Categorical Models of Intuitionistic Theories of Sets and Classes

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Abstract

The thesis consists of three sections, developing models of intuitionistic set theory in suitable categories. First, the categorical framework in which models are constructed is reviewed, and the theory of all such models, called Basic Intuitionistic Set Theory (BIST), is stated; second, we give a notion of an *ideal* over a category, with which one can build a model of BIST in which a given topos occurs as the sets; and third, a sheaf model is given of a Basic Intuitionistic Class Theory conservatively extending BIST.

Contents

Introduction

We begin with a brief sketch (elaborated in section 1 below) of the leading ideas of algebraic set theory, as it was recently presented in [2], and first proposed in [8] (see also [3, 11, 10, 6]). The basic tool of algebraic set theory is the notion of a *category with class structure*, or *class category* for short, which provides an axiomatic framework in which models of set theory are constructed. A class structure on a category $\mathcal{C}$ consists of a subcategory $\mathcal{S} \hookrightarrow \mathcal{C}$ of *small maps* and a *powerclass* functor $\mathcal{P} : \mathcal{C} \longrightarrow \mathcal{C}$. A class category may contain *universes* which are models of (untyped) set theory. Thinking of objects as classes, the small maps determine which classes are to be thought of as sets, the powerclass $\mathcal{P}(C)$ is the class of all sub*sets* of a class C, and a universe U is a sub–fixed point of $\mathcal{P}$, in the sense that $\mathcal{P}(U) \subseteq U$.

The language of elementary set theory (first–order logic with a binary "membership" relation ϵ and a "sethood" predicate S) can be interpreted in any such universe U, and the elementary theory of all such universes can be completely axiomatized by a system of set theory, called Basic Intuitionistic Set Theory (BIST), first formulated in [2]. It is noteworthy for including the unrestricted axiom scheme of Replacement in the absence of the full axiom scheme of Separation (a combination that can not occur in classical logic, where Replacement implies Separation).

The objects of a category with class structure that have a small morphism into the terminal object are called *small* objects or *sets*. These are easily shown to be a topos. In [2] it is shown that any topos whatsoever occurs as the subcategory of small objects in some category with class structure. This is achieved by defining a notion of an *ideal on a topos*. The central part (section 2) of this thesis consists in a modification of this notion. It is shown that a useful notion of ideal on a topos can be obtained by considering certain sheaves on the topos under the coherent (or finite epimorphic families) covering. Namely, these are those sheaves that occur as colimits of filtered diagrams of representables, in which every morphism is a monomorphism. Following a suggestion by André Joyal, these sheaves are characterized as satisfying a "small diagonal" condition with respect to maps with representable fibers. The subcategory of such ideals then forms a category with class structure in which one can solve for fixed points of the powerobject functor.

If the powerclass of an ideal C is thought of as the class of all sub*sets* of C, then the powerobject of C in the category of sheaves can be thought of as the "hyper class" of all sub*classes* of C, since ideals are closed under subsheaves. The first step in a comparison between these two kinds of powerobjects is carried out in section 3, where it is shown that there is a model in the category of sheaves of a Morse-Kelley style theory of sets and classes which is a conservative extension of BIST. In analogy with BIST, this theory has only a restricted axiom of Separation. That is to say, it is not in general the case that the intersection between a class and a set is again a set. The subobject of classes for which this is the case is however easily definable in sheaves as a particular exponent of the universe $\mathcal{U}$. One therefore has the option of restricting to this exponent in order to obtain a class theory with unrestricted separation but restricted comprehension.

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Conventions

- A punctuation mark following a quantifier indicates that the quantifier has maximal scope. That is to say, the scope of a quantifier is the largest (well–formed) formula following the punctuation mark. Thus the scope of the quantifier in $\forall x.\, \phi \wedge \psi$ is $\phi \wedge \psi$, the scope of the quantifier in $(\forall x.\, \phi) \wedge \psi$ is ϕ.

- The unique existence quantifier $\exists! x.\, \phi$ is written $\exists^{=1} x.\, \phi$, since the exclamation mark appears in front of a formula in some contexts to indicate shorthand notation.

- Standard shorthand notation for restricted quantifiers is used. Thus $\forall x \in \phi.\, \psi$ for $\forall x.\, \phi \to \psi$ and $\exists x \in \phi.\, \psi$ for $\exists x.\, \phi \wedge \psi$. In cases involving infix notation of relation symbols, such as $x \epsilon y$, we write $\forall x \epsilon y.\, \phi$ for $\forall x.\, x \epsilon y \to \phi$, and similarly for the existential quantifier. In the context of a set theory, we may speak of a *bounded quantifier*, by which we always mean one of the form $\forall x \epsilon y$ or $\exists x \epsilon y$, where ϵ is the membership relation of the set theory. A *bounded formula*, or Δ_0–*formula*, is a formula where all quantifiers are bounded. The symbol $\in$ is overloaded, as it is used in some set theories as the membership predicate.

- We write $\phi(x)$ for a formula ϕ with the free variable x distinguished and thereafter $\phi(y)$ for $\phi(\frac{y}{x})$. This is not to say that there are not other free variables in ϕ, nor that x actually occurs in ϕ.

- The language, or canonical signature, of a category is usually considered together with its canonical interpretation. For that reason, symbols in the language are usually chosen to be the same symbols that denote the corresponding objects, morphisms, or subobjects in the category. This also applies when theories are developed with particular interpretations in mind. The symbol ϵ, for instance, is used both as the membership predicate in class logic and as the membership subobjects of class categories.

- Scott brackets always indicate interpretations of one form or another. A formula–in–context enclosed by Scott brackets, say $[\![x{:}A \mid \phi]\!]$, thus always denotes a subobject of some category. Which category and which interpretation is being considered should be clear by context, but subscript may sometimes be used to avoid confusion. In such cases, $[\![]\!]_{\mathcal{C}}$ denotes the canonical interpretation of the language of $\mathcal{C}$.

1 Categories of Classes

A *cartesian* category is a category with finite limits. A *regular* category is a cartesian category with images in which covers are stable under pullback. A *coherent* category is a regular category in which, for every object A, $Sub(A)$ has finite unions which are stable under inverse image functors $f^* : Sub(B) \longrightarrow Sub(A)$. A coherent category is *positive* if it has finite, disjoint coproducts. A *Heyting* category is a coherent category in which, for every morphism $f : A \longrightarrow B$, the inverse image functor $f^* : Sub(B) \longrightarrow Sub(A)$ has a right adjoint. See [7, A1]. Typed first-order logic can be interpreted in any Heyting category, and the so-called *internal logic* of a small Heyting a category is a typed first–order theory which can be used to obtain results concerning the category (see e.g. [7, D1.3.10]). We consider the extension of first-order logic by *power types* containing collections or sets, so to speak, that obey familiar axioms. Correspondingly, we consider Heyting categories equipped with *power objects*, the relevant properties of which is characterized in terms of a notion of a *small map*. The intuition is that a map is small if its fibers are sets.

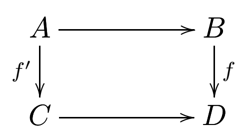
We begin by giving the axioms for a system of small maps on a Heyting category with power objects, as it can be found in [2], and by showing that the set theory we consider is sound with respect to such categories. For the opposite direction of the correspondence, we then show that it is also complete. We end section 1 by considering special cases such as natural numbers, and, what will become of more importance in section 2 and 3, universes, that is, objects that model untyped set theory.

1.1 Class Category

Let $\mathcal{C}$ be a positive Heyting category, i.e. a Heyting category with finite disjoint coproducts that are stable under pullback (see [7, A1.4.4]. A *system of small maps* on $\mathcal{C}$ is a collection of morphisms of $\mathcal{C}$ satisfying the following closure conditions:

(S1) Every identity map $Id_A : A \longrightarrow A$ is small, and the composite $g \circ f : A \longrightarrow C$ of any two small maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is again small.

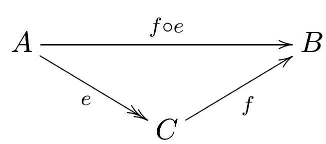
(S2) The pullback of a small map along any map is small. Thus in an arbitrary pullback diagram,



f' is small if f is small.

(S3) Every diagonal $\Delta : A \longrightarrow A \times A$ is small.

(S4) If $f \circ e$ is small and e is regular epic, then f is small, as indicated in the diagram:



(S5) Copairs of small maps are small. Thus if $f : A \longrightarrow C$ and $g : B \longrightarrow C$ are small, then so is $(f, g) : A + B \longrightarrow C$.

Proposition 1.1.1 *In the context of axioms S1 and S2, axiom S3 is equivalent to the condition that regular monomorphisms are small, and to the condition that if a composition $g \circ f$ is small, then the first component f is small.*

Proof [2] $\dashv$

A relation $r : R \rightarrowtail A \times B$ is defined to be a *small relation* if the second projection $\pi_2 \circ r : R \rightarrowtail A \times B \longrightarrow B$ is a small map. We make the small relations representable in requiring that $\mathcal{C}$ has a *power structure* consisting of, for every object A in $\mathcal{C}$, an object PA and a small relation $\epsilon_A \rightarrowtriangle A \times PA$ such that the following two axioms are satisfied:

(P1) For any small relation $R \stackrel{m}{\rightarrowtail} A \times B$, there exists a unique *classifying map* $\rho : B \longrightarrow PA$ such that the following is a pullback:

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_A \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow \\ A \times B & \underset{Id \times \rho}{\longrightarrow} & A \times PA \end{array}$$

(P2) The internal *subset* relation $\subseteq_A \rightarrowtail PA \times PA$ (defined as $[\![x{:}PA, y{:}PA \mid \forall z{:}A.\, z\epsilon_A x \rightarrow z\epsilon_A y]\!]$) is a small relation.

If $\mathcal{C}$ is a Heyting category in which each object A has a designated object PA and a designated relation $\epsilon_A \rightarrowtriangle A \times PA$, then we say that $\mathcal{C}$ has a *pre-power structure*. We call a positive Heyting category $\mathcal{C}$ with a system of small maps and a power structure a *class category* or a *category of classes*[1] (class categories and their properties were defined and studied in [2]). It is possible for a category to be equipped with distinct systems of small maps and power structures, but we shall mostly refer to a class category just as $\mathcal{C}$, assuming a system of small maps and a power structure to be implicitly fixed. Since, by **P1**, a morphism $f : A \longrightarrow B$ in a class category $\mathcal{C}$ is a small map just in case its graph is a small relation, which again holds just in case its graph has a classifying morphism $\rho : B \longrightarrow PA$ such that:

$$\begin{array}{ccc} Grph(f) & \longrightarrow & \epsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times B & \underset{Id \times \rho}{\longrightarrow} & A \times PA \end{array}$$

[1] This usage does in general not correspond to the usage of these terms elsewhere, e.g. in [2] or [10].

we can recover the system of small maps from the power structure on $\mathcal{C}$. When we need to be more specific, we will therefore refer to a category $\mathcal{C}$ as a class category relative to a power structure $\mathcal{P}$, and write ($\mathcal{C}$,$\mathcal{P}$) (On the other hand, a system of small maps on a positive Heyting category $\mathcal{C}$ determines the power structure, if it exists, up to isomorphism, so we could also say that $\mathcal{C}$ is a class category relative to a system of small maps $\mathcal{S}$, and write $(\mathcal{C}, \mathcal{S})$). Heyting functors between class categories that preserve the small map and power object structure will be called *logical* functors. We remark, finally, that a class category as defined in [2] is required to have a universal object (see 1.5.1)

1.2 Class Logic

Any topos is a class category where all maps are small. We describe a variant of topos logic (see [7]) adapted to suit class categories in general (see also [10]): A *class signature* Σ is defined by specifying (i) a set of type constants, Σ_C; (ii) a set of typed function symbols, Σ_F; and (iii) a set of typed relation symbols, Σ_R.

- The set of Σ–types is inductively defined by
  - The type constants are Σ–types.
  - 1 is a Σ–type.
  - If V and W are Σ–types, the so is $V \times W$ and PV.
- Each function symbol $f \in \Sigma_F$ comes with a typing $f : V \to W$, and each relation symbol $R \in \Sigma_R$ comes with a typing $R : V$, where V and W are Σ–types.
- For each Σ–type V, we assume a sufficiently large collection,$\mathcal{V}_V$, of variables of type V.
- We define the terms t of type V (with respect to Σ) written $t : V$ inductively for all V (simultaneously):
  - If $x \in \mathcal{V}_V$, then $x : V$.
  - $* : 1$
  - If $x : V$ and $y : W$, then $\langle x, y\rangle : V \times W$
  - If $z : V \times W$, then $\pi_1 z : V$ and $\pi_2 z : W$.
  - If $f : V \to W$ is in Σ_F and $t : V$, then $f(t) : W$

- The set of Σ-formulas is inductively defined by:
  - $\top$ and $\bot$ are Σ-formulas.
  - If $R : V$ is a relation constant and $t : V$ is a term, then $R(t)$ is a Σ-formula.
  - If $s, t : V$ are terms, then $s =_V t$ is a Σ-formula (which we mostly write without bothering with the subscript).
  - If $s : V$ and $t : PV$ are terms, then $s \epsilon_V t$ is a Σ-formula (which we sometimes write without bothering with the subscript).
  - If ϕ and ψ are Σ-formulas, then so is $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\forall x{:}V.\, \phi$, and $\exists x{:}V.\, \phi$.

A *class theory* (CT) is a first-order theory over a class signature Σ which contains the following formulas (for $u, v{:}PA$, the expression $u \subseteq_A v$ abbreviates $\forall x{:}A.\, x\epsilon_A u \to x\epsilon_A v$):

CT1. (Extensionality) For all Σ-types A
$\forall u, v{:}PA.\, (\forall x{:}A.\, x\epsilon_A u \leftrightarrow x\epsilon_A v) \to u = v$

CT2. (Empty set) For all Σ-types A
$\exists u{:}PA.\, \forall x{:}A.\, x\epsilon_A u \to \bot$

CT3. (Pairing) For all Σ-types A
$\forall x, y{:}A.\, \exists u{:}PA.\, \forall z{:}A.\, z\epsilon_A u \leftrightarrow z = x \vee z = y$

CT4. (Union) For all Σ-types A
$\forall p{:}PPA.\, \exists u{:}PA.\, \forall z{:}A.\, z\epsilon_A u \leftrightarrow \exists v{:}PA.\, v\epsilon_{PA} p \wedge z\epsilon_A v$

CT5. (Power set) For all Σ-types A
$\forall u{:}PA.\, \exists p{:}PPA.\, \forall v{:}PA.\, v\epsilon_{PA} p \leftrightarrow v \subseteq u$

CT6. (Replacement) For every formula $\phi(x, y, \vec{z})$, with all free variables shown and where $x : A$ and $y : B$ and $\vec{z} : \vec{Z}$,

$$\forall \vec{z}{:}\vec{Z}.\, \forall u{:}PA.\, (\forall x\epsilon_A u.\, \exists! y{:}B.\, \phi(x, y, \vec{z})) \\ \to (\exists v{:}PB.\, \forall y{:}B.\, y\epsilon_B v \leftrightarrow \exists x\epsilon_A u.\, \phi(x, y, \vec{z}))$$

(In words: if a relation is functional when restricted to some set, then the image of that set under the relation is a set.)

