \in b(z)]\!] \leq [\![y, z \mid y \in a(z)]\!]$$

But this means that

$$[\![y, z \mid y \in a(z)]\!] \; = \; [\![y, z \mid y \in b(z)]\!]$$

whence $a = b$.

For the smallness conditions we make use of the following lemma, the proof of which is straightforward.

Lemma 20. *For any formula φ with at most the variable x free, the following are equivalent:*

1. *the condition "$\{x \mid \varphi\}$ is small" is valid,*
2. *the subobject $[\![x \mid \varphi]\!] \rightarrowtail U$ is small.*

Similarly, if φ has an additional parameter u, then the condition "$\{x \mid \varphi\}$ is small" is valid if and only if the relation $[\![u, x \mid \varphi]\!] \rightarrowtail U \times U$ is small, and similarly for several parameters.

(empty set) $\emptyset = \{x \mid \bot\}$.

Clearly $[\![x \mid \bot]\!]$ is small.

(pairs) $\{a, b\} = \{x \mid x = a \vee x = b\}$.

Singletons $\{a\}$ are small, since the diagonal $U \to U \times U$ is small. And generally, the union of small subobjects $A, B \rightarrowtail U$ is small, by the axioms of coproducts and quotients.

(powerset) $P(a) = \{x \mid x \subseteq a\}$.

The subset relation $[\![x, y \mid x \subseteq y]\!] \rightarrowtail \mathcal{P}U \times \mathcal{P}U$ is small by the powerset axiom.

(union) $\bigcup a = \{x \mid \exists y.\ y \in a \wedge x \in y\}$ if $\mathsf{S}(a)$ and $y \in a \to \mathsf{S}(y)$.

We need to show that the relational product $\in_U \circ \in_{\mathcal{P}U}$ of the small realtions $\in_U \rightarrowtail U \times \mathcal{P}U$ and $\in_{\mathcal{P}U} \rightarrowtail \mathcal{P}U \times \mathcal{P}\mathcal{P}U$ is again small. But this holds in general, by an easy diagram chase.

(intersection) $\bigcap a = \{x \mid \forall y.\ y \in a \to x \in y\}$ if $\mathsf{S}(a)$ and $y \in a \to \mathsf{S}(y)$.

It seems to be somewhat easier to show the equivalent condition:

$$\mathsf{S}(a) \wedge \forall y \in a (y \subseteq b) \to \S x \in b\ \forall y \in a.\ x \in y$$

This follows from the fact that Π functors along small maps exist and preserve small maps.

(replacement) $\{F(x) \mid x \in a\}$ if $\mathsf{S}(a)$ and F is a functional relation.

First we show that for any relation $F \rightarrowtail U \times U$ and any $a : 1 \to \mathcal{P}U$, the interpretation of the following formula is true:

$$\forall x \in a\ \exists ! y.\ F(x,y) \to \S y\ \exists x \in a.\ F(x,y)$$

The lefthand side means that the relation $[\![x, y \mid F(x,y)]\!] \rightarrowtail [\![x \mid x \in a]\!] \times U$ is the graph of a (unique) morphism,

$$f : [\![x \mid x \in a]\!] \to U$$

The image factorization of this map f is then:

$$[\![x \mid x \in a]\!] \twoheadrightarrow [\![y \mid \exists x \in a.\ F(x,y)]\!] \rightarrowtail U$$

And since $[\![x \mid x \in a]\!]$ is small, so is $[\![y \mid \exists x \in a.\ F(x,y)]\!]$. Thus the righthand side holds as well.

But now for arbitrary $a : Z \to \mathcal{P}U$, we can pull the entire problem back to $\mathcal{C}/Z$ and use the case just shown, since pullback preserves both the class structure and the internal logic.

The existence of an infinite set in $\mathcal{C}$ clearly implies the internal validity of infinity. □

Remark 21. If *all* monomorphisms are small in the category $\mathcal{C}$, so that $\mathcal{C}$ models the full class structure mentioned in remark 8, then the full axiom scheme of separation is also satisfied. Specifically, for all formulas φ (including ones with the predicate S and in which quantified variables are unbounded):

(separation) if a is a set, then so is $\{x \in a \mid \varphi\}$.

We refer to the system resulting from bIST by adding full separation as Intuisionistic Set Theory (IST).

3.2 Class completeness of bIST

In the foregoing section we saw that the elementary set theory bIST is sound with respect to models in categories of classes. One of the virtues of our approach is that this theory is also complete with respect to such models, and this is quite easily shown. Thus our goal in this section is to prove the following:

Theorem 22. *If an elementary formula φ (in the language $\{\mathsf{S}, \in\}$) is valid in every category of classes $\mathcal{C}$, then it is provable in the elementary set theory bIST.*

In fact, we shall prove the stronger statement that there exists a *single* category of classes $\mathcal{C}_0$ such that, for any formula φ:

$$\mathcal{C}_0 \models \varphi \quad \text{implies} \quad \text{bIST} \vdash \varphi$$

The category $\mathcal{C}_0$ is familiar to logicians as consisting of the definable classes over the theory bIST, together with the definable functional relations between them as morphisms. Category theorists are well-acquainted with $\mathcal{C}_0$ as the *syntactic category* of the first-order theory bIST, a standard construction, for details of which in general cf. [**?**], D1.4.

Definition 23. The category $\mathcal{C}_0$ consists of the following data:

objects $\{x_1, \ldots, x_n | \varphi\}$ are formulas in context $x_1, \ldots, x_n | \varphi$, identified up to α-equivalence.

arrows $[f] : \{x|\varphi\} \to \{y|\psi\}$ are equivalence classes of formulas in context $x, y | f(x, y)$ that are "provably functional relations", i.e. in bIST:

$$\begin{aligned} f(x,y) &\vdash \varphi(x) \wedge \psi(y) \\ \psi(y) &\vdash \exists x. f(x,y) \\ f(x,y) \wedge f(x,y') &\vdash y = y' \end{aligned}$$

with two such f and g identified if $\vdash f \leftrightarrow g$.

identity $1_{\{x|\varphi\}} = [x = x' \wedge \varphi(x)] : \{x|\varphi(x)\} \to \{x'|\phi(x')\}$

composition $[g(y,z)] \circ [f(x,y)] = [\exists y. f(x,y) \wedge g(y,z)]$

We use various obvious notational devices to make things more readable, like dropping the context of variables where it can be inferred, and displaying variables for substitutions.

Lemma 24. *The syntactic category $\mathcal{C}_0$ of the first-order theory bIST is a Heyting category with stable, disjoint coproducts. Thus $\mathcal{C}_0$ satisfies axiom (C).*

Proof. We just need to show the coproducts, since the rest is standard (cf. [**?**] D1.4.10). For these we can set:

$$\{x \mid \varphi(x)\} + \{y \mid \psi(y)\} = \{w, z \mid (w = 0 \wedge \varphi(z)) \vee (w = 1 \wedge \psi(z))\}$$

□

Now define a map $[f] : \{x|\varphi\} \to \{y|\psi\}$ in $\mathcal{C}_0$ to be *small* if in bIST,

$$\psi(y) \vdash \S x. f(x, y)$$

Lemma 25. *With these small maps, $\mathcal{C}_0$ satisfies axiom (S).*

Proof. For (S1) we need to show that the small maps form a subcategory. An identity map

$$[x = x' \wedge \varphi(x)] : \{x|\varphi(x)\} \to \{x'|\varphi(x')\}$$

is small because in bIST:

$$\varphi(x) \vdash \S x'. x = x' \wedge \varphi(x)$$

For composition, suppose we have the arrows:

$$\begin{aligned} [f(x, y)] &: \{x|\varphi\} \to \{y|\psi\} \\ [g(y, z)] &: \{y|\psi\} \to \{z|\vartheta\} \end{aligned}$$

and we know that:

$$\begin{aligned} \psi(y) &\vdash \S x. f(x, y) \\ \vartheta(z) &\vdash \S y. g(y, z) \end{aligned}$$

Then by UnionRep one has:

$$\vartheta(z) \vdash \S x. \exists y. f(x, y) \wedge g(y, z)$$

(S2) requires the diagonal $\Delta_\varphi : \{x|\varphi\} \to \{x|\varphi\} \times \{x|\varphi\}$ to be small. But

$$\Delta_\varphi = [x, y, y'|\varphi(x) \wedge x = y \wedge x = y']$$

which is clearly small.