CT7. (Binary intersection) For all Σ-types A
$\forall u, v{:}PA.\, \exists w{:}PA.\, \forall x{:}A.\, x\epsilon_A w \leftrightarrow x\epsilon_A u \wedge x\epsilon_A v$

More readable axioms can be obtained by extending the term formation rules by suitable clauses. E.g. by ‘If $s : PV$ and $t : PV$ are terms, then $s \cap t : PV$ is a term’. Axiom **CT7** would then read “For all Σ–types A, $\forall u, v{:}PA.\, \forall x{:}A.\, x\epsilon_A u \cap v \leftrightarrow x\epsilon_A u \wedge x\epsilon_A u$. See the development of class logic in [10]. We shall return to the issue of separation in section 3.2.

1.3 Soundness

Let Σ be a class signature. A Σ-*structure* is an interpretation in a class category of the type constants and of the function and relation symbols (respecting the typing). Such a structure determines an interpretation of all Σ-types and -formulas in familiar first-order ways, as soon as we add the requirement that if V is a Σ-type and $[\![V]\!] = C$, then $[\![PV]\!] = PC$, and $[\![\epsilon_V]\!] = \epsilon_C \rightarrowtail C \times PC$. We must show that any Σ-structure satisfies the axioms CT1-CT7. This is, essentially, already shown in [2], but we include the proofs for the sake of completeness. For simplicity, we use the same name for a Σ-type and the object it denotes in the structure, i.e. $[\![A]\!] = A$. We write $[\![x{:}A|\phi]\!] \cong A$ to signify that the subobject interpreting $x{:}A|\phi$ in the structure is the top object in the subobject lattice of the object interpreting the type A.

Lemma 1.3.1 (Extensionality) *Let a Σ–structure be given.*

$$[\![u, v{:}PA \mid (\forall x{:}A.\, x\epsilon_A u \leftrightarrow x\epsilon_A v) \rightarrow u = v]\!] \cong PA \times PA$$

Proof Suppose the generalized element $\langle s, t\rangle : Y \longrightarrow PA \times PA$ factors through $[\![u, v{:}PA \mid \forall x{:}A.\, x\epsilon_A u \leftrightarrow x\epsilon_A v]\!]$. Then $[\![x{:}A, y{:}Y \mid x\epsilon s(y)]\!] = [\![x{:}A, y{:}Y \mid x\epsilon t(y)]\!]$ as subobjects of $A \times Y$, so $s = t$ by P1. ⊣

Lemma 1.3.2 *In any class category $\mathcal{C}$, coproduct inclusion maps are small, and arrows with source the initial object* 0 *are small.*

Proof In any positive coherent category, coproduct inclusion maps can be obtained by pullbacks:

$$\begin{array}{ccc} A & \xrightarrow{\quad a \quad} & 1 \\ \downarrow & \lrcorner & \downarrow \\ A + B & \xrightarrow[a+b]{} & 1 + 1 \\ \uparrow & \urcorner & \uparrow \\ B & \xrightarrow[b]{} & 1 \end{array}$$

It follows that coproduct inclusions are monic and, in a class category, that they are small. Since coproducts are disjoint, any arrow $0 \longrightarrow A$ is a pullback of a coproduct inclusion map, $A \longrightarrow A + A$, and therefore small.$\dashv$

Lemma 1.3.3 (Empty set) *Let a Σ–structure be given.*

$$[\![\ |\ \exists u{:}PA.\, \forall x{:}A.\, x\epsilon_A u \to \bot]\!] \cong 1$$

Proof By Lemma 1.3.2, $0 \longrightarrow A \times 1$ is a small relation, and we get a witness global point $\emptyset_A : 1 \longrightarrow PA$:

$$\begin{array}{ccc} 0 & \longrightarrow & \epsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times 1 & \xrightarrow[Id \times \emptyset_A]{} & A \times PA \end{array}$$

.

Lemma 1.3.4 (Binary Intersection) *Let a Σ–structure be given.*

$$[\![u, v{:}PA \mid \exists w{:}PA.\, \forall x{:}A.\, x\epsilon_A w \leftrightarrow x\epsilon_A u \wedge x\epsilon_A v]\!] \cong PA \times PA$$

.

Proof Let generalized elements $\beta, \gamma : Y \rightrightarrows PA$ be given. Then we have corresponding small relations $B \rightarrowtriangle A \times Y$, $C \rightarrowtriangle A \times Y$. Their intersection $D := B \cap C$ is again a small relation, giving us a classifying map $\delta : Y \longrightarrow PA$, which is a witness. $\dashv$

Binary intersection together with extensionality gives us a binary intersection map $\cap : PA \times PA \longrightarrow PA$ in $\mathcal{C}$. Similarly, since a binary union of small relations is again a small relation (by S5 and S4), $\mathcal{C}$ will satisfy a binary union axiom, giving rise to a binary union map $\cup : PA \times PA \longrightarrow PA$, such that $[\![x{:}A, u, v{:}PA | x \ \epsilon\ u \cup v]\!] = [\![x{:}A, u, v{:}PA | x\epsilon u \vee x\epsilon v]\!]$ as subobjects of $A \times PA \times PA$. Another useful map is the singleton map, classifying the diagonal:

$$\begin{array}{ccc} A & \longrightarrow & \epsilon \\ \Delta \downarrow & \lrcorner & \downarrow \\ A \times A & \xrightarrow[Id \times \{-\}]{} & A \times PA \end{array}$$

$[\![x, y{:}A \mid x = y]\!] = [\![x, y{:}A \mid x\epsilon\{y\}]\!]$ as subobjects of $A \times A$.

Lemma 1.3.5 (Pairing) *Let a* Σ*–structure be given.*

$$[\![x, y{:}A \mid \exists u{:}PA.\, \forall z{:}A.\, z\epsilon_A u \leftrightarrow z = x \vee z = y]\!] \cong A \times A$$

Proof Compose $\{-\} \times \{-\} : A \times A \longrightarrow PA \times PA$ and $\cup : PA \times PA \longrightarrow PA$. $\dashv$

Lemma 1.3.6 (Union) *Let a* Σ*–structure be given.*

$$[\![p{:}PPA \mid \exists u{:}PA.\, \forall z{:}A.\, z\epsilon_A u \leftrightarrow \exists v{:}PA.\, v\epsilon_{PA} p \wedge z\epsilon_A v]\!] \cong PPA$$

Proof We need to show that the relational product $\epsilon_A \circ \epsilon_{PA}$ (i.e. the subobject $[\![x{:}A, w{:}PPA \mid \exists u{:}PA.\, x\epsilon_A u \wedge u\epsilon_{PA} w]\!]$) of the small relations $\epsilon_A \rightarrowtail A \times PA$ and $\epsilon_{PA} \rightarrowtail PA \times PPA$ is again small. But this holds in a class category by diagram chase. We draw the diagram in which to chase:

$$\begin{array}{ccccccc}
[\![a, u, \alpha \mid a\epsilon u \wedge u\epsilon\alpha]\!] & \longrightarrow & [\![a, u, \alpha \mid u\epsilon\alpha]\!] & \longrightarrow & \epsilon_{PA} & \searrow & \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow & & \\
[\![a, u, \alpha \mid a\epsilon u]\!] & \longrightarrow & A \times PA \times PPA & \longrightarrow & PA \times PPA & \longrightarrow & PPA \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow & & \\
\epsilon_A & \longrightarrow & A \times PA & \longrightarrow & PA & &
\end{array}$$

Lemma 1.3.7 (Powerset) *Let a* Σ*–structure be given.*

$$[\![u{:}PA \mid \exists p{:}PPA.\, \forall v{:}PA.\, v\epsilon_{PA} p \leftrightarrow v \subseteq u]\!] \cong PPA$$

Proof Use the classifying map of $\subseteq: \rightarrowtriangle PA \times PA$ as a witness. $\dashv$

We have the following result from [2]:

Theorem 1.3.8 *Any slice of a class category is again a class category, and pullback functors preserve the small map and power object structure.*

Proof We briefly indicate the relevant definitions, as the details of the proof can be found in [2]: For any object X of $\mathcal{C}$, the small maps in $\mathcal{C}/X$ are just the maps that are small as maps in $\mathcal{C}$. The power object in $\mathcal{C}/X$ of an object $a : A \longrightarrow X$ are the 'fiber wise' power objects $\pi_1 : [\![x : X, u : PA \mid \forall y.\, \epsilon u.\ a(y) = x]\!] \longrightarrow X$ $\dashv$

Theorem 1.3.8 will of course be a useful tool throughout this text, allowing us to greatly simplify many proofs. For a detailed example of how such simplification works, see Example A.2.2.

Lemma 1.3.9 (Replacement) *Let a Σ–structure be given. For any Σ–formula $\phi(x, y, \vec{z})$,*

$$\begin{aligned}&[\![u{:}PA, \vec{z}{:}\vec{Z} \mid (\forall x\epsilon_A u.\, \exists^{=1} y{:}B.\, \phi(x, y, \vec{z}))]\!] \\ &\qquad \leq [\![u{:}PA, \vec{z}{:}\vec{Z} \mid (\exists v{:}PB.\, \forall y{:}B.\, y\epsilon_B v \leftrightarrow \exists x{:}A.\, x\epsilon_A u \wedge \phi(x, y, \vec{z}))]\!]\end{aligned}$$

Proof Let a formula $\phi(x, y)$ be given. By Theorem 1.3.8, we may assume without loss that ϕ has no parameters $\vec{z}$ and we do only the case of a global point. Suppose $\delta : 1 \longrightarrow PA$ factors through $[\![u{:}PA|(\forall x\epsilon_A u.\, \exists! y{:}B.\, \phi(x, y))]\!]$. Now, δ is the classifying map of a small subobject $D \rightarrowtail A$, and ϕ determines a functional relation on $D \times B$, and so an arrow $f : D \longrightarrow B$. By S4, the reg. epi-mono factorization of f yields a small subobject $C \rightarrowtail B$, the classifying map γ of which is a witness of the consequent statement. $\dashv$

We summarize the result so far:

Theorem 1.3.10 (Soundness) *Any Σ-structure satisfies the class theory axioms* CT1 – CT7*.*

1.4 Internal Characterization and Completeness

1.4.1 Internal Characterization

We have now shown that class logic is sound with respect to class categories. This is the first part of showing that we have, in class logic, a characterization of the internal logic of class categories [10]. From section 1.3 we now know that the internal logic of a class category is a class theory. Conversely, we need to show that if the internal logic of a category is a class theory, then that category is a class category. It follows (1.4.4) that the syntactic category of a class theory is a class category, and that class theories are sound and complete with respect to interpretations in categories of classes.

The class axioms of section 1.2 serve to characterize class categories, in the following sense: Let $\mathcal{C}$ be a small Heyting category, and let $\Sigma_{\mathcal{C}}$ denote the canonical signature of $\mathcal{C}$ (which we also refer to as *the language of* $\mathcal{C}$). By the *theory of* $\mathcal{C}$, $\mathbb{T}_{\mathcal{C}}$, we mean the collection of $\Sigma_{\mathcal{C}}$–sentences that are true under the canonical $\Sigma_{\mathcal{C}}$–structure. (See [7, D1.3.10]. We also say, for such a sentence ϕ, that it is *true in* $\mathcal{C}$, and write $\mathcal{C} \models \phi$.) Let a pre-power structure

$\mathcal{P}$ for $\mathcal{C}$ be given. That is to say, for each object A in $\mathcal{C}$, we have assigned an object which we denote PA and a subobject of $A \times PA$ which we denote ϵ_A. Relative to this pre-power structure, we can regard $\Sigma_{\mathcal{C}}$ as a class signature.

Covention 1.4.2 In what follows, let $\text{Ƨ}x{:}A.\,\phi$ abbreviate

$$\exists u{:}PA.\, \forall x{:}A.\, x \epsilon u \leftrightarrow \phi$$

where $u{:}PA$ is not free in ϕ (read 'set many $x{:}A$ such that ϕ').

Theorem 1.4.3 *Let $\mathcal{C}$ be a small Heyting category, and let $\mathcal{P}$ be a pre-power structure on $\mathcal{C}$. If $\mathbb{T}_{\mathcal{C}}$ is a class theory (when $\Sigma_{\mathcal{C}}$ is considered as a class signature relative to $\mathcal{P}$), then $(\mathcal{C}, \mathcal{P})$ is a class category.*

Proof Define a map $f : A \longrightarrow B$ of $\mathcal{C}$ to be small if there exists a 'fibre map' $f^{-1} : B \longrightarrow PA$ such that

$$[\![x{:}A, y{:}B \mid x \epsilon_A f^{-1}(y)]\!] = [\![x{:}A, y{:}B \mid f(x) = y]\!]$$

as subobjects of $A \times B$. We shall show that the axioms S1-S5 and P1-P2 are satisfied. We shall often reason informally in the internal logic of $\mathcal{C}$, and in doing so we shall make informal use of comprehension terms. E.g. the value of $f^{-1}(y)$ for a small map f at y might be denoted by $\{x{:}A \mid f(x) = y\}$. We start with some useful consequences of the class theory axioms:

- It is a consequence of Replacement (and Extensionality) that any morphism $f : A \longrightarrow B$ has an *internal direct image map* $Pf : PA \longrightarrow PB$ such that $[\![y \epsilon_B Pf(u)]\!] = [\![\exists x \epsilon_A u.\, f(x) = y]\!]$ as subobjects of $B \times PA$. (We note for later that this describes the arrow part of the power functor $\mathcal{P} : \mathcal{C} \to \mathcal{C}$, on a class category $\mathcal{C}$, which sends an object to its power object. See [2] for detail. $\mathcal{P}$ preserves monomorphisms.)
- It is a further consequence of Replacement that if $d : D \rightarrowtail A$ is in a subobject (considered as an equivalence class) $[\![x{:}A \mid \phi(x)]\!] \rightarrowtail A$, then $Pd : PD \rightarrowtail PA$ is in the subobject $[\![u{:}PA \mid \forall x \epsilon_A u.\, \phi(x)]\!] \rightarrowtail PA$.

(S1) Consider an isomorphism $f : A \longrightarrow B$ for objects A and B in $\mathcal{C}$, and give its inverse the name g. We reason in the internal logic of $\mathcal{C}$: For each $x{:}A$ there exists a singleton $\{x\}$ of type PA by the Pairing axiom (class theory axiom 3), which is unique by Extensionality (class theory axiom 1). Hence we may define a map $x \mapsto \{x\}$, which composed with g is the required fibre map.

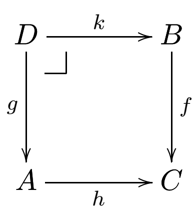
Let small maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be given. We reason in the internal logic of $\mathcal{C}$: Let $z{:}C$ be given. Since g is small, there is a fibre $g^{-1}(z)$ in PB. For each $y \epsilon_B g^{-1}(z)$ there is a fibre $f^{-1}(y)$ so by Replacement we have the set $\{f^{-1}(y){:}PA \mid y \epsilon_B g^{-1}(z)\}$. We apply the Union axiom to get the fibre of $g \circ f$ at z.