(S3) concerns pullbacks, which in $\mathcal{C}_0$ are constructed as indicated in the following diagram:

$$\begin{tikzcd}
\{x,y|\varphi \wedge \psi \wedge \exists z.f(x,z) \wedge g(y,z)\} \arrow[r] \arrow[d, "p_1"'] & \{y|\psi\} \arrow[d, "{[g(y,z)]}"] \\
\{x|\varphi\} \arrow[r, "{[f(x,y)]}"'] & \{z|\vartheta\}
\end{tikzcd}$$

We need the first projection p_1 to be small if f is. Thus we can assume:

$$\vartheta(z) \vdash \S y.g(y,z)$$

and we need to show:

$$\varphi(x) \vdash \S x', y.\ x = x' \wedge \varphi(x) \wedge \psi(y) \wedge \exists z.f(x,z) \wedge g(y,z)$$

There is clearly an isomorphism in $\mathcal{C}_0$:

$$\{x', y|x = x' \wedge \varphi(x) \wedge \psi(y) \wedge \exists z.f(x,z) \wedge g(y,z)\} \cong \{y|\exists z.f(x,z) \wedge g(y,z)\}$$

and so, by Replacement, it suffices to show:

$$\varphi(x) \vdash \S y.\exists z.f(x,z) \wedge g(y,z)$$

But this now follows thus:

$$\begin{aligned}
\varphi(x) &\vdash \exists z.f(x,z) \wedge \vartheta(z) \\
&\vdash \exists z.f(x,z) \wedge \S y.g(y,z) \\
&\vdash \S y.\exists z.f(x,z) \wedge g(y,z)
\end{aligned}$$

The remaining cases (S4) and (S5) are left to the reader. □

The *powerobjects* in $\mathcal{C}_0$ are defined in the expected way by,

$$\mathcal{P}\{x|\varphi\} = \{y|\mathsf{S}(y) \wedge \forall x.x \in y \rightarrow \varphi\}$$

with the membership relation given by the evident arrow,

$$\{x,y|\varphi(x) \wedge x \in y \wedge y \in \mathcal{P}\{x|\varphi\}\} \rightarrowtail \{x|\varphi\} \times \mathcal{P}\{x|\varphi\}\,.$$

Lemma 26. *$\mathcal{C}_0$ satisfies axiom (P).*

Proof. For (P1), suppose we have a relation,

$$\{x, y|\rho\} \rightarrowtail \{x|\varphi\} \times \{y|\psi\}$$

that is small,

$$\psi(y) \vdash \S x.\rho(x, y)$$

Then the arrow,

$$[y, z|\mathsf{S}(z) \wedge \forall x.x \in z \leftrightarrow \rho(x, y)] : \{y|\psi\} \longrightarrow \mathcal{P}\{x|\varphi\}$$

has the required characteristic property.

(P2) follows easily from the powerset axiom. □

Finally, $\mathcal{C}_0$ has the *universal object*,

$$U = \{u|u = u\}$$

Lemma 27. *$\mathcal{C}_0$ satisfies axiom (U).*

Proof. For any object $\{x|\varphi\}$, there is a canonical monomorphism,

$$i_\varphi = [\varphi(x) \wedge x = u] : \{x|\varphi\} \rightarrowtail U$$

□

We have now shown that $\mathcal{C}_0$ is a category with class structure. It remains only to consider validity of formulas of bIST in $\mathcal{C}_0$. The canonical interpretation of bIST in $\mathcal{C}_0$ with respect to U yields, for each formula in context $x_1, \dots, x_n|\varphi$, a subobject,

$$[\![x_1, \dots, x_n \mid \varphi]\!] \rightarrowtail U^n$$

On the other hand, there is the object determined by φ, with its canonical mono,

$$i_\varphi : \{x_1, \dots, x_n|\varphi\} \rightarrowtail U^n$$

An easy induction on φ shows that these are the same subobject of U^n. For the record:

Lemma 28. *For any formula in context $x_1, \ldots, x_n | \varphi$,*

$$[\![x_1, \ldots, x_n \mid \varphi]\!] = \{x_1, \ldots, x_n | \varphi\} \rightarrowtail U^n$$

Finally, to prove the theorem, take any formula φ that is valid in $\mathcal{C}_0$,

$$U \models \varphi$$

then for suitable context x we have,

$$[\![x \mid \varphi]\!] \cong \{u | u = u\}$$

canonically. Whence, by the foregoing lemma,

$$\vdash \varphi$$

in bIST, as required to complete the proof of the theorem.

4 The model in ideals

Our next objective is to prove that *every* topos occurs as the category of sets in a category of classes. To this end, we shall show how to construct the required category of classes out of a given topos, as the *category of ideals* in the topos. The sets will turn out to be exactly the principal ideals, and thus essentially the same as the original topos.

In order to perform this construction, we first need to have a suitable ordering of the objects of a topos. This ordering will be given by a distinguished subcategory of monos $A \hookrightarrow B$ called "inclusions", and will be a partial order with finite joins $A \cup B$. Observe that the objects of a topos that is already of the form $\mathcal{S}_\mathcal{C} \hookrightarrow \mathcal{C}$ are naturally ordered as the small subobjects of the universe U, so the possibility of such an ordering is clearly necessary.

4.1 Toposes with inclusions

Definition 29. (i) A *partially-ordered topos* ("po-topos") is a topos equipped with a *structural system of inclusions*: i.e. a subcategory of distinguished monos, written $A \hookrightarrow B$, such that

(a) the inclusions partially order the objects,

(b) every subobject $S \rightarrowtail E$ is represented by a unique inclusion $S \cong A \hookrightarrow E$,

(c) inclusions are preserved by a choice of product and covariant powerobject functors.

(ii) A $\cup$-*topos* is a po-topos having binary joins $A \cup B$ with respect to the partial ordering $A \hookrightarrow B$ of inclusion.

Proposition 30. *For any small category $\mathbb{C}$, the presheaf topos* $\mathbf{Sets}^{\mathbb{C}^{\mathrm{op}}}$ *is a po-topos.*

Proof. As inclusions $P \hookrightarrow Q$ of presheaves we take the *natural inclusions*, i.e. natural transformations $\vartheta : P \to Q$ such that every component $\vartheta_C : PC \to QC$ is an inclusion in **Sets**. These evidently compose and include the identities. Moreover, every monomorphism $m : M \rightarrowtail P$ is equivalent as a subobject to exactly one natural inclusion, namely that with components the pointwise images $[MC] \hookrightarrow PC$.

Given a choice of products in **Sets** that preserve inclusions (as can be assumed), the pointwise products in $\mathbf{Sets}^{\mathbb{C}^{\mathrm{op}}}$ then clearly preserve natural inclusions. The power-objects $\mathcal{P}Q$ must be chosen in a special way, however, namely:

$$\mathcal{P}Q(C) = \downarrow(yC \times Q) ,$$

where the lower segment is taken with respect to the inclusion ordering. The action of $\mathcal{P}Q$ on any $C' \to C$ is by pullback of inclusions along the evident resulting arrow $yC' \times Q \to yC \times Q$. Observe that one indeed has the required natural isomorphism:

$$\downarrow(yC \times Q) \cong \mathrm{Sub}(yC \times Q)$$

since every subobject is represented by a unique inclusion. Given $Q \hookrightarrow Q'$, we then have $yC \times Q \hookrightarrow yC \times Q'$ (since products preserve inclusions), whence plainly $\downarrow(yC \times Q) \subseteq \downarrow(yC \times Q')$. Thus covariant $\mathcal{P}$ indeed preserves inclusions. $\square$

Lemma 31. *Let $\mathcal{E}$ be a po-topos, and $\mathcal{F} \to \mathcal{E}$ a full subcategory satisfying the following conditions:*

1. *$\mathcal{F}$ is a topos,*

2. $\mathcal{F}$ is replete in $\mathcal{E}$, i.e. $F \in \mathcal{F}$ and $F \cong F'$ implies $F' \in \mathcal{F}$,

3. $\mathcal{F}$ is closed under finite limits and exponentials in $\mathcal{E}$.

4. the canonical comparison arrow $i : \Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}$ is monic.

Then $\mathcal{F}$ is a po-topos.

Proof. Since $\mathcal{F}$ is closed under finite limits, an arrow is monic there if and only if it is so in $\mathcal{E}$; thus we can take an inclusion $F' \hookrightarrow F$ in $\mathcal{F}$ to be simply an inclusion in $\mathcal{E}$. Every $\mathcal{F}$-monic then has a (unique) representing inclusion, since this is so in $\mathcal{E}$ and $\mathcal{F}$ is replete.

We can plainly take the finite products inherited from $\mathcal{E}$. For power-objects, first observe that since the classifying arrow of $\mathrm{true}_{\mathcal{F}} : 1 \to \Omega_{\mathcal{F}}$ is the mono $i : \Omega_{\mathcal{F}} \rightarrowtail \Omega_{\mathcal{E}}$, for every $F \in \mathcal{F}$ there is a mono $i^F : \Omega_{\mathcal{F}}^F \rightarrowtail \Omega_{\mathcal{E}}^F$. Now define $\mathcal{P}_{\mathcal{F}}(F)$ by taking the unique inclusion indicated in the factorization:

$$\begin{tikzcd}
(\Omega_{\mathcal{F}})^F \arrow[r, tail] \arrow[d] & (\Omega_{\mathcal{E}})^F \arrow[d, "\cong"] \\
\mathcal{P}_{\mathcal{F}}(F) \arrow[r, hook] & \mathcal{P}_{\mathcal{E}}(F)
\end{tikzcd}$$

Since the left-hand vertical is plainly also iso, and the exponential $(\Omega_{\mathcal{F}})^F$ is in $\mathcal{F}$, we have that $\mathcal{P}_{\mathcal{F}}(F)$ is also in $\mathcal{F}$. The evident resulting covariant functor $\mathcal{P}_{\mathcal{F}}$ then clearly preserves inclusions since $\mathcal{P}_{\mathcal{E}}$ does so.