(S2) We show first that our axioms guarantee Cartesian products, in the sense that

$$\mathcal{C} \models \forall u{:}PA. \forall v{:}PB. \exists t{:}P(A \times B). \forall z{:}(A \times B).\, z \epsilon_{(A\times B)} t \leftrightarrow \pi_1(z) \epsilon_A u \wedge \pi_2(z) \epsilon_B v$$

We continue to reason in $\mathcal{C}$: Let $u{:}PA$ and $v{:}PB$ be given. Fix a of type A. We may define a functional relation $b{:}B, z{:}(A \times B) \mid b = \pi_2(z) \wedge a = \pi_1(z)$ (think of it as a function $f_a : B \to A \times B$ defined by $b \mapsto < a, b >$). By Replacement, $w_a = \{< a, b > \mid b \epsilon v\}$ is a set (the image of v under f_a). $a \mapsto w_a$ is again a function (again we define it as a functional relation), so by Replacement, $\{w_a \mid a \epsilon u\}$ is a set of type $PP(A \times B)$. By Union, finally, $\bigcup_{a \epsilon u} w_a = \{< a, b > \mid a \epsilon_A u,\ b \epsilon_B v\}$ is a set of type $P(A \times B)$.

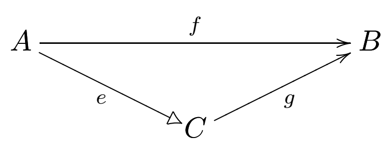
Now, consider a given pullback



where f is small. By (S1), we may without loss of generality treat D as the subobject $[\![a{:}A, b{:}B \mid h(a) = f(b)]\!]$. We define a map $A \longrightarrow P(A \times B)$, which factors through PD, by $a \mapsto \{a\} \times f^{-1}(h(a))$, thereby obtaining the required fibre map of g.

(S3) The fibre map $\langle Id, Id \rangle^{-1} : A \times A \longrightarrow PA$ is constructed by forming singletons and taking the intersection.

(S4) Given a commuting triangle:



where f is small and e is a cover, the fibre map $g^{-1} : B \longrightarrow PC$ is obtained by composing $f^{-1} : C \longrightarrow PA$ and the direct image map $Pe : PA \longrightarrow PC$.

(S5) First, we must show that $\mathcal{C}$ has disjoint coproducts which are stable under pullbacks. We borrow the construction from [10]: For A and B in $\mathcal{C}$, let the coproduct be defined as

$$[\![u{:}PA, v{:}PB | (\exists x\epsilon_A u.\, \forall y\epsilon_A u.\, y = x \wedge v = \emptyset_B) \vee (\exists x\epsilon_B u.\, \forall y\epsilon_B v.\, y = x \wedge u = \emptyset_A)]\!]$$

with inclusion morphisms defined by $x : A \mapsto \langle \{x\}, \emptyset_B \rangle$ and $y : B \mapsto \langle \{x\}, \emptyset_B \rangle$. For given small maps $f : A \longrightarrow C$ and $g : B \longrightarrow C$, the fibre map $[f, g]^{-1}$ is constructed by composing the fibre maps with the internal direct image maps of the inclusion maps, and then using (the restriction to $A + B$ of) the binary union map $(PA \times PB) \times (PA \times PB) \longrightarrow (PA \times PB)$.

(P1) To see that the subobject $\epsilon_A \rightarrowtail A \times PA$ is a small relation, consider the map $u \mapsto u \times \{u\}$. To classify a small relation $R \rightarrowtail A \times X$, compose the fibre map of the second projection with the internal direct image map of the first projection. The classifying map is unique by Extensionality, and, finally, any relation that has a classifying map is a small relation by S2.

(P2) By the Power set axiom.

1.4.4 Completeness

Let a class theory $\mathbb{T}$ over a signature Σ be given. A $\mathbb{T}$–*model* $\mathcal{M}_\mathbb{T}$ is a Σ–structure in which all sentences of $\mathbb{T}$ are true. Denote by $\mathcal{C}_\mathbb{T}$ the syntactic category of $\mathbb{T}$ (see e.g. [7, D1.4]).

We wish to show completeness for class theories with respect to models in class categories. To that effect, we will show that the syntactic category of a class theory can be given a class category structure.

Theorem 1.4.5 *$\mathcal{C}_\mathbb{T}$ is a class category.*

Proof First, we need to define a power structure on $\mathcal{C}_\mathbb{T}$: For any object $\ulcorner x{:}A \mid \phi \urcorner$ in $\mathcal{C}_\mathbb{T}$, define its power object to be $\ulcorner u{:}PA \mid \forall x\epsilon_A u.\, \phi \urcorner$ (here we are using the existence of product types and pairing terms in order to assume without loss of generality that the context consists of only one term). The epsilon subobject is defined to be $\ulcorner x{:}A, u{:}PA \mid x\epsilon_A u \wedge \forall x\epsilon_A u.\, \phi \urcorner$. By Theorem 1.4.3, it now remains to check that certain sentences in the internal logic of $\mathcal{C}_\mathbb{T}$ are true (under the canonical interpretation). But this is a straightforward translation job. For example, an instance of Extensionality:

$$\forall u, v{:}P\Phi.\, \forall x{:}\Phi.\, (x\epsilon_\Phi u \leftrightarrow x\epsilon_\Phi v) \rightarrow u = v$$

where $\Phi = \ulcorner x{:}A \mid \phi \urcorner$, is, as a subobject of $\ulcorner \mid \top \urcorner$, equal to the interpretation in the generic model of the Σ-statement:

$$\forall u, v{:}PA.\, (\forall x{:}A.\, x \epsilon_A u \to \phi) \wedge (\forall x{:}A.\, x \epsilon_A v \to \phi)$$

$$\to ((\forall x{:}A.\, \phi \to (x \epsilon_A u \leftrightarrow x \epsilon_A v)) \to u = v)$$

which is provable in $\mathbb{T}$. (A more readable way of writing the statement above might be:

$\forall u, v \subseteq \phi.\, (\forall x \in \phi.\, x \epsilon u \leftrightarrow x \epsilon v) \to u = v.$)

We remark that in [2], this is proved by defining a morphism $[\sigma] : \ulcorner x{:}A | \phi \urcorner \longrightarrow \ulcorner y{:}B \mid \psi \urcorner$ to be small if $\psi \vdash_{\mathrm{BIST}} \text{Ƨ} x{:}A.\, \sigma$. $\dashv$

As a consequence (see e.g. [7, D1.4]), we have that class logic is complete with respect to models in categories of classes:

Corollary 1.4.6 (Completeness) *Let $\mathbb{T}$ be a class theory over a signature Σ. For any Σ–sentence ϕ, $\mathbb{T} \vdash \phi$ iff for every $\mathbb{T}$–model $M_{\mathbb{T}}$, $M_{\mathbb{T}} \models \phi$.*

1.5 Universes and Numbers

1.5.1 The untyped set theory BIST

So far we have been considering a certain typed set theory where a set is of a different type than its elements. We wish now to consider the, perhaps more familiar, situation where a set and its elements are all of the same type. Intuitively, this would seem to mean that we have a class A of elements which contains the class of all sets of elements of A, so that $PA \subseteq A$, so to speak. We specify first what it means for this situation to occur in a class theory, and then we give the set theory which is modeled as a result.

A *universal* object in a class category $\mathcal{C}$ is an object $\mathcal{U}$ such that for every object A in $\mathcal{C}$, there exists a monomorphism $A \rightarrowtail \mathcal{U}$. In particular, there exists a monomorphism $P\mathcal{U} \rightarrowtail \mathcal{U}$. A choice of such a monomorphism makes $\mathcal{U}$ a *universe*, that is, an object U together with a monomorphism $\iota : PU \rightarrowtail U$. A universe in $\mathcal{C}$ allows us to interpret untyped, or single-typed, set theories in $\mathcal{C}$. In particular, we can model set theories which, in addition to the membership predicate, contain a "sethood" predicate S:

Definition 1.5.2 In any class category $\mathcal{C}$ with a universe (U, ι), the *canonical* BIST *structure* with respect to (U, ι) interprets the sethood predicate S by the monomorphism $\iota : PU \rightarrowtail U$, and the "membership" relation $\in$ is interpreted as the composite monomorphism

$$\begin{array}{c} \epsilon_U \\ \big\downarrow \\ U \times PU \\ {\scriptstyle \langle Id\times\iota\rangle}\big\downarrow \\ U \times U \end{array}$$

From Theorem 1.3.10, it follows that the following set theory, called *Basic Intuitionistic Set Theory*, or BIST, holds in any class category with a universe (under the interpretation just described). A direct proof of this fact can be found in [2]. We state this result below for reference. In the following presentation, we make use of the shorthand notation of $\text{Ƨ}x.\,\phi$ for $\exists y.\, S(y) \wedge (\forall x.\, x \in y \leftrightarrow \phi)$, where y is not free in ϕ. $\text{Ƨ}y \in x.\,\phi$ is short for $\text{Ƨ}y.\, y \in x \wedge \phi$, and the expression $x \subseteq y$ stands for $S(x) \wedge S(y) \wedge \forall z \in x.\, z \in y$.

BIST1. (Membership) $y \in x \to S(x)$

BIST2. (Extensionality) $S(x) \wedge S(y) \wedge (\forall z.\, z \in x \leftrightarrow z \in y) \to x = y$

BIST3. (Empty Set) $\text{Ƨ}z.\,\bot$

BIST4. (Pairing) $\text{Ƨ}z.\, z = x \vee z = y$

BIST5. (Union) $S(x) \wedge (\forall y \in x.\, S(y)) \to \text{Ƨ}z.\, \exists y \in x.\, z \in y$

BIST6. (Replacement) $S(x) \wedge (\forall y \in x.\, \exists! z.\, \phi) \to \text{Ƨ}z.\, \exists y \in x.\, \phi$

BIST7. (Power Set) $S(x) \to \text{Ƨ}y.\, y \subseteq x$

BIST8. (Binary Intersection) $S(x) \wedge S(y) \to \text{Ƨ}z.\, z \in x \wedge z \in y$

Theorem 1.5.3 *For any class category $\mathcal{C}$, the set theory* BIST *is sound with respect to the canonical* BIST*structures in $\mathcal{C}$.*

We shall turn to the question of infinity shortly. First, we show that a restricted form of separation is derivable in BIST. For this purpose, we introduce the shorthand notation $!\phi$ (read *simply phi*) to stand for the formula

$$\text{Ƨ}z.\, z = \emptyset \wedge \phi$$

where z is not free in ϕ. (We say that a formula ϕ is *simple* if $!\phi$ is true, where context determines what counts as true—usually provable in BIST or valid in a model of BIST.) Separation for simple formulas is provable in BIST, and there are some nice closure properties for formulas that are simple in BIST. We state these results below, but postpone the proofs until section 3.2 (see also [1]).

Proposition 1.5.4 (!-Sep) BIST $\vdash (S(x) \wedge \forall y \in x. !\phi) \rightarrow \text{Ƨ}y.\, y \in x \wedge \phi$

Lemma 1.5.5 *The following hold in* BIST*:*

1. $!\bot$
2. $!\phi \wedge !\psi \rightarrow !(\phi \wedge \psi)$
3. $!\phi \wedge !\psi \rightarrow !(\phi \vee \psi)$
4. $(S(x) \wedge \forall y \in x. !\phi) \rightarrow\ !(\exists y \in x. \phi)$
5. $(S(x) \wedge \forall y \in x. !\phi) \rightarrow !(\forall y \in x. \phi)$
6. $!\phi \wedge !\psi \rightarrow !(\phi \rightarrow \psi)$
7. $\neg(\phi \wedge \psi) \wedge !(\phi \vee \psi) \rightarrow !\phi \wedge !\psi$

The following form of Δ_0 separation therefore holds.

Proposition 1.5.6 (Δ_0-Sep) *In* BIST*, separation holds for* S*–predicate free* Δ_0 *formulas in the context of a "well–typing", in the following sense: For a* Δ_0 *formula* ϕ *in which the* S*–predicate does not occur, let* $x_1, \ldots, x_n$*, be a list of all the variables occurring on the right hand side of an* ϵ *in* ϕ*. Construct a formula* ψ_n *by induction on* n *as follows:* $\psi_0 = \top$*. If* x_i *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge S(x_i)$*. If* x_i *is bound by a quantifier* $\forall x_i \epsilon t.$ *or* $\exists x_i \epsilon t.$ *and* t *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge S(t) \wedge \forall s \epsilon t.\, S(s)$*. If* t *itself is bound by a formula* $\forall t \epsilon u.$ *or* $\exists t \epsilon u.$ *and* u *is free in* ϕ*, then* $\psi_i = \psi_{i-1} \wedge S(u) \wedge \forall s \epsilon u.\, S(s) \wedge \forall p \epsilon s.\, S(p)$*. If* u *is bound as well, then continue in the same way. We have then that:*

$$\text{BIST} \vdash S(x) \wedge \psi_n \rightarrow \text{Ƨ}y \epsilon x.\, \phi$$

and if x_n *is free in* ϕ *that:*

$$\text{BIST} \vdash S(x) \wedge (\forall y \epsilon x.\, S(y)) \wedge \psi_{n-1} \rightarrow \text{Ƨ}x_n \epsilon x.\, \phi$$

The reader who is familiar with the presentation of BIST in [2] will have noticed that we have replaced the axiom of Intersection therein by a axiom of Binary Intersection (**BIST8**). The justification is in the following corollary:

Corollary 1.5.7 BIST $\vdash \forall w, x. (S(x) \wedge \forall y \in x. y \subseteq w) \rightarrow \text{Ƨ}z. z \in w \wedge \forall y \in x. z \in y$

Proposition 1.5.8 *In order to have unrestricted Δ_0 separation, it is sufficient to add to* BIST *an axiom stating that the S–predicate is simple:*

$$\forall x. !S(x)$$

Proof See Remark 3.2.10.

In addition to being sound with respect to models of the appropriate kind in class categories with universal objects, BIST is also complete with respect to such models. A detailed proof of this can be found in [2]. Again, the trick is to put a class category structure on the syntactic category $\mathcal{C}_{\text{BIST}}$. If $\ulcorner x \,|\, \phi \urcorner$ is an object, then $\ulcorner u \,|\, S(u) \wedge \forall x. x \in u \rightarrow \phi \urcorner$ is the power object, and a map $[\ulcorner x, y \mid \psi \urcorner]$ is small iff BIST $\vdash \forall y. \text{Ƨ}x. \psi$. The universal object is $\ulcorner x \mid \top \urcorner$. Here we should remark, again, that in [2], BIST contains a more general intersection axiom. However, the proof that the syntactic category is a class category in [2] makes use only of binary intersection, and consequently, since [2] also shows that the general intersection axiom is sound with respect to class categories, the more general intersection axiom is implied by binary intersection in the context of the other axioms (Corollary 1.5.7). We state the completeness result for reference:

Theorem 1.5.9 BIST *is sound and complete with respect to canonical* BIST *structures in class categories.*

Proof [2] $\dashv$

For further details concerning the relation between the internal logic of a class category with a universal object and the set theory modeled by this universal object, we refer to [10].

1.5.10 Infinity

Let $\mathcal{C}$ be a class category. A *small object* in $\mathcal{C}$ is an object A such that the unique morphism $A \longrightarrow 1$ is a small map. Denote by $\mathcal{C}_S$ the full subcategory of small objects of $\mathcal{C}$.