□

Corollary 32. *Every Grothendieck topos is a po-topos.*

Corollary 33. *Every small topos is equivalent to a po-topos.*

Proof. Take the replete image of the yoneda embedding into presheaves. □

4.1.1 Unions

Now consider unions $P \cup Q$ of presheaves on a small category $\mathbb{C}$. Call two presheaves P and Q *disjoint* if $PC \cap QC = \emptyset$ for all C. If this is the case, then we can define a presheaf,

$$(P \cup Q)(C) = PC \cup QC$$

with the evident action on arrows, making $P \cup Q \cong P + Q$. This clearly makes $P \cup Q$ the join of P and Q with respect to inclusion. In particular:

Proposition 34. *Any two representable functors yC and yD have a union, and*

$$yC \cup yD = yC + yD\ .$$

It would now be nice to simply apply the method of corollary 33 to infer that any small topos is equivalent to a $\cup$-topos. Unfortunately, however, the union of representables is usually not (isomorphic to a) representable. We therefore have to adjust the unions by considering sheaves for a suitable topology. The construction proceeds in three steps:

1. Sheaves for the finite-epi topology on $\mathcal{E}$ have the property that the (sheafified) yoneda embedding preserves all finite coproducts and epimorphisms.

2. Let

$$Y = \coprod_E yE$$

be the *sheaf* coproduct of all the representables, and consider those $A \hookrightarrow Y$ such that $A \cong yE$ for some $E \in \mathcal{E}$. The full subcategory $\mathcal{Y} \to \mathrm{Sh}(\mathcal{E})$ of all such sheaves A is equivalent to $\mathcal{E}$ and, moreover, has inclusions with joins.

3. There is a topos structure on the sheaves that respects the inclusion ordering, but it does not induce the topos structure on the subcategory $\mathcal{Y}$; specifically, the power objects do not agree. We therefore pass to a "larger" yet still equivalent subcategory of sheaves,

$$\mathcal{Y} \hookrightarrow \mathcal{Y}' \hookrightarrow \mathrm{Sh}(E)$$

constructed from a cumulative hierarchy over Y.

For the details of the rather lengthy, sheaf-theoretic argument, the reader is referred to [**?**]. We summarize the result with the following.

Proposition 35. *Every small topos is equivalent to a $\cup$-topos.*

4.2 Category of ideals in a topos

Throughout this section, let $\mathcal{E}$ be a fixed topos with a system of structural inclusions with joins, i.e. a $\cup$-topos in the sense of definition 29 of the foregoing subsection. By an *ideal* in $\mathcal{E}$ we then mean an order ideal with respect to the inclusion ordering, i.e. a non-empty collection $\mathbf{C}$ of objects of $\mathcal{E}$, such that $A, B \in \mathbf{C}$ and $A' \hookrightarrow A$ implies $A \cup B \in \mathbf{C}$ and $A' \in \mathbf{C}$. A *morphism of ideals* consists of an order-preserving map,

$$\mathbf{f} : \mathbf{C} \to \mathbf{D}$$

together with a family of epimorphisms in $\mathcal{E}$,

$$\mathbf{f}_C : C \twoheadrightarrow \mathbf{f}(C) \qquad \text{for all } C \in \mathbf{C}$$

satisfying the "naturality" condition that, whenever $C' \hookrightarrow C$ in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, two heads, "\mathbf{f}_C"'] & \mathbf{f}(C) \\ C' \arrow[u, hook] \arrow[r, two heads, "\mathbf{f}_{C'}"'] & \mathbf{f}(C') \arrow[u, hook] \end{tikzcd}$$

With the obvious identities and composition, these morphisms form the *category of ideals* in the topos $\mathcal{E}$, denoted:

$$\mathbf{Idl}(\mathcal{E})$$

We shall usually write simply $\mathbf{f}$ for the morphism $(\mathbf{f}, (\mathbf{f}_C)_{C \in \mathbf{C}})$.

Note that because epi-inclusion factorizations in $\mathcal{E}$ are unique, the values $\mathbf{f}(C)$ and $\mathbf{f}_C$ determine the values $\mathbf{f}(C')$ and $\mathbf{f}_{C'}$ for all $C' \hookrightarrow C$. Indeed, locally (i.e. on the segment below any fixed $C \in \mathbf{C}$) the mapping $\mathbf{f}$ is essentially the same as the direct image functor

$$(\mathbf{f}_C)_! : \mathrm{Sub}(C) \to \mathrm{Sub}(\mathbf{f}(C))$$

This implies the following.

Lemma 36. *Every morphism of ideals* $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ *preserves unions,*

$$\mathbf{f}(A \cup B) = \mathbf{f}(A) \cup \mathbf{f}(B)$$

for all $A, B \in \mathbf{C}$*. Moreover,* $\mathbf{f}$ *is "locally surjective" in the sense that for every* $C \in \mathbf{C}$ *and* $D \hookrightarrow \mathbf{f}(C)$*, there is some* $C' \hookrightarrow C$ *with* $\mathbf{f}(C') = D$*.*

Next, observe that taking principal ideals determines a functor,

$$\downarrow : \mathcal{E} \to \mathbf{Idl}(\mathcal{E})$$

as follows: for any $f : A \to B$ in $\mathcal{E}$, we define:

$$\downarrow(f)(A' \hookrightarrow A) \;=\; f_!(A') \hookrightarrow B$$

where $f_!(A')$ is the image of A' under f, given by the unique epi-inclusion factorization, as indicated in:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A' \arrow[u, hook] \arrow[r, two heads, "f'"'] & f_!(A') \arrow[u, hook]
\end{tikzcd}$$

Moreover, we can then let $\downarrow(f)_{A'} = f'$, where f' is the indicated epi part of the factorization.

Proposition 37. *The principal ideal functor is full and faithful.*

Proof. Given any morphism of ideals $\mathbf{f} : \downarrow(A) \to \downarrow(B)$, consider the composite map:

$$T(\mathbf{f}) = i \circ \mathbf{f}_A : A \twoheadrightarrow \mathbf{f}(A) \hookrightarrow B$$

where $i : \mathbf{f}(A) \hookrightarrow B$ is the canonical inclusion. Then by naturality, the value of $\mathbf{f}$ on every $A' \hookrightarrow A$ is just $T(\mathbf{f})_!(A')$, and $\mathbf{f}_{A'} = \downarrow(T(\mathbf{f}))_{A'} : A' \to T(\mathbf{f})_!(A')$. Thus $\mathbf{f} = \downarrow(T(\mathbf{f}))$. Since clearly $T(\downarrow(f)) = f$ for any $f : A \to B$, this proves the proposition. $\square$

Our objective in this subsection is to prove the following.

Theorem 38. *For any topos $\mathcal{E}$, the category $\mathbf{Idl}(\mathcal{E})$ is a category of classes with $\mathcal{E}$ as the full subcategory of sets, under the principal ideal embedding,*

$$\downarrow : \mathcal{E} \cong \mathcal{S}_{\mathcal{C}} \hookrightarrow \mathbf{Idl}(\mathcal{E})$$

Thus, in particular, the small objects in $\mathbf{Idl}(\mathcal{E})$ are exactly the principal ideals.

The proof requires a rather lengthy verification of the axioms for class structure. We shall give an outline in the form of a series of lemmas, leaving the detailed verification to the interested reader (a worthwhile exercise, as some of the proofs work out quite pleasantly).

Lemma 39. *The category $\mathbf{Idl}(\mathcal{E})$ of ideals satisfies axiom (C), i.e. it is a regular category with coproducts and dual images.*

Proof. The terminal ideal is $\downarrow(1)$.

The product of two ideals $\mathbf{A}$ and $\mathbf{B}$ is:

$$\mathbf{A} \times \mathbf{B} = \{C \hookrightarrow A \times B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$$

which is an ideal because, if $C \hookrightarrow A \times B$ and $C' \hookrightarrow A' \times B'$, then we have:

$$C \cup C' \hookrightarrow (A \times B) \cup (A' \times B') \subseteq (A \cup A') \times (B \cup B')$$

Given $\mathbf{f}, \mathbf{g} : \mathbf{A} \rightrightarrows \mathbf{B}$, their equalizer is (the evident inclusion into $\mathbf{A}$ of):

$$E(\mathbf{f}, \mathbf{g}) = \{A \in \mathbf{A} \mid \mathbf{f}(A) = \mathbf{g}(A), \mathbf{f}_A = \mathbf{g}_A\}$$

which can be shown to be an ideal.