Proposition 1.5.11 *$\mathcal{C}_S$ is a topos with the class category structure it inherits from $\mathcal{C}$.*

Proof In [2]. $\dashv$

Definition 1.5.12 A *class category with infinity* is a class category with a small object $\mathcal{N}$ equipped with morphisms $0 : 1 \longrightarrow \mathcal{N} \longleftarrow \mathcal{N} : s$ such that $\mathcal{N}$ is a natural number object (n.n.o) in $\mathcal{C}_S$.

By [7, D5.1.3], this condition is equivalent to saying that $\mathcal{C}$ contains a small object I equipped with a monomorphism $t : I \rightarrowtail I$ and a well–supported small subobject $A \rightarrowtail I$ disjoint from t. An object satisfying Definition 1.5.12 is called a *natural number set* (n.n.s.) in [10]. The category of ideals, which we will describe in the next section, is an example of a class category with an n.n.o. and an n.n.s. that do not coincide (ibid).

Definition 1.5.13 We extend a class signature to a *class signature with infinity* by adding a type $\mathcal{N}$ to the set of Σ–types, and two function symbols $0 : 1 \to \mathcal{N}$ and $s : \mathcal{N} \to \mathcal{N}$ to the set Σ_F.

A *class theory with infinity* is a class theory over a class signature with infinity to which we have added the following axioms, the conjunction of which we shall refer to as CT8.(Infinity):

N1 $\exists u{:}P\mathcal{N}.\, \forall x{:}\mathcal{N}.\, x \epsilon_{\mathcal{N}} u$

N2 $\forall x, y{:}\mathcal{N}.\, s(x) = s(y) \to x = y$

N3 $\forall x{:}\mathcal{N}.\, s(x) \neq 0$

N4 $\forall u{:}P\mathcal{N}.\, 0 \epsilon_{\mathcal{N}} u \wedge (\forall x{:}\mathcal{N}.\, x \epsilon_{\mathcal{N}} u \to s(x) \epsilon_{\mathcal{N}} u) \to \forall x{:}\mathcal{N}.\, x \epsilon_{\mathcal{N}} u$

For a class signature Σ with infinity, a Σ–structure is just a structure for the class signature part of Σ (as in section 1.3) in a class category with infinity, with the added requirement the the type $\mathcal{N}$ be interpreted as the object $\mathcal{N}$ etc., just as indicated by our choice of symbols.

Proposition 1.5.14 *(Soundness) Let Σ be a class signature with infinity. Any Σ–structure in a class category with infinity satisfies axiom* CT8.

Proof Immediate, since $\mathcal{N}$ is a n.n.o. in the topos $\mathcal{C}_S$ and the inclusion into $\mathcal{C}$ is Heyting and preserves the class structure. ⊣

Proposition 1.5.15 *(Internal characterization) Let $\mathcal{C}$ be a class category, and assume that $\mathcal{C}$ contains an object $\mathcal{N}$ equipped with morphisms $0 : 1 \longrightarrow \mathcal{N} \longleftarrow \mathcal{N} : s$ such that the infinity axioms above are satisfied. Then $\mathcal{C}$ is a class category with infinity.*

PROOF By N1, $\mathcal{N}$—and thereby the morphisms 0 and s as well as the power type $P\mathcal{N}$ and the membership subobject $\epsilon_{\mathcal{N}}$—are in $\mathcal{C}_S$. We may therefore restrict our attention to the small object topos, where N2-N4 describes $\mathcal{N}$ as a n.n.o. (see e.g.[7, D5]) $\dashv$

Corollary 1.5.16 *(Completeness) Let $\mathbb{T}$ be a class theory with infinity over a signature Σ. For any Σ–sentence ϕ,*
$\mathbb{T} \vdash \phi$ iff $\mathcal{M}_{\mathbb{T}} \models \phi$, for every $\mathbb{T}$–model $\mathcal{M}_{\mathbb{T}}$

PROOF A proof similar to that of Theorem 1.4.5 shows that $\ulcorner x{:}\mathcal{N} \mid \top \urcorner$ is a n.n.o. in the subcategory $\mathcal{C}_{\mathbb{T}S}$. $\dashv$

To introduce a similar axiom of infinity for BIST, we extend the language of BIST with two constant symbols, $\mathbb{N}$ and 0, and a binary relation symbol s. We extend BIST with the following axioms, the conjunction of which we call BIST9 (Infinity):

BIST N1 $0 \in N$

BIST N2 $\forall x, y.\, s(x, y) \to x \in N \wedge y \in N$

BIST N3 $\forall x \in N.\, \exists^{=1} y \in N.\, s(x, y)$

BIST N4 $\forall x, y.\, s(x, y) \to y \neq 0$

BIST N5 $\forall x, y, z.\, s(x, z) \wedge s(y, z) \to x = y$

BIST N6 $\forall u \subseteq N.\, 0 \in u \wedge (\forall x \in u.\, \forall y.\, s(x, y) \to y \in u) \to u = N$

Proposition 1.5.17 BIST+Infinity *is sound and complete with respect to class categories with infinity equipped with a universal object.*

PROOF (Soundness) Choose a monomorphism $i : \mathcal{N} \rightarrowtail \mathcal{U}$, thereby obtaining monomorphisms $\iota \circ Pi : P\mathcal{N} \rightarrowtail P\mathcal{U} \rightarrowtail \mathcal{U}$ and $\iota \circ P(op) \circ P(i \times i) : P(\mathcal{N} \times \mathcal{N}) \rightarrowtail P(\mathcal{U} \times \mathcal{U}) \rightarrowtail P\mathcal{U} \rightarrowtail \mathcal{U}$, where op is the ordered pair map $(\langle x, y \rangle \mapsto \{\{x\}, \{x, y\}\})$, see [10] or [2] for details. Composing with these maps gives the required global points interpreting the constants 0 and N and s.

(Completeness) It is straightforward to check that the object $\ulcorner x \mid x \in N \urcorner$ in $\mathcal{C}_{\mathrm{BIST}}$ satisfies CT8. $\dashv$

Remark 1.5.18 Instead of extending the language of BIST one may prefer to consider an axiom of infinity directly in the language of BIST. One can define the notions of ordered pair and function in BIST, and so such an axiom can be presented (as in [2]) by formally stating that there exists a set, an element of that set, and an injective function on that set such that the element is not in the image of the function. Soundness and completeness with respect to class categories with infinity still hold. To show completeness, one needs to consider the slice of the syntactic category over the object $\ulcorner x, y, z \mid \phi(x, y, z)\urcorner$, where $\phi(x, y, z)$ states that x is a set, y is an injective function on x, z is an element of x, and z is not in the image of y.

2 Ideals over a topos

2.1 Small maps in sheaves

As already remarked, the small objects in $\mathcal{C}$ form a topos (Proposition 1.5.11). Moreover, it is shown in [2] that every small topos occurs as the category of small objects in a category with class structure. The purpose of this section is to provide a new proof of the latter fact, using a more canonical construction that avoids some of the difficulties in the original proof. The original proof proceeded in two steps where, first, it was shown that every small topos is equivalent with a topos with a distinguished system of *inclusions*, and, second, a class category of *inclusion ideals* was defined over that topos (the definitions are reviewed in section A.1). There were mainly two drawbacks to this approach. First, the proof that equivalent toposes with such systems exist was felt to be unnecessarily complex. And second, the properties of the universal object of the resulting category of ideals would depend upon the particular choice of equivalent topos with inclusions. Now, it can be shown that a useful notion of ideal can be defined directly on a given topos, without going via systems of inclusions, and that a corresponding class category of ideals can be constructed as a subcategory of sheaves on that topos. Ivar Rummelhoff has pursued this idea using the inductive completion ([7, C4.2]), see [10]. In addition to being more direct, this construction also allows one to gain a better insight into the properties of various universes containing the original topos. In this section, we will present the construction in a sheaf setting. A comparison between this construction and the original construction found in [2] can be found in section A.1.

The idea, then, is to use the category of sheaves over a given small topos. A candidate system of small maps is proposed for which the representables are the small objects, and a full subcategory of sheaves is identified in which this system satisfies the small maps conditions S1–S5. We then identify a power structure, and show how one can find universes which contain the original small topos.

Let a small topos (or, for this subsection, just a pretopos) $\mathcal{E}$ be given. Consider the category $\mathrm{Sh}(\mathcal{E})$ of sheaves on $\mathcal{E}$, for the coherent covering [7, A2.1.11(b)]. Recall that the Yoneda embedding $y : \mathcal{E} \hookrightarrow \mathrm{Sh}(\mathcal{E})$ is a full and faithful Heyting functor [7, D3.1.17].

We intend to build a class category in $\mathrm{Sh}(\mathcal{E})$ where the representables are the small objects. First, we define a system $\mathcal{S}$ of small maps on $\mathrm{Sh}(\mathcal{E})$ by including in $\mathcal{S}$ the morphisms of $\mathrm{Sh}(\mathcal{E})$ with "representable fibers" in the following sense:

Definition 2.1.1 (Small Map) A morphism $f : A \longrightarrow B$ in $\mathrm{Sh}(\mathcal{E})$ is *small* if for any morphism with representable domain $g : yD \longrightarrow B$, there exists an object C in $\mathcal{E}$, and morphisms f', g' in $\mathrm{Sh}(\mathcal{E})$ such that the following is a pullback:

$$\begin{array}{ccc} yC & \xrightarrow{f'} & yD \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

Thus, in this sense, *small maps pull representables back to representables.*

Proposition 2.1.2 *$\mathcal{S}$ satisfies axioms S1, S2, and S5.*

Proof S1 and S2 follow easily from the Two Pullback lemma (also known as the Pasting lemma).

For S5, the pullback of, say, $yD \xrightarrow{h} C$ along $(f, g) : A + B \longrightarrow C$ is the coproduct of the pullback of h along f and of h along g. But this is representable, since representables are closed under finite coproducts in $\mathrm{Sh}(\mathcal{E})$. ⊣

We move to consider S3. A *directed diagram* (in a category $\mathcal{C}$) is a functor $I \longrightarrow \mathcal{C}$ where I is a directed preorder. A small directed diagram in $\mathcal{C}$ in which (the image of) every morphism is a monomorphism in $\mathcal{C}$ we shall call an *ideal diagram.* An ideal diagram has no non-trivial parallel pairs, and is therefore also a filtered diagram (and every small filtered diagram in which the image of every morphism is a monomorphism can be represented as an ideal diagram).

Definition 2.1.3 (Ideal over $\mathcal{E}$) An object A in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$ is an *ideal over* $\mathcal{E}$ if it can be written as a colimit of an ideal diagram $I \longrightarrow \mathcal{E}$ of representables,

$$A \;\cong\; \varinjlim_I (yC_i)$$

We denote the full *subcategory of ideals* in $\mathbf{Sets}^{E^{\mathrm{op}}}$ by $\mathrm{Idl}(\mathcal{E})$.

Lemma 2.1.4 *Every ideal is a sheaf.*

Proof Since an ideal diagram is a filtered diagram, filtered colimits commute with finite limits, being a sheaf is a finite limit condition, and all representables are sheaves, all such presheaves are also sheaves. $\dashv$

In accordance with a conjecture by André Joyal, it now turns out that the ideals over $\mathcal{E}$ are exactly the sheaves for which S3 holds, i.e. for which the diagonal $A \rightarrowtail A \times A$ is small. The following proof of this fact is due to Steve Awodey:

Lemma 2.1.5 *Any sheaf F can be written as a colimit (in $\mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$) of representables $\varinjlim_I(yC_i)$ where I has the property that for any two objects i, j in I, there is an object k in I and morphisms $i \longrightarrow k$ and $j \longrightarrow k$.*

Proof We may write a sheaf F as the colimit of the composite functor $\int F \xrightarrow{\pi} \mathcal{E} \xrightarrow{y} \mathbf{Sets}^{\mathcal{E}^{\mathrm{op}}}$, where $\int F$ is the category of elements of F, and π is the forgetful functor. The objects in $\mathrm{Sh}(\mathcal{E})$ can be characterized as the functors $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$ which preserve monomorphisms and finite products. It follows that $\int F$ has the required property, since for any two objects (A, a), (B, b) in $\int F$ (with $a \in FA$, $b \in FB$),

$$(A, a) \longrightarrow (A + B, \langle a, b \rangle) \longleftarrow (B, b)$$

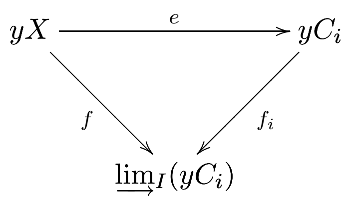
(By the coproduct $A + B$, we mean the coproduct in $\mathcal{E}$, hence the product $A \times B$ in $\mathcal{E}^{\mathrm{op}}$, which is sent to the product $FA \times FB$ in $\mathbf{Sets}$.) $\dashv$

Theorem 2.1.6 *For any sheaf F, the following are equivalent:*

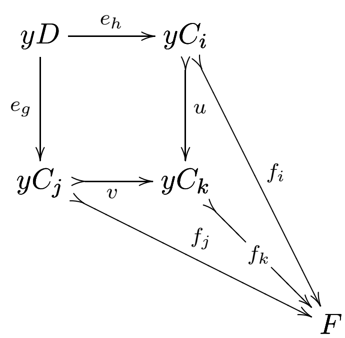
1. *F is an ideal.*
2. *The diagonal $F \rightarrowtail F \times F$ is a small map.*
3. *For all arrows with representable domain $yC \xrightarrow{f} F$, the image of f in sheaves is representable, $f : yC \twoheadrightarrow yD \rightarrowtail F$, for some D in $\mathcal{E}$.*

Proof (1)⇒(2):

We write F as an ideal diagram of representables, $F = \varinjlim_I(yC_i)$. Note that the pullback of any arrow $f : A \longrightarrow F \times F$ along $\Delta : F \longrightarrow F \times F$ is the equalizer of the pair $\pi_1 f, \pi_2 f : A \rightrightarrows F$. Thus let $g, h : yD \rightrightarrows F$ be given, and we must verify that their equalizer $e : E \rightarrowtail yD$ is representable. Recall that, in $\mathbf{Sets}^{E^{\mathrm{op}}}$, if we are given a colimit $\varinjlim_I(yC_i)$ and an arrow $f : yX \longrightarrow \varinjlim_I(yC_i)$, f factors through the base of the colimiting cocone, i.e.



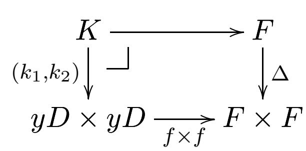
for some i (where f_i is an arrow of the colimiting cocone). Hence we may factor h as $yX \xrightarrow{e_h} yC_i \xrightarrow{f_i} \varinjlim_I(yC_i)$ and g as $yX \xrightarrow{e_g} C_j \xrightarrow{f_j} \varinjlim_I(yC_i)$. Since the diagram is directed, there is a C_k and arrows u, v such that the two triangles in the following commute:



Since f_k is monic, the equalizer $e : E \rightarrowtail yD$ of $h = f_k u e_h$ and $g = f_k v e_g$ is precisely the equalizer of ue_h and ve_g. But Yoneda preserves and reflects equalizers, so we may conclude that the equalizer of h and g is representable, $E \cong yC$.

(2)⇒(3):

Let $yD \xrightarrow{f} F$ be given. The kernel pair k_1, k_2 of f can be described as the pullback:



Since $yD \times yD \cong y(D \times D)$ is representable and the diagonal of F is small, K is representable ($K \cong yK$, with some abuse of notation). Hence we may rewrite the kernel pair as

$$yK \underset{yk_2}{\overset{yk_1}{\rightrightarrows}} yD \xrightarrow{f} F$$

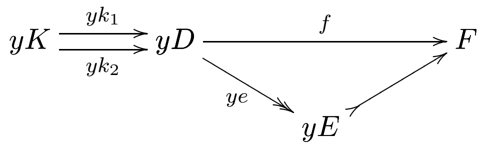
The kernel pair is an equivalence relation in $\widehat{\mathcal{E}}$. Since Yoneda is full and faithful and cartesian, $K \underset{k_2}{\overset{k_1}{\rightrightarrows}} D$ is an equivalence relation in $\mathcal{E}$. Since $\mathcal{E}$ is effective, there is a coequalizer

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} D \overset{e}{\twoheadrightarrow} E$$

such that k_1 and k_2 is the kernel pair of e. Since Yoneda preserves pullbacks and regular epis into $\mathrm{Sh}(\mathcal{E})$,

$$yK \underset{yk_2}{\overset{yk_1}{\rightrightarrows}} yD \overset{ye}{\twoheadrightarrow} yE$$

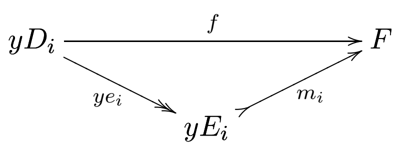
is a coequalizer diagram in $\mathrm{Sh}(\mathcal{C})$. This gives us, then, the required epi–mono factorization:



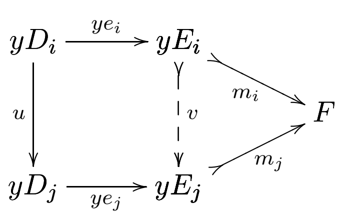
(3)⇒(1):

Step 1: To construct an ideal diagram of representables.

We write F as a colimit $F = \varinjlim_I (yD_i)$, in accordance with Lemma 2.1.5 (so that I is the category of elements of F). Now, for each $i \in I$, factor in sheaves the cocone arrow $yD_i \xrightarrow{f_i} F$:



For $yD_i \xrightarrow{u} yD_j$ in the diagram I, consider the diagram:



Since $f_i = f_j u$, it follows that f_i factors through yE_j, which gives us the mono v, making the triangle in the diagram commute (to see this, the diagram must be considered in $\mathrm{Sh}(\mathcal{E})$, where e_i is a cover). Since m_j is monic, the square commutes.

The new diagram I' of the yE_i and v thus obtained is directed, since I has the property described in Lemma 2.1.5 and any parallel pair of arrows collaps by the construction.

Step 2: To show $F \cong \varinjlim_{I'}(yE_i)$

Observe that the ye_i's in the diagram above give us a morphism $e : \varinjlim_I yD_i \longrightarrow \varinjlim_{I'} yE_i$, while the m_i's give us a monomorphism $\varinjlim_{I'} yE_i \rightarrowtail F$, such that the following commutes:

$$\begin{array}{ccc} \varinjlim_I yD_i & \xrightarrow{\quad e \quad} & \varinjlim_{I'} yE_i \\ & {\scriptstyle\cong}\searrow \qquad \swarrow{\scriptstyle m} & \\ & F & \end{array}$$

Thus m is also an isomorphism. ⊣

In order to ensure that S3 is satisfied, we therefore narrow our attention from $\mathrm{Sh}(\mathcal{E})$ to the full subcategory of ideals, denoted $\mathrm{Idl}(\mathcal{E})$. We shall see that no further restriction is needed. First, we verify that $\mathrm{Idl}(\mathcal{E})$ is a positive Heyting category:

Lemma 2.1.7 $\mathrm{Idl}(\mathcal{E})$ *is closed under (presheaf) subobjects and finite limits.*

Proof We use the description of ideals as sheaves with small diagonal. That $\mathrm{Idl}(\mathcal{E})$ is closed under subobjects follows from S2.

$1 \xrightarrow{\Delta} 1 \times 1$ is iso, hence small.

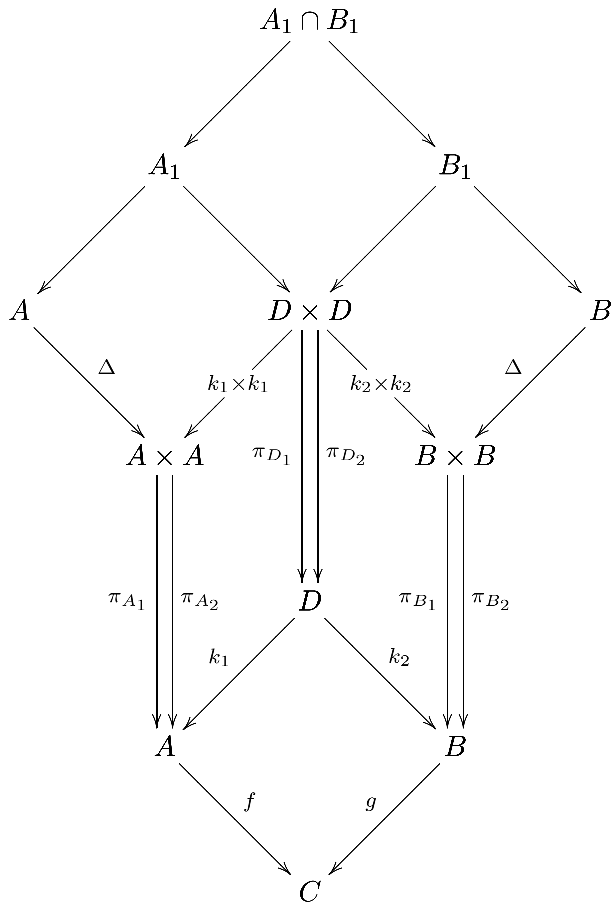
If A, B are ideals and C is any sheaf, we consider the pullback:

$$\begin{array}{ccc} D & \xrightarrow{k_2} & B \\ {\scriptstyle k_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Now, if we pull the diagonals back:

$$\begin{array}{ccc} A_1 & \longrightarrow & A \\ {\scriptstyle \alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ D \times D & \xrightarrow[k_1 \times k_1]{} & A \times A \end{array} \qquad \begin{array}{ccc} B_1 & \longrightarrow & B \\ {\scriptstyle \beta}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ D \times D & \xrightarrow[k_2 \times k_2]{} & B \times B \end{array}$$

By a diagram chase, the diagonal of D is $A_1 \cap B_1$, which is small since smallness is preserved by pullback and composition. We draw the diagram in which to chase:



Where all squares not involving projections are pullback squares.

Lemma 2.1.8 Idl($\mathcal{E}$) *is closed under finite coproducts (of sheaves), and inclusion maps are small.*

PROOF $0 \longrightarrow 0 \times 0$ is iso, so small.

Now, the terminal object 1 in Sh($\mathcal{E}$) is representable, and so is $1+1$, since Yoneda preserves finite coproducts. The inclusion $i_1 : 1 \longrightarrow 1+1$ is therefore small. But coproducts in Sh($\mathcal{E}$) being disjoint, the following is a pullback:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_1} \\ A+B & \xrightarrow[!_A + !_B]{} & 1+1 \end{array}$$

So by S2, the inclusion map i_A is small.

The diagonal of $A + B$ can be regarded as the disjoint union of the diagonal of A and of B:

$$\begin{array}{ccccc} A & \xrightarrow{p_A} & A+B & \xleftarrow{p_B} & B \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ A \times A & \dashrightarrow & (A+B)\times(A+B) & \dashleftarrow & B \times B \\ & {\scriptstyle p_{A\times A}}\searrow & \uparrow{\scriptstyle \cong} & \swarrow{\scriptstyle p_{B\times B}} & \\ & & (A\times A)+(A\times B)+(B\times A)+(B\times B) & & \end{array}$$

By smallness of coproduct inclusions and isos, and applying S5, if A, B are ideals then so is $A + B$. ⊣

Proposition 2.1.9 Idl($\mathcal{E}$) *is positive Heyting, and with the structure inherited from* Sh($\mathcal{E}$)*.*

Proof We have done finite limits and finite coproducts. For a morphism $f : A \longrightarrow B$ of ideals, $Im(f)$ is an ideal, since there is a monomorphism $Im(f) \rightarrowtail B$. The cover $e : A \twoheadrightarrow Im(f)$ is the coequalizer of its kernel pair in Sh($\mathcal{E}$), the kernel pair is the same in Idl($\mathcal{E}$), so e is also a regular epimorphism in Idl($\mathcal{E}$).

For dual images, since Idl($\mathcal{E}$) is closed under subobjects and finite limits can be taken in sheaves, dual images can also be taken in sheaves.

Lemma 2.1.10 *S4 is satisfied in* Idl($\mathcal{E}$)*.*

Proof Let $A \xrightarrow{a} B \xrightarrow{b} C$ be given, and assume $b \circ a$ is small. Let $yG \longrightarrow C$ be given, and consider the following two pullback diagram:

$$\begin{array}{ccccc} yD & \twoheadrightarrow & E & \longrightarrow & yG \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[a]{} & B & \xrightarrow[b]{} & C \end{array}$$

By Theorem 2.1.6, the image of a representable is a representable in Idl($\mathcal{E}$). Hence E in the diagram above is (isomorphic to) a representable. ⊣

We summarize the results of this subsection:

Theorem 2.1.11 *For any pretopos $\mathcal{E}$, the full subcategory* $\mathrm{Idl}(\mathcal{E}) \hookrightarrow \mathrm{Sh}(\mathcal{E})$ *of ideals is a positive Heyting category with a system of small maps satisfying axioms S1–S5.*

We conclude by noting a characterizing feature of $\mathrm{Idl}(\mathcal{E})$ which we shall make extensive use of in 2.2.4:

Lemma 2.1.12 $\mathrm{Idl}(\mathcal{E})$ *has colimits of ideal diagrams ("*ideal colimits*").*

Proof Any such diagram is an ideal diagram of representables, see [7, section C2]. ⊣

Proposition 2.1.13 *If $\mathcal{C}$ is a category with ideal colimits, and $F : \mathcal{E} \longrightarrow \mathcal{C}$ is a functor which preserves monomorphisms, then there is a unique (up to natural isomorphism) extension $\tilde{F} : \mathrm{Idl}(\mathcal{E}) \longrightarrow \mathcal{C}$ of F such that $\tilde{F}$ is* continuous*, in the sense of preserving ideal colimits, and such that the following commutes:*

$$\begin{array}{ccc} \mathrm{Idl}(\mathcal{E}) & \xrightarrow{\tilde{F}} & \mathcal{C} \\ {\scriptstyle y}\uparrow & \nearrow_{F} & \\ \mathcal{E} & & \end{array}$$

In particular, the power object functor $\mathcal{P} : \mathrm{Idl}(\mathcal{E}) \to \mathrm{Idl}(\mathcal{E})$ which sends an object to its power object and a morphism to its corresponding direct image map is continuous.

Proof Write $E = \varinjlim_I (yC_i)$ and set $\tilde{F}(E) = \varinjlim_I (FC_i)$. ⊣

2.2 Power objects and universes in $\mathrm{Idl}(\mathcal{E})$

2.2.1 Power objects

It remains to establish the existence of a power structure on $\mathrm{Idl}(\mathcal{E})$ which corresponds to the small map structure we have chosen. In this section, we require $\mathcal{E}$ to be a topos, for we shall use the power objects in $\mathcal{E}$ to build power objects for ideals. We shall rely heavily on the characterization of $\mathrm{Idl}(\mathcal{E})$ as the colimits of ideal diagrams of representables, on the fact that ideal diagrams in $\mathbf{Sets}^{E^{\mathrm{op}}}$ commute with finite limits, and on the fact that Yoneda preserves and reflects finite limits into $\mathrm{Idl}(\mathcal{E})$. Now, the *power functor* $P : \mathcal{E} \to \mathcal{E}$ which sends an object A to its power object PA and a morphism $f : A \longrightarrow B$ to the direct image morphism $Pf : PA \longrightarrow PB$, preserves monomorphisms. Therefore, if we have an ideal $A = \varinjlim_{i\in I}(yA_i)$ in $\mathrm{Sh}(\mathcal{E})$, we can apply the power object functor to obtain another ideal $PA = \varinjlim_{i\in I}(yPA_i)$.

Lemma 2.2.2 *Let $A = \varinjlim_{i\in I}(yA_i)$ where I is an ideal diagram. Then the ideal $PA := \varinjlim_{i\in I}(yPA_i)$ together with the relation $\epsilon_A := \varinjlim_{i\in I}(y\epsilon_{A_i}) \rightarrowtriangle A \times PA$ satisfies axiom P1.*

Proof First, we should complete the definition of the subobject ϵ_A. In any class category, and in any topos in particular, any monomorphism $u : A \rightarrowtail B$ leads to the following being a pullback:

$$\begin{array}{ccc} \epsilon_A & \longrightarrow & \epsilon_B \\ \downarrow & \lrcorner & \downarrow \\ A \times PA & \xrightarrow[u\times Pu]{} & B \times PB \end{array}$$

Now, $A = \varinjlim_I(yA_i)$. Say, for the sake of having an example, that $u : yA_i \rightarrowtail yA_j$ is an arrow of that diagram. Since I is filtered, $\varinjlim_I(yA_i) \times \varinjlim_I(yPA_i) \cong \varinjlim_I(y(A_i \times PA_i))$, and the pullback

$$\begin{array}{ccc} y\epsilon_{A_i} & \xrightarrow{\epsilon_u} & y\epsilon_{A_j} \\ \downarrow & \lrcorner & \downarrow \\ y(A_i \times PA_i) & \xrightarrow[u\times Pu]{} & y(A_j \times PA_j) \end{array}$$

serves to illustrate what the arrows are in the diagram $\epsilon_A := \varinjlim_{i\in I}(y\epsilon_{A_i})$, and what the monomorphism $\epsilon_A \rightarrowtail A \times PA$ is. It follows from the construction that ϵ_A is a small relation.