The regular epis $\mathbf{e} : \mathbf{A} \twoheadrightarrow \mathbf{B}$ in $\mathbf{Idl}(\mathcal{E})$ are characterized by the mapping $\mathbf{A} \ni A \mapsto \mathbf{e}(A) \in \mathbf{B}$ being surjective. Given any morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, we can take the image (the coequalizer of the kernel pair) of $\mathbf{f}$ to be the subcollection:

$$\{\mathbf{f}(A) \mid A \in \mathbf{A}\} \subseteq \mathbf{B}$$

This is an ideal because $\mathbf{f}$ preserves unions. These images are easily seen to be stable under pullbacks using the following description of the latter, which we include for good measure.

Given $\mathbf{f} : \mathbf{A} \to \mathbf{C}$ and $\mathbf{g} : \mathbf{B} \to \mathbf{C}$, the pullback consists of (the evident projection morphisms on) the object:

$$\mathbf{A} \times_{\mathbf{C}} \mathbf{B} = \{D \hookrightarrow A \times B \mid A \in \mathbf{A}, B \in \mathbf{B}, \mathbf{f}(A) = \mathbf{g}(B), \mathbf{f}_A \circ d_1 = \mathbf{g}_B \circ d_2\}$$

where $d_1 : D \to A$ and $d_2 : D \to B$ are the two projections of $D \hookrightarrow A \times B$, as indicated in the following commutative diagram:

$$\begin{tikzcd}
D \arrow[rr, "d_2"] \arrow[dd, "d_1"'] \arrow[dr, leftrightarrow] & & B \arrow[d, "g_B"] \\
& A \times B \arrow[ur] \arrow[dl] & g(B) \arrow[d, hook] \\
A \arrow[r, twoheadrightarrow, "f_A"'] & f(A) \arrow[r, hook] \arrow[ur, dash, "="'] & C
\end{tikzcd}$$

The coproduct of ideals $\mathbf{A}$ and $\mathbf{B}$ is simply the ideal:

$$\{A + B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$$

with the inclusion morphisms $A \mapsto A + 0$ and $B \mapsto 0 + B$.

Finally, the dual image along $\mathbf{f} : \mathbf{C} \to \mathbf{D}$ of a subideal $\mathbf{A} \rightarrowtail \mathbf{C}$ is calculated as follows. Without loss of generality, we can assume that $\mathbf{A} \subseteq \mathbf{C}$. Then let:

$$\mathbf{f}_*(\mathbf{A}) = \{D \in \mathbf{D} \mid \mathbf{f}(C) \subseteq D \Rightarrow C \in \mathbf{A},\ \text{for all } C \in \mathbf{C}\}$$

To see that this works, note that the condition in the braces is equivalent to:

$$\mathbf{f}^*(\downarrow(D)) \subseteq \mathbf{A}$$

□

Definition 40. A morphism of ideals $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is *small* if its mapping part has a right adjoint $\mathbf{f} \dashv \mathbf{f}^{-1}$. Thus, explicitly, if for every $B \in \mathbf{B}$ there is some $\mathbf{f}^{-1}(B) \in \mathbf{A}$ such that for all $A \in \mathbf{A}$:

$$\mathbf{f}(A) \hookrightarrow B \quad \text{iff} \quad A \hookrightarrow \mathbf{f}^{-1}(B)$$

When this holds, the right adjoint $\mathbf{f}^{-1}$ is called the *inverse image* of $\mathbf{f}$.

Observe that a morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is therefore small just in case, for every $B \in \mathbf{B}$, the set,

$$\{A \in \mathbf{A} \mid f(A) \subseteq B\}$$

has a greatest element. The proof of the following is now almost immediate.

Lemma 41. *The following characterizations hold in the category $\mathbf{Idl}(\mathcal{E})$:*

1. *The* small objects *are exactly the principal ideals $\downarrow E$ for $E \in \mathcal{E}$.*
2. *Every morphism $\mathbf{f} : \downarrow E \to \downarrow F$ between small objects is of the form:*

   $$\mathbf{f} = \downarrow f$$

   for a unique $f : E \to F$ in $\mathcal{E}$, and is therefore small.
3. *The small subobjects $\mathbf{C}' \rightarrowtail \mathbf{C}$ are exactly those isomorphic to subobjects of the form $\downarrow C \subseteq \mathbf{C}$ for some $C \in \mathbf{C}$.*
4. *A morphism $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is small if, whenever $S \rightarrowtail \mathbf{B}$ is a small subobject, then $f^*(S) \rightarrowtail \mathbf{A}$ is also small.*

Lemma 42. *The small maps so defined satisfy axiom (S).*

Proof. (S1) It is clear that these small maps for a subcategory, since adjoints compose.

(S2) Suppose we have the situation:

$$\begin{tikzcd}
\mathbf{A} \times_{\mathbf{C}} \mathbf{B} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] & \mathbf{B} \arrow[d, "\mathbf{g}"] \\
\mathbf{A} \arrow[r, "\mathbf{f}"'] & \mathbf{C}
\end{tikzcd}$$

with $\mathbf{g}$ small. To show $\mathbf{p}$ small, we need to find $\mathbf{p}^{-1}(A) \in \mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ for each $A \in \mathbf{A}$. Consider the pullback diagram:

$$\begin{tikzcd}
T'' \arrow[r, two heads] \arrow[d, two heads] & T \arrow[d, two heads, "\mathbf{g}_T"] \\
T' \arrow[r, two heads] \arrow[d, hook] & \mathbf{g}(T) \arrow[d, hook] \\
A \arrow[r, two heads, "\mathbf{f}(A)"'] & \mathbf{f}(A)
\end{tikzcd}$$

in which $T = \mathbf{g}^{-1}(\mathbf{f}(A))$. It follows that the subobject $T'' \subseteq T' \times T$ is in the pullback $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$. We then set:

$$\mathbf{p}^{-1}(A) = T''$$

(S3) Given $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ and $T \hookrightarrow A \times B$ in $\mathbf{C} \times \mathbf{C}$, we take the evident pullback:

$$\begin{tikzcd}
T' \arrow[rr] \arrow[d, hook] & & T \arrow[d, hook] \\
A \cap B \arrow[r, tail, "\Delta_{A \cap B}"'] & (A \cap B) \times (A \cap B) \arrow[r, tail] & A \times B
\end{tikzcd}$$

We then set:

$$\Delta^{-1}(T) = T'$$

(S4) Given the following situation, with $\mathbf{g}$ small:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, two heads, "\mathbf{e}"] \arrow[dr, "\mathbf{g}"'] & \mathbf{B} \arrow[d, "\mathbf{f}"] \\
& \mathbf{C}
\end{tikzcd}$$

for $C \in \mathbf{C}$, we set:

$$\mathbf{f}^{-1}(C) = \mathbf{e}(\mathbf{g}^{-1}(C))$$

(S5) Given a coproduct diagram, with $\mathbf{f}$ and $\mathbf{g}$ small:

$$\begin{tikzcd}
\mathbf{A} \arrow[r] \arrow[dr, "\mathbf{f}"'] & \mathbf{A}+\mathbf{B} \arrow[d, "{[\mathbf{f},\mathbf{g}]}"] & \mathbf{B} \arrow[l] \arrow[dl, "\mathbf{g}"] \\
& \mathbf{C} &
\end{tikzcd}$$

for $C \in \mathbf{C}$, we set:

$$[\mathbf{f},\mathbf{g}]^{-1}(C) = \mathbf{f}^{-1}(C) + \mathbf{g}^{-1}(C)$$

$\square$

Definition 43. To define the *small powerobjects*, given any ideal $\mathbf{C}$, we set:

$$\mathcal{P}\mathbf{C} = \{S \hookrightarrow \mathcal{P}C \mid C \in \mathbf{C}\}$$

The *elementhood relation* is defined by:

$$\in_{\mathbf{C}} = \{S \hookrightarrow \in_C \mid C \in \mathbf{C}\}$$

with the canonical mono $\in_{\mathbf{C}} \rightarrowtail \mathbf{C} \times \mathcal{P}\mathbf{C}$ defined by factoring out the image S' indicated in the following:

$$\begin{tikzcd}
S \arrow[r, "\cong"] \arrow[d, hook] & S' \arrow[d, hook] \\
\in_C \arrow[r, tail] & C \times \mathcal{P}C
\end{tikzcd}$$

Lemma 44. *These small powerobjects satisfy axiom (P).*

Proof. (P1) Suppose we have a small relation $\mathbf{R} \rightarrowtail \mathbf{C} \times \mathbf{A}$, with small second projection,

$$\mathbf{r} : \mathbf{R} \rightarrowtail \mathbf{C} \times \mathbf{A} \to \mathbf{A}$$

Without loss of generality, we can assume that $\mathbf{R} \subseteq \mathbf{C} \times \mathbf{A}$. We then define the characteristic map:

$$\mathbf{f} : \mathbf{A} \to \mathcal{P}\mathbf{C}$$

as follows. For $A \in \mathbf{A}$, take $\mathbf{r}^{-1}(A) \in \mathbf{R}$, which therefore has the form $\mathbf{r}^{-1}(A) \subseteq C \times A$ for some $C \in \mathbf{C}$ and $A \in \mathbf{A}$. Now factor the corresponding characteristic map $f : A \to \mathcal{P}C$, to get:

$$f : A \twoheadrightarrow S \hookrightarrow \mathcal{P}C$$

We then set:

$$\mathbf{f}(A) = S$$

and we let $\mathbf{f}_A$ be the epi in the above displayed factorization.