Let a $yC \rightarrowtail A$ be a small subobject of the ideal $A = \varinjlim_I(yA_i)$. The inclusion arrow $yC \rightarrowtail \varinjlim_I(yA_i)$ factors through some colimiting cocone morphism $yA_i \rightarrowtail A$, and we get the following diagram, in which $\gamma : 1 \longrightarrow PA_i$ classifies $C \rightarrowtail A_i$:

$$\begin{array}{ccccc} yC & \longrightarrow & y\epsilon_{A_i} & \rightarrowtail & \varinjlim_I(\epsilon_{yA_i}) \\ \downarrow \ \lrcorner & & \downarrow \ \lrcorner & & \downarrow \\ yA_i \times 1 & \xrightarrow{Id\times\gamma} & yA_i \times yPA_i & \rightarrowtail & \varinjlim_I(yA_i \times yPA_i) \\ \downarrow \ \lrcorner & & \downarrow & \overset{=}{\swarrow} & \\ \varinjlim_I(yA_i) \times 1 & \longrightarrow & \varinjlim_I(yA_i \times yPA_i) & & \end{array}$$

from which we can conclude that the global point $1 \xrightarrow{\gamma} yPA_i \rightarrowtail \varinjlim_I(yPA_i)$ classifies $yC \rightarrowtail A$. We see that nothing prevents this argument from going through in the slightly more general case when yC is a small relation $yC \rightarrowtail A \times yD$, for some $D \in \mathcal{E}$, so that we get a classifying map $\rho : yD \longrightarrow PA$ such that:

$$\begin{array}{ccc} yC & \longrightarrow & \epsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times yD & \xrightarrow[Id \times \rho]{} & A \times PA \end{array}$$

For the general situation, consider a small relation $R \rightarrowtriangle A \times X$, and write X as a ideal diagram of representables, $X = \varinjlim_J(yC_j)$. Since $\pi_2 : R \longrightarrow X$ pulls representables back to representables, and since pullbacks commute with filtered colimits, we obtain a reindexing of R as a colimit of a diagram over J of representables by considering the pullback

$$\begin{array}{ccc} \varinjlim_J(\pi_2^*(yC_j)) & \longrightarrow & \varinjlim_J(yC_j) \\ \cong\downarrow & \lrcorner & \downarrow = \\ R & \xrightarrow[\pi_2]{} & X \end{array}$$

This allows to consider each index $j \in J$ separately, and build a cocone over $(yC_j)_{j \in J}$ with PA as vertex by using the classifying maps

$$\begin{array}{ccc} \pi_2^*(yC_j) & \longrightarrow & \epsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times yC_j & \longrightarrow & A \times PA \end{array}$$

thus obtaining the classifying map $X \longrightarrow PA$. $\dashv$

It follows from Lemma 2.2.2 and Theorem 2.1.11 that for ideal diagrams I and J, if $\varinjlim_I(yC_i) \cong \varinjlim_J(yD_j)$, then $\varinjlim_I(yPC_i) \cong \varinjlim_J(yPD_j)$, as the system of small maps determine the power objects up to isomorphism in a class category (P2 not needed). Hence power objects for ideals may be defined in the manner of Lemma 2.2.2.

It remains to verify axiom P2. We need to construct an internal power set map $\mathcal{P} : PA \longrightarrow PPA$, that is, a classifying map for $\subseteq_A \rightarrowtriangle PA \times PA$. If $A = \varinjlim_I(yA_i)$, then $PA = \varinjlim_I(yPA_i)$ and $PPA = \varinjlim_I(yPPA_i)$. In any class category, $\mathcal{E}$ in particular, the following square commutes for any $f : A \longrightarrow B$:

$$\begin{array}{ccc} PA & \xrightarrow{\mathcal{P}_A} & PPA \\ {\scriptstyle Pf}\downarrow & & \downarrow{\scriptstyle PPf} \\ PB & \xrightarrow[\mathcal{P}_B]{} & PPB \end{array}$$

and if f is a monomorphism, then the square is a pullback. This allows us to construct the power set map $\mathcal{P} : PA \longrightarrow PPA$ directly out of the maps $\mathcal{P}_{A_i} : PA_i \longrightarrow PPA_i$ for $i \in I$. Correspondingly, in any class category, $\mathcal{E}$in particular, if $f : A \rightarrowtail A_j$ is a monomorphism, then

$$\begin{array}{ccc} \subseteq_{A_i} & \rightarrowtail & \subseteq_{A_j} \\ \downarrow & \lrcorner & \downarrow \\ PA_i \times PA_i & \xrightarrow[Pf \times Pf]{} & PA_j \times P_j \end{array}$$

is a pullback, and we can define the subobject $\varinjlim_I(y \subseteq_{A_i}) \rightarrowtail \varinjlim_I(yA_i) \times \varinjlim_I(yA_i) = PA \times PA$. It is now straighforward to verify that

$$\begin{array}{ccc} \varinjlim_I(\subseteq_{A_i}) & \longrightarrow & \varinjlim_I(\epsilon_{PA_i}) \\ \downarrow & \lrcorner & \downarrow \\ \varinjlim_I(PA_i) \times \varinjlim_I(PA_i) & \xrightarrow[Id \times \mathcal{P}]{} & \varinjlim_I(PA_i) \times \varinjlim_I(PPA_i) \end{array}$$

is a pullback, and the verification that $\varinjlim_I(\subseteq_{A_i}) \cong \subseteq_A$ is a similar diagram chase.

To summarize, then:

Theorem 2.2.3 *The full subcategory* $\mathrm{Idl}(\mathcal{E}) \hookrightarrow \mathrm{Sh}(\mathcal{E})$ *of ideals is a class category with respect to the small maps given in Lemma 2.2.2 .*

2.2.4 Universes

We move to find a universe in $\mathrm{Idl}(\mathcal{E})$. We are particularly interested in universes U which include $\mathcal{E}$, in the sense that for every representable yC there is a monomorphism $yC \rightarrowtail U$. This will allow us to conclude that every topos occurs, up to equivalence, as the small objects of a class category with a universal object.

Since the powerobject functor $\mathcal{P}$ is continuous (Proposition 2.1.13), we can find fixed points for it (further details can be found in both [2] and [10]). For one example, we compose $\mathcal{P}$ on $\mathrm{Idl}(\mathcal{E})$ with the continuous functor $C \mapsto A+C$ for a fixed A in $\mathrm{Idl}(\mathcal{E})$, to obtain the functor G_A defined by $C \mapsto A + P(C)$. To construct a universal object, we wish for every representable to have a monomorphism into our universe, so take as our starting point $A := \coprod_{C \in \mathcal{E}} yC$ (where the coproduct is taken in sheaves). This is an ideal, for it is the colimit of the ideal diagram of finite coproducts of representables, which themselves are representable, with arrows the coproduct inclusions.

Now consider the ideal diagram of ideals

$$A \xrightarrow{i_A} A + PA \xrightarrow{Id_A + Pi_A} A + P(A + PA) \longrightarrow \dots$$

Where i_A is the coproduct inclusion. Call the colimit U. Then, since the functor G_A is continuous,

$$A + PU \cong U$$

so we have a universe, U, consisting of the class A of atoms and the class PU of sets (with respect to the powerobject endofunctor $\mathcal{P}$, U is the free $\mathcal{P}$-algebra over A).

U is not yet a universal object, however. We obtain, finally, our category with class structure containing $\mathcal{E}$ as the small objects by cutting out the part of $\mathrm{Idl}(\mathcal{E})$ we need:

Proposition 2.2.5 *If $\mathcal{C}$ is a class category and U is a universe in $\mathcal{C}$, then the full subcategory $\downarrow(U)$ of objects A in $\mathcal{C}$ such that there exists a monomorphism $A \rightarrowtail U$ is a class category with the structure it inherits from $\mathcal{C}$ and with U as its universal object.*

Proof We can demonstrate the existence of a *encoded ordered pair* map $U \times U \longrightarrow U$ e.g. by reasoning in $\mathcal{C}$: Let $\langle x, y \rangle : U \times U$ be given. Since the language of $\mathcal{C}$ is a class logic, there exists a unique $z : PPU$ such that $z = \{\{x\}, \{x, y\}\}$. The encoded ordered pair map is then the monomorphism obtained by composing this monomorphism $U \times U \rightarrowtail PPU$ with the inclusions $PPU \rightarrowtail PU \rightarrowtail U$. It is now straightforward to verify that $\downarrow(U)$ is closed under the Heyting and class category structure. ⊣

Theorem 2.2.6 *Every topos occurs, up to equivalence, as the small objects in a class category with a universal object.*

Remark 2.2.7 The particular universe constructed to prove Theorem 2.2.6 models BIST enhanced with a decidable sethood axiom, namely:

$$\forall x. S(x) \vee \neg S(x)$$

since $PU \rightarrowtail U$ is a coproduct inclusion. As a further consequence, separation holds for all bounded formulas. That is to say, if ϕ is a bounded (i.e. Δ_0) formula, then the (universally quantified) statement

$$S(x) \to \text{Ƨ}y \in x. \phi$$

is modeled by this universe. (It also satisfies some other conditions like $\in$-induction, but we will not pursue this here. See [10].) Moreover, if the underlying topos $\mathcal{E}$ is boolean, then this universe models the principle of excluded middle for all simple formulas (that is, formulas ϕ such that $!\phi$ holds), including, then, all bounded formulas. See Remark 3.2.10 for proof of these claims. The construction of this universe is a particular example of obtaining universes by solving for fixed points of suitable functors. This, and the properties of the universes obtained, is studied in [10].

Remark 2.2.8 For any topos $\mathcal{E}$, universes U in $\mathrm{Idl}(\mathcal{E})$ which contain $\mathcal{E}$ satisfy a stronger set theory than BIST:

Coll is the axiom scheme of Collection which says that for any total relation R on a set A, there is a set B contained in the "range" of R:

(**Coll**) $\quad S(z) \wedge (\forall x \epsilon z. \exists y. \phi) \to (\exists w. S(w) \wedge (\forall x \epsilon z. \exists y \epsilon w. \phi) \wedge (\forall y \epsilon w. \exists x \epsilon z. \phi))$

It follows from results in [2] that BIST+Coll is sound and complete with respect to class categories of the form $\downarrow U$ in $\mathrm{Idl}(\mathcal{E})$ for toposes $\mathcal{E}$ (where U contains $\mathcal{E}$)(see also section A.1).

3 Sheaf models of theories of sets and classes

Section 1 studied the correspondence between class categories and class logic—and between class categories with a universe and the set theory BIST. Section 2 showed that every (small) topos embeds logically and conservatively in a class category with a universe, and, therefore, that BIST is a conservative extension of higher order logic. In this section, we shall use the fact that every (small) class category embeds logically and conservatively in a topos. Hence higher order logic is a conservative extension of BIST (and of set theories that strengthen BIST, such as ZF). The fact (from section 1) that we can study a set theory in terms of its class category is thereby useful in finding conservative models of theories of sets and classes extending BIST.

3.1 Theories of sets and classes

The language of Von Neumann–Bernays–Gödel class theory (NBG) is a two–sorted first–order language, where we usually use upper case letters for the *class* variables, and lower case letters for the *set* variables. It contains two "membership" predicates, $\in$ and ε, which takes sets, respectively classes, on the right and sets on the left. We give the following informally presented axioms for NBG, based on the presentation in [4] but omitting choice axioms:

ZF axioms All axioms of ZF except Separation and Replacement.

Class Extensionality Classes which have the same elements are equal. I.e. $\forall X, Y. (\forall z. z\varepsilon X \leftrightarrow z\varepsilon Y) \rightarrow X = Y$

Class Separation The intersection of a class and a set is a set.

Class Replacement For every functional class of ordered pairs if the domain is a set, then the image is a set.

Class Comprehension If ϕ is a formula where all class variables are free, then there is a class $\{x|\phi\}$ (where x is a set variable).

Morse–Kelley (MK) class theory strengthens NBG by replacing the axiom scheme **Class Comprehension** with a full, unrestricted comprehension scheme, that is, the same axiom scheme without the restriction on the formulas ϕ. While NBG is conservative over ZF, MK proves the consistency of ZF. (Another difference is that NBG is finitely axiomatizable, while MK is not, but we shall not be concerned with that issue. More on theories of sets and classes can be found in [5].)

We saw in section 1.5.1 that any class category with a universe models the set theory BIST. In section 1.4.4 it was stated that the (small) syntactic category of any set theory including BIST is a class category. Any small class category can be embedded into its category of sheaves by the Yoneda embedding, which is Heyting (into sheaves). Similarly, the class category $\mathrm{Idl}(\mathcal{E})$, for a small topos $\mathcal{E}$, can be embedded into sheaves on $\mathcal{E}$ by the inclusion functor $\mathrm{Idl}(\mathcal{E}) \rightarrow \mathrm{Sh}(\mathcal{E})$ (which sends objects and morphisms to themselves) which is Heyting. We wish to use the higher order structure present in those categories of sheaves to model NBG– or MK–style class theories extending set theories such as BIST or ZF. These class theories will be conservative extensions of the set theory we start out with, which will reflect itself in that we have to choose between full separation or full comprehension. In the case where we choose full separation, and hence have to restrict comprehension, we cannot make direct use of the power objects present in these topoi of sheaves, but need to define suitable subobjects containing, so to speak, the classes which can be used for separation. We shall call them *simple classes*, in analogy with the simple formulas of section 1.5.1. In this section, 3.2 reviews separation and the notion of simplicity in a class category, as it can also be found in [10]; while the new sheaf models of theories of sets and classes extending BIST and ZF are developed in 3.3.

3.2 Simplicity in a class category

Let $\mathcal{C}$ be a class category. If a monomorphism $m : A \rightarrowtail B$ is small, then so is every other monomorphism to which it is isomorphic in $\mathcal{C}/B$.

Definition 3.2.1 A subobject $A \triangleright\!\!\longrightarrow B$ in a class category $\mathcal{C}$ is *simple* if it is (represented by) a small monomorphism.

Proposition 3.2.2 *There is a morphism $\top : 1 \longrightarrow P1$ such that for any monomorphism $f : A \rightarrowtail B$, f is small iff there exists a (necessarily unique) morphism $\rho : B \longrightarrow P1$ such that the following is a pullback:*

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\rho]{} & P1 \end{array}$$

In other words, $\top : 1 \longrightarrow P1$ is a simple subobject classifier.

Proof The diagonal $1 \longrightarrow 1 \times 1$ is small, and so we may define $\top$ as:

$$\begin{array}{ccc} 1 & \xrightarrow{o} & \epsilon_1 \\ \downarrow & \lrcorner & \downarrow \\ 1 \times 1 & \xrightarrow[Id \times \top]{} & 1 \times P1 \end{array}$$

I.e. $\top$ is the singleton map $\{-\} : 1 \longrightarrow P1$. It follows that

$$\begin{array}{ccc} \epsilon_1 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \langle Id, \top \rangle} \\ 1 \times P1 & \xrightarrow[=]{} & 1 \times P1 \end{array}$$

is a pullback square, whence the subobject $\epsilon_1 \rightarrowtriangle 1 \times P1$ is instantiated by $1 \xrightarrow{\langle Id,\top\rangle} 1 \times P1$ (in the sense that the monomorphism is an element of the subobject, considered as an equivalence class). Let $f : A \rightarrowtail B$ be given. Note that any arrow with source 1 is small by Proposition 1.1.1. If f is a pullback of $\top$ along some arrow, then f is small by S2. Conversely, suppose f is small. Then $\langle !, f\rangle : A \rightarrowtail 1 \times B$ is a small relation, and we obtain the two pullback diagram:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle\langle 1,f\rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle\langle Id,\top\rangle} \\ 1\times B & \xrightarrow[\langle Id,\rho\rangle]{} & 1\times P1 \\ {\scriptstyle\pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle\pi_2} \\ B & \xrightarrow[\rho]{} & P1 \end{array}$$

. ⊣

Definition 3.2.3 For any class category $\mathcal{C}$:

- For any object A in $\mathcal{C}$ and any formula ϕ in the language of $\mathcal{C}$, we write $\text{Ƨ}x{:}A.\,\phi$ (read "there are set many $x{:}A$ such that ϕ') in the internal language of $\mathcal{C}$ as a shorthand for

  $$\exists u{:}PA.\,\forall x{:}A.\,x\epsilon_A u \leftrightarrow \phi$$

  where $u{:}PA$ is not free in ϕ.

- For any ϕ in the language of $\mathcal{C}$, we write $!\phi$ (read 'simply phi') as a shorthand for

  $$\text{Ƨ}z{:}1.\,\phi$$

  where $z{:}1$ is not free in ϕ.

Remark 3.2.4 We have now used the same symbols for very similar definitions in the language of a class category and in BIST. However, apart from it being clear from context which definition is intended, we also have that in a class category $\mathcal{C}$ with a universe U where $\iota : PU \rightarrowtail U$ interprets the predicate S and $\epsilon_U \rightarrowtail U \times PU \rightarrowtail U \times U$ interprets the predicate $\in$,
$\mathcal{C} \models (\text{Ƨ}x{:}U.\,\phi) \leftrightarrow (\exists y{:}U.\,S(y) \wedge \forall x{:}U.\,x \in y \leftrightarrow \phi)$ and
$\mathcal{C} \models (!\phi) \leftrightarrow (\exists y{:}U.\,S(y) \wedge \forall z{:}U.\,z \in y \leftrightarrow z = \emptyset \wedge \phi)$
so (since the right hand side is the corresponding definition in BIST) in the context of an interpretation of the language of BIST in a class category, the definitions match.

Proposition 3.2.5 *For any subobject $R \rightarrowtail A$ in $\mathcal{C}$, R is simple iff $\mathcal{C} \models \forall x{:}A.\, !R(x)$.*

Proof If R is simple we have a morphism $r : A \longrightarrow P1$ such that

$$\begin{array}{ccc} R & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \{-\} \\ A & \xrightarrow[r]{} & P1 \end{array}$$

But then $\mathcal{C} \models \forall x{:}A.\, \forall y{:}1.\, y \epsilon_1 r(x) \leftrightarrow R(x)$.

If $\mathcal{C} \models \forall x{:}A.\, \text{Ƨ} y{:}1.\, R(x)$, then, by extensionality, that defines the required morphism $r : A \longrightarrow P1$. ⊣

We say that a formula ϕ (of $\mathcal{C}$) is simple (in $\mathcal{C}$) if its canonical interpretation is a simple subobject, which is to say, then, that the formula $!\phi$ is true (in $\mathcal{C}$).

Proposition 3.2.6 *The simple formulas in $\mathcal{C}$ are closed under conjunction, disjunction, implication, and bounded quantification. Morover, if the disjunction of two disjoint formulas is simple, then both formulas are simple.*

Proof Let ϕ and ψ in the language of $\mathcal{C}$ be given, and assume that their (canonical) interpretations $[\![\phi]\!] \rightarrowtail X$ and $[\![\psi]\!] \rightarrowtail X$ are simple subobjects. $[\![\phi]\!] \cap [\![\psi]\!] \rightarrowtail X$ is simple by S2 and S1, and $[\![\phi]\!] \cup [\![\psi]\!] \rightarrowtail X$ is simple by S5 and S4. By a straightforward diagram chase, $[\![x, u \mid x \epsilon u \wedge \phi]\!] \rightarrowtail X \times PX$ is a small relation, so $[\![u \mid \exists x \epsilon u.\, \phi]\!] \rightarrowtail PX$ is a simple subobject by S4.

Note that the ϵ_A and $=_A$ and $\subseteq_A$ subobjects are simple in $\mathcal{C}$ for all objects A, and that substituting a term for a variable preserves simplicity (by S2). Now, as observed in [10], if we choose monomorphisms f and g from the subobjects interpreting ϕ and ψ respectively, $[\![x{:}X \mid \phi \to \psi]\!] = [\![x{:}X \mid Pf(f^{-1}(x)) \subseteq_X Pg(g^{-1}(x))]\!]$ as subobjects of X, and the latter, then, is simple. Similarly $[\![u{:}PX \mid \forall x \epsilon_X u.\, \phi]\!] = [\![u{:}PX \mid u = Pf(f^*(u))]\!]$ as subobjects of PX, where we by f^* mean the internal inverse image map.

Finally, let two disjoint subobjects P and Q be given, and assume that their union is simple. Since the subobjects are disjoint, the union is the coproduct $P + Q$, and P and Q are simple by composing with the small inclusion maps.

Corollary 3.2.7 *Let $[\![\psi]\!] \rightarrowtail X \times Y$ be a small relation and ϕ a simple formula (all in $\mathcal{C}$). Then $\forall x \in \psi.\, \phi$ (i.e. $\forall x{:}X.\, \psi \to \phi$) is a simple formula,* *as is $\exists x \in \psi.\, \phi$ (so simple formulas are closed under this form of restricted quantification).*

PROOF Since ψ is a small relation, is has a classifying map $\rho : Y \longrightarrow PX$ such that $[\![x, y \mid \psi]\!] = [\![x, y \mid x \epsilon_X \rho(y)]\!]$ as subobjects of $X \times Y$. By S2, substitution of terms preserves simplicity, and by Proposition 3.2.6, $\forall x \epsilon_X u.\, \phi$ is simple, so $\forall x \epsilon_X \rho(y).\, \phi$ is simple, as is, then, $\forall x \in \psi.\, \phi$. ⊣

Corollary 3.2.8 *For any object A and formulas ϕ and ψ in a class category $\mathcal{C}$, the following are valid in $\mathcal{C}$:*

1. $!\bot$
2. $!\phi \wedge !\psi \rightarrow !(\phi \wedge \psi)$
3. $!\phi \wedge !\psi \rightarrow !(\phi \vee \psi)$
4. $\forall u{:}PA.\, (\forall y \epsilon_A u.\, !\phi) \rightarrow !(\exists y \epsilon_A u.\, \phi)$
5. $\forall u{:}PA.\, (\forall y \epsilon_A u.\, !\phi) \rightarrow !(\forall y \epsilon_A u.\, \phi)$
6. $!\phi \wedge !\psi \rightarrow !(\phi \rightarrow \psi)$
7. $\neg(\phi \wedge \psi) \wedge !(\phi \vee \psi) \rightarrow !\phi \wedge !\psi$
8. $Ƨx{:}A.\, \psi \wedge (\forall x{:}A.\, \psi \rightarrow !\phi) \rightarrow !(\forall x{:}A.\, \psi \rightarrow \phi)$
9. $Ƨx{:}A.\, \psi \wedge (\forall x{:}A.\, \psi \rightarrow !\phi) \rightarrow !(\exists x{:}A.\, \psi \wedge \phi)$

PROOF For (1), all morphisms with domain 0 are small. All remaining proofs are a matter of applying Theorem 1.3.8 enough times to allow Proposition 3.2.6 to be applied directly. The proof of (3) is Example A.2.2. ⊣

Finally, the reason why we care about simplicity is this:

Proposition 3.2.9 (Simple Separation) *For all objects A in $\mathcal{C}$, and all formulas ϕ in $\Sigma_{\mathcal{C}}$:*
$\mathcal{C} \models \forall u{:}PA.\, (\forall x \epsilon u.\, !\phi \rightarrow Ƨx{:}A.\, x \epsilon u \wedge \phi)$

PROOF Thanks to Theorem 1.3.8, we may assume without loss that there are no additional parameters, and we need to do only the case of a global point:

Suppose $\alpha : 1 \longrightarrow PA$ factors through $[\![u : PA \mid \forall x \epsilon u.\ !\phi(x)]\!]$. Then

$$[\![x : A \mid x \epsilon \alpha \wedge \phi(x)]\!] \rightarrowtail [\![x : A \mid x \epsilon \alpha]\!]$$

is small, and so $[\![x : A \mid x \epsilon \alpha \wedge \phi]\!] \rightarrowtriangle A$ is a small subobject. ⊣

Remark 3.2.10 The promised proofs of Proposition 1.5.4 and Lemma 1.5.5 are now immediate consequences of completeness for BIST with respect to class categories with universes, Corollary 3.2.8, and Proposition 3.2.9. For Proposition 1.5.8, in the syntactic category of BIST plus the axiom

$$\forall x. !S(x)$$

the subobject $[\![x{:}U \mid S(x)]\!]$ is simple (Remark 3.2.4, Proposition 3.2.5), so the inclusion $PU \rightarrowtail U$ is small, whence the subobject $\in \rightarrowtail U \times U$ is a small relation as well as a simple subobject, and so any Δ_0-formula is simple (by Corollary 3.2.8) since it consists of simple predicates and restricted quantification.

As for the claims in Remark 2.2.7: First, the subobject $[\![x{:}U \mid S(x)]\!]$ is simple since it is a coproduct inclusion; and second, if $\mathcal{E}$ is boolean then $P1 \cong 1 + 1$, and this object is then both a simple subobject classifier and a complemented subobject classifier.

3.3 'Class' power objects and theories of sets and classes

3.3.1 'Small', 'simple', and 'full' power objects

Let $\mathcal{C}$ be a class category. Let $\mathcal{G}$ be a topos, and let $z : \mathcal{C} \hookrightarrow \mathcal{G}$ be an embedding such that:

- z is full and faithful and Heyting.
- Every object in $\mathcal{G}$ is a colimit of a diagram in $z\mathcal{C}$.

The two main examples we have in mind are:

Example 3.3.2 $\mathcal{C}$ is any small class category, $\mathcal{G}$ is $\mathrm{Sh}(\mathcal{C})$, and z is the Yoneda embedding.

Example 3.3.3 $\mathcal{C}$ is $\mathrm{Idl}(\mathcal{E})$, for some topos $\mathcal{E}$, or $\downarrow\mathcal{U}$ for some universe $\mathcal{U}$ in $\mathrm{Idl}(\mathcal{E})$, and z is the inclusion $\mathrm{Idl}(\mathcal{E}) \to \mathrm{Sh}(\mathcal{E})$.

Lemma 3.3.4 *If $C \in \mathcal{C}$, then $\mathcal{E}/zC$ and $\mathcal{C}/C$ and $z/C : \mathcal{C}/C \longrightarrow \mathcal{E}/zC$ inherit the listed properties.*

Proof The class structure is preserved by slicing by [2] ⊣

In what follows, we shall denote the power objects in $\mathcal{C}$ by $P_s A$ for an object A in $\mathcal{C}$, reserving the notation PB, for an object B in $\mathcal{G}$, for the topos power objects in $\mathcal{G}$. We shall talk of *small* and of *full* power objects, respectively. We continue to denote the 'membership' relations in $\mathcal{C}$ by ϵ, while the topos 'membership' relations are denoted ε. This allows us to abuse notation a bit by dropping the z in most cases, so that we can just think of $\mathcal{C}$ as a full subcategory of $\mathcal{G}$ which is positive Heyting with the structure it inherits from $\mathcal{G}$.

For any object A in $\mathcal{C}$, since PA is the full power object, there exists a unique $\kappa_A : P_sA \longrightarrow PA$ such that the following is a pullback in $\mathcal{G}$:

$$\begin{array}{ccc} \epsilon_A & \longrightarrow & \varepsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times P_sA & \xrightarrow[Id\times\kappa_A]{} & A \times PA \end{array}$$

Lemma 3.3.5 *For any A in $\mathcal{C}$, κ_A is monic.*

PROOF We reason in $\mathcal{G}$. Let $u, v{:}P_sA$ be given, and assume $\kappa u = \kappa v$. Let $x{:}A$ be given, and assume $x\epsilon u$. Then $x\varepsilon\kappa u$, so $x\varepsilon\kappa v$. But then $x\epsilon v$. By symmetry, $\forall x{:}U.\, x\epsilon u \leftrightarrow x\epsilon v$. But this implies $u = v$ by extensionality in $\mathcal{C}$ and z being Heyting. ⊣

Definition 3.3.6 Let $\mathcal{G}$, $\mathcal{C}$, and z be as above.

- The *simple class power object*, or just *simple power object*, SA of an object A in $\mathcal{G}$ is defined to be the exponential $(P_s1)^A$:

$$SA := (P_s1)^A$$

- The 'membership' relation $\eta_A \rightarrowtail A \times SA$ is defined by the pullback square:

$$\begin{array}{ccc} \eta_A & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \top \\ A \times SA & \xrightarrow[eval]{} & P_s1 \end{array}$$

.

For any object G in $\mathcal{G}$, we then have a unique $\nu_G : SG \longrightarrow PG$ such that the following is a pullback square:

$$\begin{array}{ccc} \eta_G & \longrightarrow & \varepsilon_G \\ \downarrow & \lrcorner & \downarrow \\ G \times SG & \xrightarrow[Id\times\nu_G]{} & G \times PG \end{array}$$

Lemma 3.3.7 *For all G in $\mathcal{G}$, ν_G is monic.*

Proof The following is a pullback:

$$\begin{array}{ccc} 1 & \xrightarrow{=} & 1 \\ {\scriptstyle \{-\}}\downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ P_s 1 & \underset{\kappa_1}{\rightarrowtail} & P1 \end{array}$$

So we have

$$\begin{array}{ccccccccc} 1 & \xleftarrow{=} & 1 & \longleftarrow & \eta_G & \longrightarrow & \varepsilon_G & \longrightarrow & 1 \\ {\scriptstyle \top}\downarrow & \llcorner & \downarrow{\scriptstyle \{-\}} & \llcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ P1 & \underset{\kappa_1}{\longleftarrow} & P_s 1 & \underset{eval}{\longleftarrow} & G \times P_s 1^G & \underset{Id \times \nu_G}{\longrightarrow} & G \times P1^G & \underset{eval}{\longrightarrow} & P1 \end{array}$$

We see that ν_G is the transpose of the composite $\kappa_1 \circ eval : G \times P_s 1^G \longrightarrow P_s 1$, therefore monic. ⊣

Note that for any object G in $\mathcal{G}$, we then have the following commuting square:

$$\begin{array}{ccc} G \times SG & \overset{Id \times \nu_G}{\rightarrowtail} & G \times PG \\ {\scriptstyle eval}\downarrow & & \downarrow{\scriptstyle eval} \\ P_s 1 & \underset{\kappa_1}{\rightarrowtail} & P1 \end{array}$$

Definition 3.3.8 For any object G in $\mathcal{G}$, let $!_G \rightarrowtriangle PG$ denote the subobject determined by $\nu_G : SG \rightarrowtail PG$. This subobject is then definable as

$$[\![u{:}PG \mid\, !_G(u)]\!] := [\![u{:}PG \mid \forall x{:}G.\, \exists y{:}P_s 1.\, eval(x,u) = \kappa_1(y)]\!]$$

(for intuition, think of $P1$ as the "object of truth values" and think of $P_s 1$ as those truth values that are sets. Then this states that SG are those classes u such that for all $x : G$, the truth value of $x \varepsilon u$ is a set).