(P2) The subset relation $\subseteq_{\mathbf{C}} \rightarrowtail \mathcal{P}\mathbf{C} \times \mathcal{P}\mathbf{C}$ is given by the subideal:

$$\subseteq_{\mathbf{C}} = \{S \hookrightarrow \subseteq_C \hookrightarrow \mathcal{P}C \times \mathcal{P}C \mid C \in \mathbf{C}\}$$

with the evident inclusion. To see that the second projection

$$\mathbf{q} : \subseteq_{\mathbf{C}} \to \mathcal{P}\mathbf{C} \times \mathcal{P}\mathbf{C} \to \mathcal{P}\mathbf{C}$$

is small, take any $S \hookrightarrow \mathcal{P}C$ in $\mathcal{P}\mathbf{C}$, and form the pullback:

$$\begin{tikzcd}
S' \arrow[rr] \arrow[d, hook] & & S \arrow[d, hook] \\
\subseteq_C \arrow[r, hook] & \mathcal{P}C \times \mathcal{P}C \arrow[r, "p_2"'] & \mathcal{P}C
\end{tikzcd}$$

We then set:

$$\mathbf{q}^{-1}(S) = S'$$

$\square$

To conclude the proof of theorem 38, it now remains only to show that $\mathbf{Idl}(\mathcal{E})$ has a universal object.

Lemma 45. *The total ideal,*

$$\mathbf{U} = \{E \mid E \in \mathcal{E}\}$$

satisfies axiom (U).

Proof. Obviously, $\mathbf{C} \subseteq \mathbf{U}$ for every ideal $\mathbf{C}$. $\square$

4.3 Topos models of bIST

We conclude this section by observing that the set theory bIST can now be modelled in *any* topos $\mathcal{E}$ with a natural numbers object. This fact is more surprising and subtle than it may at first seem, for a similar statement holds in a very straightforward sense for weaker elementary set theories like bounded Zermelo. Such theories, in which all quantifiers can be bounded, can be interpreted quite directly using the internal logic of $\mathcal{E}$, with the objects of $\mathcal{E}$ as the sets. By contrast, the theory bIST involves unbounded variables, which range over all sets, particularly in the axiom of replacement. In order to interpret such formulas, we need to have a structure with an "object of sets" U (respectively $\mathcal{P}U$) over which unbounded variables are interpreted. This role is played by the universal ideal $\mathbf{U} = \mathcal{E}$ in the category $\mathbf{Idl}(\mathcal{E})$.

Proposition 46. *For any topos $\mathcal{E}$, the category $\mathbf{Idl}(\mathcal{E})$ of ideals in $\mathcal{E}$ has a model of* bIST$^-$*, namely the universal ideal $\mathcal{E}$ itself:*

$$\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \mathrm{bIST}^-$$

Moreover, if $\mathcal{E}$ has a natural numbers object, then the axiom of infinity is also satisfied:

$$\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \mathrm{bIST}$$

Proof. The first statement follows simply from the fact that $\mathbf{Idl}(\mathcal{E})$ is a category with class structure by theorem 38, and every such category models bIST$^-$ by proposition 19. To verify the axiom of infinity if $\mathcal{E}$ has a natural numbers object,

$$1 \xrightarrow[o]{} N \xrightarrow[s]{} N$$

consider the structure,

$$\downarrow 1 \xrightarrow[\downarrow o]{} \downarrow N \xrightarrow[\downarrow s]{} \downarrow N$$

in $\mathbf{Idl}(\mathcal{E})$. We then have a mono:

$$\downarrow N + \downarrow 1 \xrightarrow[\cong]{} \downarrow (N+1) \rightarrowtail_{\downarrow [o,s]} \downarrow N$$

since the functor $\downarrow : \mathcal{E} \to \mathbf{Idl}(\mathcal{E})$ preserves coproducts and monos. $\square$

A warning is probably in order regarding how to understand this proposition. It implies that every topos $\mathcal{E}$ satisfies e.g. the axiom of replacement *with respect to the canonical interpretation of* bIST *in* $\mathbf{Idl}(\mathcal{E})$. This is not the same as saying that replacement will always be satisfied whenever $\mathcal{E}$ occurs in an ambient category in such a way that that axiom can be interpreted. The example of the sets of rank less than or equal to $\omega+\omega$, written $\mathbf{Sets}_{\omega+\omega}$, among all sets **Sets** is instructive: by the foregoing proposition, we have:

$$\mathbf{Sets}_{\omega+\omega} \models_{\mathbf{Idl}(\mathbf{Sets}_{\omega+\omega})} \text{Replacement}$$

But it is of course *not* the case that:

$$\mathbf{Sets}_{\omega+\omega} \models_{\mathbf{Sets}} \text{Replacement}$$

since the set $\{\mathcal{P}^n(\omega) \mid n \in \omega\}$ has rank $\omega+\omega+1$. How can replacement hold in the model in $\mathbf{Idl}(\mathbf{Sets}_{\omega+\omega})$? Briefly, the class function $n \mapsto \mathcal{P}^n(\omega)$, to which we would apply replacement, does not exist in that model; thus there is no natural numbers object for classes (only one for sets).

This makes it plain that the question of which elementary formulas are satisfied by a topos $\mathcal{E}$ depends not only on $\mathcal{E}$, but also on the ambient category of classes used to interpret the formulas. Indeed, the models $\mathbf{Idl}(\mathcal{E})$ have some rather special properties, as the following proposition also shows.

Proposition 47. *For any topos $\mathcal{E}$, the ideals model satisfies the axiom scheme of collection,*

$$\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \text{Coll}$$

which says that for any total relation R on a set A, there is a set B contained in the "range" of R,

$$B \subseteq \{y \mid \exists x \in A. R(x,y)\ \}$$

such that the restriction of R to $A \times B$ is still total on A.

More formally, the canonical model of bIST *in the category* $\mathbf{Idl}(\mathcal{E})$ *of ideals in $\mathcal{E}$ satisfies all instances of the following axiom scheme:*

$$\text{(Coll)} \qquad \mathsf{S}(z) \wedge (\forall x \in z. \exists y. \varphi) \rightarrow \\ \exists w. (\mathsf{S}(w) \wedge (\forall x \in z. \exists y \in w. \varphi) \wedge (\forall y \in w. \exists x \in z. \varphi))$$

Proof. Consider the following diagram in $\mathbf{Idl}(\mathcal{E})$:

$$\begin{tikzcd}
\mathbf{A} \arrow[dd, equal] & \mathbf{R}' \arrow[l, two heads, "\mathbf{p}'"'] \arrow[r, two heads, "\mathbf{q}'"] \arrow[d, tail] & \mathbf{B} \arrow[dd, tail] \\
 & \mathbf{R} \arrow[dl, "\mathbf{p}"'] \arrow[d, tail] \arrow[dr, "\mathbf{q}"] & \\
\mathbf{A} & \mathbf{A} \times \mathbf{U} \arrow[l] \arrow[r] & \mathbf{U}
\end{tikzcd}$$

in which $\mathbf{A} = \downarrow A$ is small. Given such an $\mathbf{R}$ with first projection $\mathbf{p}$ epic, we seek small $\mathbf{B}$ so that the restriction $\mathbf{R}'$ of $\mathbf{R}$ still has epic first projection $\mathbf{p}'$. But for *any* epimorphism of ideals $\mathbf{e} : \mathbf{X} \twoheadrightarrow \downarrow A$, there is a small $\mathbf{E} \subseteq \mathbf{X}$ with epic restriction $\mathbf{e}' : \mathbf{E} \subseteq \mathbf{X} \twoheadrightarrow \downarrow A$, namely let $\mathbf{E} = \downarrow E$ for some $E \in \mathbf{X}$ with $\mathbf{e}(E) = A$. Applying this fact to $\mathbf{p}$, we take small $\mathbf{R}' \subseteq \mathbf{R}$, and then let $\mathbf{B}$ be the image of $\mathbf{q}$ restricted to $\mathbf{R}'$, as indicated in the above diagram. $\square$

It is now resonable to ask, to what extent the completeness theorem 22 for bIST with respect to categories $\mathcal{C}$ with class structure really requires the full range of such categories. Could consideration be restricted to just the "standard" models of the form $\mathbf{Idl}(\mathcal{E})$ for toposes $\mathcal{E}$? Observe that such models do indeed suffice to violate the stronger separation scheme involving also the set predicate $\mathsf{S}(x)$, since the map $\mathcal{P}\mathbf{U} \rightarrowtail \mathbf{U}$ need not be small (as the reader can show). In the next and final section, we shall show that simply adding the axiom scheme of collection to the theory bIST is indeed enough to make it complete with respect to models in ideals.

Remark 48. With respect to the set theory IST with full separation mentioned in remark 21, the situation is as follows. It can be shown that for

Grothendieck and realizability toposes one can find systems of inclusions satisfying the *axiom of boundedness*, which requires that every small set of objects has an upper bound in the inclusion ordering. If a structural system of inclusions satisfies this axiom, then one can modify the construction of $\mathbf{Idl}(\mathcal{E})$ by requiring that all ideals be *small directed*, in the sense that every small subset of an ideal $\mathbf{I}$ has an upper bound in $\mathbf{I}$. One can show that the corresponding model $\mathbf{Idl}_{\mathrm{sd}}(\mathcal{E})$ then validates the axioms of IST.

5 Topos completeness of bIST

Throughout this section, we shall use the following terminological conventions for brevity: by a *logical* functor $L : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with class structure we mean a functor that preserves the structure mentioned in the axioms (C), (S), (P), and (U). Specifically, such a functor L preserves finite limits and coproducts, regular epimorphisms and dual images, small maps and powerobjects, and the universe $\mathcal{U}$. The term *class category* will refer to a category with class structure and an infinite object. All toposes will be assumed to have natural numbers objects (NNO). We shall consider such toposes in relation to the set theory bIST (with the axiom of infinity), and class categories, but everything we say will apply as well to arbitrary toposes, the theory bIST^-, and arbitrary categories with class structure. Finally, let us write,

$$\mathrm{bIST_C} = \mathrm{bIST} + \mathrm{Coll}$$

for basic intuitionistic set theory with the axiom scheme of collection, as displayed in proposition 47.

We have already seen that every topos $\mathcal{E}$ gives rise to a class category $\mathbf{Idl}(\mathcal{E})$ in which $\mathrm{bIST_C}$ has a model. We aim to show now that such models in ideals actually suffice for provability in $\mathrm{bIST_C}$, in the following sense.

Theorem 49. *For any formula φ, if $\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \varphi$ for all toposes $\mathcal{E}$, then:*

$$\mathrm{bIST_C} \vdash \varphi$$

Moreover, the direct forcing semantics $\mathcal{E} \Vdash \varphi$ over a topos $\mathcal{E}$, as defined in [?], can be shown to agree with the internal class semantics $\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \varphi$ in the category $\mathbf{Idl}(\mathcal{E})$ of ideals in $\mathcal{E}$. We therefore have the following result, which was already announced in *ibid.*:

Corollary 50. *The set theory* $\mathrm{bIST_C}$ *is complete for forcing semantics over topoi. Specifically, for any formula* φ*, if* $\mathcal{E} \Vdash \varphi$ *for all toposes* $\mathcal{E}$*, then:*

$$\mathrm{bIST_C} \vdash \varphi$$

The proof of the theorem proceeds by the following three steps:

Step 1: By theorem 22, bIST is known to be complete for models in class categories. The same holds for $\mathrm{bIST_C}$ with respect to models in class categories $\mathcal{C}$ that satisfy the axiom of collection.

Step 2: Any class category $\mathcal{C}$ satisfying the axiom of collection has a faithful logical functor,

$$\mathcal{C} \rightarrowtail \mathcal{C}'$$

into another one $\mathcal{C}'$ that is "saturated" with small objects.

Step 3: The saturated class category $\mathcal{C}'$ has a faithful logical functor,

$$\mathcal{C}' \rightarrowtail \mathbf{Idl}(\mathcal{E})$$

into the category of ideals in a topos $\mathcal{E}$.

The topos $\mathcal{E}$ in step 3 is simply the subcategory $\mathcal{S}_{\mathcal{C}'}$ of small objects in $\mathcal{C}'$. Step 2 is required to ensure there are enough such objects.

5.1 Class categories with collection

Recall the set theoretic axiom scheme of collection (called "strong collection" in [**?**]):

$$\text{(Coll)} \qquad \mathsf{S}(z) \wedge (\forall x \in z.\exists y.\varphi) \rightarrow \\ \exists w.(\mathsf{S}(w) \wedge (\forall x \in z.\exists y \in w.\varphi) \wedge (\forall y \in w.\exists x \in z.\varphi))$$

which is supposed to hold for *arbitrary* formulas φ. It is not hard to show that this scheme is satisfied by the model $\mathcal{U}$ of bIST in a category of classes $\mathcal{C}$ if the following condition is satisfied:

Definition 51. A class category $\mathcal{C}$ is said to *have collection* if, in every slice category $\mathcal{C}/I$, given a relation $R \rightarrowtail A \times Y$ with A small and epic first projection,

$$\begin{tikzcd} R \arrow[r, tail] \arrow[dr] & A \times Y \arrow[d] \\ & A \end{tikzcd}$$

the "object of collection sets",

$$S = \{w \in \mathcal{P}Y \mid \forall x \in A.\exists y \in w.R(x,y) \wedge \forall y \in w.\exists x \in A.R(x,y)\} \rightarrowtail \mathcal{P}Y$$

has global support. That is, the unique map $S \to 1$ is epic.

The class category completeness theorem 22 for bIST also holds for $\mathrm{bIST_C}$ with regard to class categories with collection:

Lemma 52. *If an elementary formula φ (in the language $\{\mathsf{S}, \in\}$) is valid in every category of classes $\mathcal{C}$ with collection, then it is provable in the elementary set theory* $\mathrm{bIST_C}$.

Proof. It is clear that the syntactic category of classes $\mathcal{C}_0$ constructed in the proof of theorem 22 has collection if that scheme is added to the theory bIST. □

Thus we already have the required Step 1. The other two steps will involve construction of suitable class categories. One of the great virtues of the algebraic approach to set theory is that the models it produces are closed under the typical algebraic constructions of finite limits and filtered colimits. We shall make use of this fact to construct models with special properties as limits and colimits of suitable diagrams. First we must verify that such constructions are indeed permitted.

Lemma 53. *1. If $\mathcal{C}$ is a class category with collection, so is the slice category $\mathcal{C}/C$ for any object C.*

2. If $(\mathcal{C}_i)_{i \in I}$ is a family of class categories, for any index set I, then the product category,

$$\prod_{i \in I} \mathcal{C}_i$$

is also a class category, and is a product in the category of class categories and logical functors. Moreover, if each $\mathcal{C}_i$ has collection, then so does $\prod_{i \in I} \mathcal{C}_i$.

3. *If $\mathcal{C} : \mathbb{J} \to \mathbf{Cat}$ is a diagram of class categories, for any small, filtered index category $\mathbb{J}$, then the colimit category,*

$$\varinjlim_{j \in \mathbb{J}} \mathcal{C}_j$$

is also a class category, and is a colimit in the category of class categories and logical functors. Moreover, if each $\mathcal{C}_j$ has collection, then so does $\varinjlim_{j \in \mathbb{J}} \mathcal{C}_j$.

Proof. Inspection. □

5.2 Saturating a class category

Definition 54. A class category $\mathcal{C}$ is *saturated* if it satisfies the following conditions:

Small covers: given any epi $C \twoheadrightarrow A$ with A small, there is a small subobject $B \rightarrowtail C$ such that the restriction $B \rightarrowtail C \twoheadrightarrow A$ is epic.

$$\begin{tikzcd} B \arrow[r, tail, dotted] \arrow[dr, two heads] & C \arrow[d, two heads] \\ & A \end{tikzcd}$$

Small generators: given any $B \rightarrowtail C$, if every small subobject $A \rightarrowtail C$ factors through B, then $B = C$.

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[dr, tail] & B \arrow[d, tail] \\ & C \end{tikzcd}$$

Lemma 55. *Every class category $\mathcal{C}$ has a faithful, logical functor,*

$$\mathcal{C} \rightarrowtail \mathcal{C}^*$$

into a class category $\mathcal{C}^*$ with small generators. Moreover, if $\mathcal{C}$ has collection, then so does $\mathcal{C}^*$.

Proof. First, suppose we have any $B \rightarrowtail C$ in $\mathcal{C}$, and consider the image $B^* \rightarrowtail C^*$, in the slice category $\mathcal{C}/\mathcal{P}(C)$, under the pullback functor,

$$* : \mathcal{C} \to \mathcal{C}/\mathcal{P}(C)$$

along $\mathcal{P}(C) \to 1$. In $\mathcal{C}/\mathcal{P}(C)$ we have a generic point $g : 1 \to (\mathcal{P}(C))^* \cong \mathcal{P}(C^*)$, which therefore corresponds to a generic small subobject $G \rightarrowtail C^*$. If there is a factorization $G \rightarrowtail B^* \rightarrowtail C^*$, then there is also one $g : 1 \to (\mathcal{P}B)^* \rightarrowtail (\mathcal{P}C)^*$. But this would imply $(\mathcal{P}B)^* = (\mathcal{P}C)^*$, whence $B^* = C^*$, and so $B = C$, since * reflects isomorphisms (because $\mathcal{P}(C)$ has a global section $1 \to \mathcal{P}(C)$). Thus, in sum, if $B \rightarrowtail C$ is proper in $\mathcal{C}$, then by passing to $\mathcal{C}/\mathcal{P}(C)$, there is a small subobject $G \rightarrowtail C^*$ "separating" B^* and C^*, in the sense that it does not factor through $B^* \rightarrowtail C^*$.

Now let:

$$\begin{aligned} \mathcal{C}_0 &= \mathcal{C} \\ \mathcal{C}_{n+1} &= \prod_{X \in \mathcal{C}_n} \mathcal{C}_n/\mathcal{P}(X) \\ \mathcal{C}^* &= \lim_{n \to \infty} \mathcal{C}_n \end{aligned}$$

The colimit is taken along the succesive "diagonal" functors,

$$\Delta_n : \mathcal{C}_n \rightarrowtail \prod_{X \in \mathcal{C}_n} \mathcal{C}_n/\mathcal{P}(X)$$

each of which is faithful and logical. Observe that by the argument just given, if in $\mathcal{C}_n$ one has $B \rightarrowtail C$ proper, then in $\mathcal{C}_{n+1}$ there is a small $A \rightarrowtail \Delta_n C$ that does not factor through $\Delta_n B \rightarrowtail \Delta_n C$, namely the subobject:

$$A(X) = \begin{cases} G \rightarrowtail C^* & X = C \\ 0 \rightarrowtail C^* & \text{otherwise} \end{cases}$$

Since the colimit $\mathcal{C}^*$ is taken over the sequence of faithful, logical functors Δ_n, any proper $[B] \rightarrowtail [C]$ in $\mathcal{C}^*$ comes from a proper $B \rightarrowtail C$ in some $\mathcal{C}_n$, which is therefore separated by a small $A \rightarrowtail \Delta_n C$ in $\mathcal{C}_{n+1}$. Thus in $\mathcal{C}^*$ we have the required small $[A] \rightarrowtail [C]$ separating $[B] \rightarrowtail [C]$.

Finally, if $\mathcal{C}$ has collection, then so does $\mathcal{C}^*$ by lemma 53. $\square$

Lemma 56. *Every class category $\mathcal{C}$ has a faithful, logical functor,*

$$\mathcal{C} \rightarrowtail \mathcal{C}^\sharp$$

into a class category $\mathcal{C}^\sharp$ in which 1 is projective. Moreover, if $\mathcal{C}$ has collection, then $\mathcal{C}^\sharp$ has collection and small covers.

Proof. We shall construct $\mathcal{C}^\sharp$ as a colimit of slices $\mathcal{C}/X$ of $\mathcal{C}$ over a suitable index category $\mathbb{J}$. This is an instance of a general construction, not further analysed here, which embedds any small regular category into one in which the terminal object 1 is projective.

To define the index category $\mathbb{J}$, we begin with a jointly faithful, set-indexed family $(F_i : \mathcal{C} \to \mathbf{Sets})_{i \in I}$ of regular functors. Thus each functor $F_i : \mathcal{C} \to \mathbf{Sets}$ preserves finite limits and regular epimorphisms, and the resulting canonical functor,

$$F = (F_i)_{i \in I} : \mathcal{C} \to \mathbf{Sets}^I$$

is both regular and faithful. Such a family exists by Deligne's theorem [**?**].

Next, we form the (covariant) *category of elements*,

$$\int_{X \in \mathcal{C}} F(X)$$

which, in this case, is defined as follows:

objects: pairs (x, X) where $X \in \mathcal{C}$ and $x \in F(X)$,

arrows: triples $f : (x, X) \to (y, Y)$ where $f : X \to Y$ and $F(f)(x) = y$.

One sees easily that $\int_{\mathcal{C}} F$ is co-filtered, since $\mathcal{C}$ has finite limits and F preserves them.

Now define $\mathcal{C}_F$ to be the colimit:

$$\mathcal{C}_F = \varinjlim_{(x,X) \in \int_{\mathcal{C}} F} \mathcal{C}/X$$

of the composite contravariant functor,

$$\begin{array}{ccccc} \int_{\mathcal{C}} F & \to & \mathcal{C} & \to & \mathbf{Cat} \\ (x, X) & \mapsto & X & \mapsto & \mathcal{C}/X \end{array}$$

where the first factor is the evident projection, and the second is the (pseudo-) functor taking $f : X \to Y$ in $\mathcal{C}$ to the pullback functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$.

Since the colimit is filtered, $\mathcal{C}_F$ is a class category by lemma 53. There is a logical functor $\pi : \mathcal{C} \to \mathcal{C}_F$, namely the canonical arrow $\pi_{(*,1)}$ to the colimit associated with the object $(*, 1) \in \int_{\mathcal{C}} F$.

Next, observe that there is a factorization,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] \arrow[dr, "F"'] & \mathcal{C}_F \arrow[d, "F'"] \\ & \mathbf{Sets} \end{tikzcd}$$

The functor $F' : \mathcal{C}_F \to \mathbf{Sets}$ is the unique (up to isomorphism) canonical one from the colimit $\mathcal{C}_F$, determined by the family of functors,

$$x^\sharp : \mathcal{C}/X \to \mathbf{Sets} \qquad \text{for } (x, X) \in \int_{\mathcal{C}} F$$

classifying the various elements $x \in F(X)$. These commute with the arrows $f^* : \mathcal{C}/Y \to \mathcal{C}/X$, since each such $f : X \to Y$ has $F(f)(x) = y$. The situation is depicted in the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & & \mathcal{C}/X \arrow[r, "\pi_{(x,X)}"] \arrow[dr, "x^\sharp"] & \mathcal{C}_F \arrow[d, "F'"] & F(X) \arrow[d, "F(f)"] & x \\ Y & \mathcal{C} \arrow[r] \arrow[ur] & \mathcal{C}/Y \arrow[u, "f^*"'] \arrow[r, "y^\sharp"'] & \mathbf{Sets} & F(Y) & y \end{tikzcd}$$

The logical functor $\pi : \mathcal{C} \to \mathcal{C}_F$ is therefore faithful, since $F : \mathcal{C} \to \mathbf{Sets}$ is faithful and $F = F' \circ \pi$.

We next show that 1 is projective in the colimit $\mathcal{C}_F$. To that end, take an epi $e : E \twoheadrightarrow 1$ in $\mathcal{C}_F$. Then $e = [e']$ and $E = [E']$ for an epi $e' : E' \to 1$ in some $\mathcal{C}/X$, associated to some $(x, X) \in \int_{\mathcal{C}} F$. This is then an epi $e' : E' \twoheadrightarrow X$ in $\mathcal{C}$, which F takes to an epi $F(e') : F(E') \twoheadrightarrow F(X)$ in $\mathbf{Sets}^I$. Since $x \in F(X)$ and $F(e')$ is epic, there is some $y \in F(E')$ with $F(e')(y) = x$. Here we are using the fact that 1 is projective in $\mathbf{Sets}^I$. But then there is an arrow $e' : (y, E') \to (x, X)$ in $\int_C F$. Now consider the pullback diagram:

$$\begin{tikzcd}
E'' \arrow[r, two heads] \arrow[d, two heads, "e''"'] & E' \arrow[d, two heads, "e'"] \\
E' \arrow[r, two heads, "e'"'] & X
\end{tikzcd}$$

We thus have $e = [e'] = [e'']$ and $E = [E'] = [E'']$. But e'' has a section, namely $\langle 1_{E'}, 1_{E'} \rangle$. Thus e has a section too.

Finally, we show that $\mathcal{C}_F$ has small covers if $\mathcal{C}$ has collection. But this follows easily, since $\mathcal{C}_F$ then also has collection by lemma 53. Then for any epi $e : C \twoheadrightarrow A$ with A small, consider the graph $E \rightarrowtail A \times C$. By collection, the "object of collection sets",

$$S = \{w \in \mathcal{P}C \mid \forall x \in A.\exists y \in w.R(x, y) \wedge \forall y \in w.\exists x \in A.R(x, y)\} \rightarrowtail \mathcal{P}C$$

has global support, $S \twoheadrightarrow 1$. Since 1 is projective, there is a global section $b : 1 \to S$, which therefore determines a small subobject $B \rightarrowtail C$ such that the composite $B \rightarrowtail C \twoheadrightarrow A$ is epic. $\square$

The foregoing two lemmas now yield the following result, which was the desired Step 2.

Proposition 57. *Every class category $\mathcal{C}$ with collection has a faithful, logical functor,*

$$\mathcal{C} \rightarrowtail \mathcal{C}'$$

into a saturated class category $\mathcal{C}'$, i.e. one that has small covers and small generators.

Proof. Given the class category $\mathcal{C}$ with collection, let $\mathcal{C}'$ be the colimit of the sequence of faithful, logical functors,

$$\mathcal{C} \rightarrowtail \mathcal{C}^* \rightarrowtail \mathcal{C}^{*\sharp} \rightarrowtail \mathcal{C}^{*\sharp*} \rightarrowtail \mathcal{C}^{*\sharp*\sharp} \rightarrowtail \ldots$$

where $*$ adds small generators as in lemma 55, and $\sharp$ adds small covers as in lemma 56. □

5.3 The derivative functor

Let $\mathcal{C}$ be a class category. In this subsection we require there to be a system of inclusions $C \hookrightarrow D$ on $\mathcal{C}$, satisfying the same conditions as in definition 29, including the existence of joins $C \cup D$ and compatibility with the product and powerobject structures. We leave it as an easy exercise for the reader to determine such a structure using the universe $\mathcal{U}$. Note that the subcategory $\mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ of small objects is then a $\cup$-topos with respect to the same inclusions.

Definition 58. Let $\mathcal{C}$ be a class category with subcategory of small objects $\mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{C}$. The *derivative* functor,

$$\mathbf{d} : \mathcal{C} \to \mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$$

is defined as follows.

$\mathbf{d}C = \{A \hookrightarrow C \mid A \text{ small}\}$

$\mathbf{d}f : \mathbf{d}C \to \mathbf{d}D$, given $f : C \to D$, is defined by factoring, as indicated in the following diagram:

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ A \arrow[u, hook] \arrow[r, dotted, two heads, "(\mathbf{d}f)_A"'] & \mathbf{d}f(A) \arrow[u, hook] \end{tikzcd}$$

Lemma 59. *For any class category $\mathcal{C}$, the derivative functor,*

$$\mathbf{d} : \mathcal{C} \to \mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$$

preserves the following structure.

(i) finite limits and coproducts

(ii) small maps

(iii) powerobjects $\mathcal{P}C$,

(iv) the universal object $\mathcal{U}$.

Proof. Routine verification. Briefly:

(i) Given $C \times D$ in $\mathcal{C}$,

$$\begin{aligned}
\mathbf{d}(C \times D) &= \{S \hookrightarrow C \times D \mid S \in \mathcal{S}_{\mathcal{C}}\} \\
&= \{S \hookrightarrow C' \times D' \mid S, C', D' \in \mathcal{S}_{\mathcal{C}},\ C' \hookrightarrow C,\ D' \hookrightarrow D\} \\
&= \mathbf{d}(C) \times \mathbf{d}(D)
\end{aligned}$$

by factoring any small $S \rightarrowtail C \times D$ into $S \rightarrowtail C' \times D' \rightarrowtail C \times D$ with $C' \rightarrowtail C$ and $D \rightarrowtail D'$ small.

The other cases are similar.

(ii) Let $f : C \to D$ be a small map. Since for any small subobject $B \rightarrowtail D$, the pullback $f^*(B) \rightarrowtail C$ is also small, we can define an inverse image for $\mathbf{d}f : \mathbf{d}C \to \mathbf{d}D$ by setting:

$$(\mathbf{d}f)^{-1}(B) = f^*(B)$$

This will plainly satisfy $\mathbf{d}f \vdash (\mathbf{d}f)^{-1}$.

(iii) For any $C \in \mathcal{C}$ and small $A \hookrightarrow \mathcal{P}C$, the subobject $\bigcup A \hookrightarrow C$ is also small, and it satisfies:

$$A \hookrightarrow \mathcal{P}X \quad \text{iff} \quad \bigcup A \hookrightarrow X$$

for all $X \hookrightarrow C$. Thus any small $A \hookrightarrow \mathcal{P}C$ can be factored as

$$A \hookrightarrow \mathcal{P}B \hookrightarrow \mathcal{P}C$$

for a small $B \hookrightarrow C$, namely $B = \bigcup A$. We therefore have:

$$\begin{aligned}
\mathbf{d}(\mathcal{P}C) &= \{A \hookrightarrow \mathcal{P}C \mid A \in \mathcal{S}_{\mathcal{C}}\} \\
&= \{A \hookrightarrow \mathcal{P}B \mid A, B \in \mathcal{S}_{\mathcal{C}},\ B \hookrightarrow C\} \\
&= \{A \hookrightarrow \mathcal{P}B \mid A \in \mathcal{S}_{\mathcal{C}},\ B \in \mathbf{d}C\} \\
&= \mathcal{P}(\mathbf{d}C)
\end{aligned}$$

(iv) For the universal object $\mathcal{U}$, the ideal

$$\mathbf{d}\,\mathcal{U} = \mathcal{S}_{\mathcal{C}}$$

is clearly universal in $\mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$.

$\square$

Lemma 60. *Let $\mathcal{C}$ be a class category and $\mathbf{d} : \mathcal{C} \to \mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$ the derivative functor.*

(i) If $\mathcal{C}$ has small covers, then $\mathbf{d}$ preserves regular epis.

(ii) If $\mathcal{C}$ has small generators, then $\mathbf{d}$ is faithful and preserves dual images.

Proof. To prove (i), suppose $\mathcal{C}$ has small covers, and take a regular epimorphism $e : C \twoheadrightarrow D$ in $\mathcal{C}$. The morphism $\mathbf{d}e : \mathbf{d}C \to \mathbf{d}D$ is then regular epic in $\mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$ if the mapping part $A \mapsto \mathbf{d}e(A)$ is surjective. Thus take $B \in \mathbf{d}D$ and consider the following diagram, in which the upper square is a pullback:

$$\begin{tikzcd}
C \arrow[r, two heads, "e"] & D \\
B' \arrow[u, hook] \arrow[r, two heads, "e'"] & B \arrow[u, hook] \\
A \arrow[u, hook] \arrow[ur, two heads, "e''"'] &
\end{tikzcd}$$

The morphism e' is regular epic since e is, so since B is small, there exists small $A \hookrightarrow B'$ with regular epic $e'' : A \twoheadrightarrow B$ by small covers. Thus we have $A \in \mathbf{d}C$ with $\mathbf{d}e(A) = B$, as required.

For (ii), suppose $\mathcal{C}$ has small generators, and consider the following situation in $\mathcal{C}$:

$$\begin{tikzcd}
C \arrow[r, "f"] & D \\
S \arrow[u, hook] & f_*S \arrow[u, hook]
\end{tikzcd}$$

We want to show:

$$\mathbf{d}(f_*S) = (\mathbf{d}f)_*\mathbf{d}S$$

While we know:

$$\begin{aligned}
\mathbf{d}(f_*S) &= \{B \hookrightarrow D \mid f^*B \hookrightarrow S\} \\
(\mathbf{d}f)_*\mathbf{d}S &= \{B \hookrightarrow D \mid A \hookrightarrow f^*B \text{ implies } A \hookrightarrow S, \text{for small } A \hookrightarrow C\}
\end{aligned}$$

Thus it suffices to show that for all $S, T \hookrightarrow C$:

$$(\text{for all small } A \hookrightarrow C, A \hookrightarrow T \text{ implies } A \hookrightarrow S) \quad \Longrightarrow \quad T \hookrightarrow S$$

But this follows easily from small generators, by considering $T \cap S$.

Finally, observe that small generation implies that the small objects generate $\mathcal{C}$. Indeed, let $f \neq g : C \rightrightarrows D$, and consider the equalizer, in the top row of the following diagram:

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, tail] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \\
A \arrow[u, dotted] \arrow[ur, tail, "i"'] & &
\end{tikzcd}$$

Since $f \neq g$, we have $\mathrm{Eq}(f,g) \neq C$. So by small generation there is some small $i : A \hookrightarrow C$ that does *not* factor through $\mathrm{Eq}(f,g)$. But then it must be that $fi \neq gi$. It thus follows that $\mathbf{d}f \neq \mathbf{d}g$. $\square$

Combining the last two lemmas now yields the following, which was the desired step 3.

Proposition 61. *If $\mathcal{C}$ is saturated, then $\mathbf{d} : \mathcal{C} \to \mathbf{Idl}(\mathcal{S}_\mathcal{C})$ is both logical and faithful.*

5.4 The ideal embedding theorem

Pulling together the results of this section, we now have proven the following embedding theorem for class categories with collection.

Theorem 62. *For any class category $\mathcal{C}$ with collection, there is a small topos $\mathcal{E}$ and a faithful logical functor $\mathcal{C} \to \mathbf{Idl}(\mathcal{E})$.*

Proof. Combine propositions 57 and 61 to get a faithful, logical functor,

$$\mathcal{C} \rightarrowtail \mathbf{Idl}(\mathcal{S}_{\mathcal{C}})$$

into the category of ideals in the subcategory $\mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ of small objects. $\square$

As a corollary, finally, we have the desired logical completeness of the set theory $\mathrm{bIST_C}$ with respect to topos models:

Theorem 63. *For any elementary formula φ in the language $\{\in, \mathsf{S}\}$ of set theory, if φ holds in ideals over every topos $\mathcal{E}$:*

$$\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \varphi$$

then it is provable in bIST*:*

$$\mathrm{bIST_C} \vdash \varphi$$

Proof. If $\mathcal{E} \models_{\mathbf{Idl}(\mathcal{E})} \varphi$ for all toposes $\mathcal{E}$, then by the foregoing embedding theorem, $\mathcal{S}_{\mathcal{C}} \models_{\mathcal{C}} \varphi$ for every class category $\mathcal{C}$ with collection and its subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects. But then $\mathrm{bIST_C} \vdash \varphi$ by theorem 52, the class category completeness theorem with collection. $\square$