We shall mostly be concerned with simple power objects for objects in $\mathcal{C}$, that is, for sheaves that are in the image of z. By Proposition 3.2.2, $R \rightarrowtriangle A \times X$ is a simple subobject in $\mathcal{C}$ iff there exists a (necessarily unique) morphism $\rho : X \longrightarrow SA$ in $\mathcal{G}$ such that the following is a pullback (in $\mathcal{G}$):

$$\begin{array}{ccc} R & \longrightarrow & \eta_A \\ \downarrow & \lrcorner & \downarrow \\ A \times X & \underset{Id \times \rho}{\longrightarrow} & A \times SA \end{array}$$

In particular, since the epsilon subobjects in $\mathcal{C}$ are simple, we have the following pullback in $\mathcal{G}$, for any object A in $\mathcal{C}$:

$$\begin{array}{ccc} \epsilon_A & \longrightarrow & \eta_A \\ \downarrow & \lrcorner & \downarrow \\ A \times P_s A & \xrightarrow[Id\times\xi_A]{} & A \times SA \end{array}$$

For any object A in $\mathcal{C}$, we therefore have three different power objects in $\mathcal{G}$, with canonical inclusion morphisms:

$$\begin{array}{ccccc} P_s A & & \xrightarrow{\kappa_A} & & PA \\ & {\scriptstyle \xi_A}\searrow & = & \nearrow{\scriptstyle \nu_A} & \\ & & SA & & \end{array}$$

Lemma 3.3.9 (Simple class extensionality) *For any object G in $\mathcal{G}$,*

$$\mathcal{G} \models \forall u, v{:}SG.\, (\forall x{:}G.\, x\eta u \leftrightarrow x\eta v) \to u = v$$

Proof By Lemma 3.3.7. ⊣

Lemma 3.3.10 (Class based replacement) *For any objects A and B in $\mathcal{C}$, for any class u of ordered pairs in $A \times B$ and any set v of objects of type A, if u is functional restricted to v, then the image of v under u is a set. Specifically:*

$$\begin{aligned} \mathcal{G} \models \forall u{:}P(A{\times}B).\, \forall v{:}P_s A.\, (\forall x \epsilon_A v.\, \exists^{=1} y{:}B.\, \langle x, y\rangle \varepsilon_{A\times B} u) \\ \to \exists t{:}P_s B.\, \forall y{:}B.\, y\epsilon_B t \leftrightarrow \exists x\epsilon_A v.\, \langle x, y\rangle \varepsilon_{A\times B} u \end{aligned}$$

Proof We basically repeat the proof of Lemma 1.3.9:

It is sufficient to check that any generalized element of $P_s A \times P(A{\times}B)$ with source in $\mathcal{C}$ that factors through $[\![v{:}P_s A, u{:}P(A \times B) | \forall x\epsilon v.\, \exists^{=1} y{:}B.\, \langle x, y\rangle \varepsilon u]\!]$ factors through $[\![v{:}P_s A, u{:}P(A \times B) | \exists t{:}P_s B.\, \forall y{:}B.\, y\epsilon t \leftrightarrow \exists x\epsilon v.\, \langle x, y\rangle \varepsilon u]\!]$. By Lemma 3.3.4, we may assume without loss that the source of our generalized element is 1, i.e. we have a global point $\langle \alpha, \beta\rangle : 1 \longrightarrow P_s A \times P(A \times B)$. We obtain thereby a small subobject $A' := [\![x\epsilon\alpha]\!]$ and a functional relation $[\![x\epsilon\alpha \wedge \langle x, y\rangle \varepsilon \beta]\!]$ on $A' \times B$ which corresponds to a morphism $A' \longrightarrow B$. Since z is full and faithful, this morphism is in $\mathcal{C}$, so the image factorization B' is the witness global point we need. ⊣

Lemma 3.3.11 (Simple classes separate sets) *For any object A in the image of z,*

$$\mathcal{G} \models \forall u{:}PA.\, !_A(u) \leftrightarrow \forall v{:}P_sA.\, \exists t{:}P_sA.\, \forall z{:}A.\, z\epsilon_A t \leftrightarrow z\epsilon_A v \wedge z\varepsilon_A u$$

Proof Let a generalized element $\alpha : X \longrightarrow PA$ be given, and suppose it factors through $!_A$. We may assume without loss that X is in $\mathcal{C}$. Then α classifies a simple subobject of $A \times X$ in $\mathcal{C}$. Now use the fact that the intersection of a small relation and a simple relation in $\mathcal{C}$ is a small relation.

In the other direction, suppose that the given generalized element $\alpha : X \longrightarrow PA$ factors through $[\![u : PA | \forall v{:}P_sA.\, \exists t{:}P_sA.\, \forall z{:}A.\, z\epsilon_A t \leftrightarrow z\epsilon_A v \wedge z\varepsilon_A u]\!]$. This amounts to saying that the transpose $\hat{\alpha} : A \times X \longrightarrow P1$ factors through $\kappa_1 : P_s1 \rightarrowtail P1$ (by the map $A \times X \longrightarrow P_s1$ that sends $\langle a, x\rangle{:}A \times X$ to the unique set $\{a\} \cap \alpha(x)$ in P_sA, and further along the direct image map $P_sA \longrightarrow P_s1$), and so α factors through $\nu_A : SA \rightarrowtail PA$.

It remains to establish a comprehension principle for simple classes. Since z is Heyting, we regard the language of $\mathcal{C}$ as simply a proper part of the language of $\mathcal{G}$. As we wish to model an NBG–type extension of the language of some object of $\mathcal{C}$, we cannot restrict our attention completely to the language of $\mathcal{C}$, but must allow for the occurence of η–predicates.

Lemma 3.3.12 (Simple class comprehension) *For any object A in the image of z, suppose we are given a formula $\phi(x, \vec{u})$ in the language of $\mathcal{G}$ such that*

1. *All free variables of ϕ are among $x, \vec{u}$, and $x : A$ and the variables in $\vec{u}$ are either typed over objects in $\mathcal{C}$ or over simple class power objects for objects in $\mathcal{C}$.*
2. *All predicate symbols and function symbols in ϕ are in the language of $\mathcal{C}$ (considered as a proper part of the language of $\mathcal{G}$) with the possible exception of η_B–predicates for objects B in $\mathcal{C}$. All bound variables are in the language of $\mathcal{C}$.*
3. *All predicate symbols in ϕ from the language of $\mathcal{C}$ denote (under the canonical interpretation) simple subobjects in $\mathcal{C}$. All quantification is bounded.*

Then

$$\mathcal{G} \models \forall \vec{u}{:}\vec{U}.\, \exists v{:}SA.\, \forall x{:}A.\, x\eta_A v \leftrightarrow \phi(x, \vec{u})$$

PROOF We consider the case with only one parameter $b : SB$, as the argument readily generalizes to a longer list of parameters. We need to show that there is a classifying map $\varrho : SB \longrightarrow SA$, or equivalently, that there is a morphism $\hat{\varrho} : A \times SB \longrightarrow P_s 1$ such that the following is a pullback:

$$\begin{array}{ccc} [\![\phi]\!] & \longrightarrow & 1 \\ \downarrow & & \downarrow \top \\ A \times SB & \xrightarrow[\hat{\varrho}]{} & P_s 1 \end{array}$$

Let a morphism $\langle f_1, f_2 \rangle : C \longrightarrow A \times SB$ be given, and assume C is in $\mathcal{C}$. Then (by 3.3.6) there is a simple subobject, suppose it has the name R in the language of $\mathcal{C}$, such that:

$$\begin{array}{ccc} [\![R]\!] & \longrightarrow & \eta_B \\ \downarrow & & \downarrow \\ A \times C & \xrightarrow[Id \times f_2]{} & A \times SB \end{array}$$

Now, the subobject obtained by the pullback:

$$\begin{array}{ccc} [\![y{:}C \mid \phi(f_1(y), f_2(y))]\!] & \longrightarrow & [\![x{:}A, u{:}SB \mid \phi(x,u)]\!] \\ \downarrow & & \downarrow \\ C & \xrightarrow[\langle f_1, f_2 \rangle]{} & A \times SB \end{array}$$

is the interpretation of $[\![y{:}C \mid \phi(f_1(y), f_2(y))]\!]$, as the diagram indicates. But it is also, then, the interpretation of the formula $[\![y{:}C \mid \phi'(f_1(y), y)]\!]$, obtained by replacing every subformula of ϕ of the form $z \eta_B f_2(y)$ by $R(z, y)$. Since R is a simple predicate, it follows from Proposition 3.2.6 that the subobject $[\![\phi']\!]$ is simple. There is, therefore, a classifying map $c : C \longrightarrow P_s 1$, such that the following is a pullback:

$$\begin{array}{ccc} [\![\phi']\!] & \longrightarrow & 1 \\ \downarrow & & \downarrow \top \\ C & \xrightarrow[c]{} & P_s 1 \end{array}$$

We may write $A \times SB$ as a colimit of objects in $\mathcal{C}$: $A \times SB = \varinjlim_I C_i$. Denote the colimiting cocone arrows by $\langle f_{i1}, f_{i2} \rangle : C_i \longrightarrow A \times SB$. For each $i \in I$, there is, by the construction above, a morphism $c_i : C_i \longrightarrow P_s1$ such that the following is a pullback:

$$\begin{array}{ccc} [\![\phi'_i]\!] & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ C_i & \xrightarrow[c_i]{} & P_s1 \end{array}$$

Where $[\![\phi'_i]\!]$ is obtained by pulling $[\![\phi]\!]$ back along $\langle f_{i1}, f_{i2}\rangle$. The c_i's form a cocone with vertex P_s1, and we denote the corresponding morphism $A \times SB \longrightarrow P_s1$ by $\hat{\varrho}$. Denote the classifying map for $[\![\phi]\!]$ by $d : A \times SB \longrightarrow P1$. Considering the diagram

$$\begin{array}{ccccccc} [\![\phi'_i]\!] & \longrightarrow & [\![\phi]\!] & \longrightarrow & 1 & \xrightarrow{=} & 1 \\ \downarrow & \lrcorner & \downarrow & & {\scriptstyle \top}\downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ C_i & \xrightarrow[\langle f_{i1}, f_{i2}\rangle]{} & A \times SB & \xrightarrow[\hat{\varrho}]{} & P_s1 & \xrightarrow[\kappa_1]{} & P1 \end{array}$$

we see that since, for every $i \in I$, the classifying map of $[\![\phi'_i]\!]$ factors as $\kappa_1 \circ c_i : C_i \longrightarrow Ps1 \rightarrowtail P1$, d must factor as $\kappa_1 \circ \hat{\varrho}$, and so the middle square is a pullback and we are done.

Scholium 3.3.13 *A formula satisfying conditions 1 and 2 in Lemma 3.3.12 has a classifying map $\varrho : \vec{U} \longrightarrow SA$ if and only if for every object C in $\mathcal{C}$ and every morphism $\rho : C \longrightarrow \vec{U}$, the formula ϕ' obtained as in the proof of Lemma 3.3.12 is interpreted as a simple subobject in $\mathcal{C}$ under the canonical interpretation.*

3.3.14 Theories of sets and classes

Any first–order theory can be conservatively extended by full intuitionistic higher–order logic, in the sense that its syntactic category can be embedded into a topos of sheaves by a conservative Heyting functor. So, too, of course, with set theories like BIST or ZF. Only in particular cases will then the new "classes" interact with the "old" sets to yield sets when intersected with sets. We illustrate this situation by giving sheaf–models of two theories of sets and classes extending BIST and ZF respectively:

Definition 3.3.15 (BICT) *Basic Intuitionistic Class Theory* (BICT) is formulated in a two typed, first-order language. We use lower case variables for the "type of elements", and upper case variables for the "type of classes". There is an element–typed "sethood" predicate S, and two binary "membership" relations $\in$ and ε which take elements to the left and elements respectively classes to the right. BICT has the following axioms:

BICT1. (BIST axioms)

All axioms of BIST except replacement, i.e. BIST**1**-BIST**5,** BIST**7**-BIST**9**.

BICT2. (Class extensionality)

$(\forall z.\, z\varepsilon X \leftrightarrow z\varepsilon Y) \rightarrow X = Y$

BICT3. (Replacement) For any formula ϕ:
$S(x) \wedge (\forall y \in x.\, \exists^{=1} z.\, \phi) \rightarrow \text{Ƨ}z.\, \exists y \in x.\, \phi$

(In light of the next axiom, we could also have stated this in terms of classes of ordered pairs.)

BICT4. (Comprehension) For any formula ϕ:

$\exists X.\, \forall z.\, z\varepsilon X \leftrightarrow \phi$

where X is not free in ϕ.

Proposition 3.3.16 *Let $\mathcal{C}$ be a small class category with universe $\mathcal{U}$. Consider the embedding of $\mathcal{C}$ by Yoneda into $\mathrm{Sh}(\mathcal{C})$. Then the structure which interprets S, $\in$, and ε as the following subobjects*

$$yP_s\mathcal{U} \overset{y\iota}{\rightarrowtail} y\mathcal{U} \qquad y\epsilon_{\mathcal{U}} \rightarrowtail y\mathcal{U} \times yP_s\mathcal{U} \overset{id\times y\iota}{\rightarrowtail} y\mathcal{U} \times y\mathcal{U} \qquad \varepsilon_{y\mathcal{U}} \rightarrowtail y\mathcal{U} \times Py\mathcal{U}$$

models BICT*.*

Proof (BICT1) Since Yoneda is Heyting.
(BICT2) By topos extensionality.
(BICT3) By Lemma 3.3.10.
(BICT4) By topos comprehension. ⊣

The result also clearly holds when $\mathcal{C}$ is a class category $\downarrow\mathcal{U}$ for a topos $\mathcal{E}$ and a universe $\mathcal{U}$ in $\mathrm{Idl}(\mathcal{E})$, and Yoneda is replaced by the inclusion into $\mathrm{Sh}(\mathcal{E})$.

Corollary 3.3.17 BICT *is a conservative extension of* BIST.

Proof Let $\mathcal{C}$ be the syntactic category $\mathcal{C}_{\mathrm{BIST}}$ of BIST. $\dashv$

It is now straightforward to show in BICT that the intersection between a class A and a set a is a set just in case $\forall x \in a.\, \text{Ƨ}z.\, z = \emptyset \wedge x \varepsilon A$. We can then define the predicate

$$!(X) \Leftrightarrow \forall x.\, \text{Ƨ}z.\, z = \emptyset \wedge x \varepsilon X$$

or equivalently

$$!(X) \Leftrightarrow \forall x.\, S(x) \rightarrow \text{Ƨ}z.\, z \in x \wedge z \varepsilon X$$

which we recognize as the subobject $Sy\mathcal{U} \rightarrowtail Py\mathcal{U}$ (in the structure of Proposition 3.3.16), of classes that intersect sets in sets. We shall not develop this any further, but instead briefly consider the structure of Proposition 3.3.16 in the case where $\mathcal{U}$ models not only BIST but ZF. There are two points of motivation for considering the case of ZF: First, insofar as one is interested in the conceptual information which can be obtained from conservatively extending set theories with (intuitionistic) higher–order logic, the main example of interest might be the classical and familiar ZF. Second, the so–called simple classes are particularly easy to discern in classical strengthenings of BIST, and so may serve to add some intuition to help with the general case.

Let therefore a boolean class category $\mathcal{B}$ with a universe $\mathcal{U}$ be given, and assume that $\mathcal{U} \models$ ZF. Let $\mathcal{E}$ be a topos and assume that we have a Heyting full and faithful functor $z : \mathcal{B} \rightarrow \mathcal{E}$ such that every object in $\mathcal{E}$ is a colimit of a diagram in the image of z. We obtain a structure for the language of NBG or MK—that is, a two–typed language with membership predicates $\in$ and ε—in $\mathcal{E}$ by interpreting $\in$ and ε as:

$$\begin{array}{ccc} z\epsilon_{\mathcal{U}} & \qquad\qquad & \varepsilon_{z\mathcal{U}} \\ \downarrow & & \Big\downarrow \\ z\mathcal{U} \times zP_s\mathcal{U} & & \\ {\scriptstyle id\times z\iota}\Big\downarrow & & \Big\downarrow \\ z\mathcal{U} \times z\mathcal{U} & & z\mathcal{U} \times Pz\mathcal{U} \end{array}$$

We then have that this structure, which we can call $\mathcal{M}$, satisfies the following sentences, that is to say, the theory $\mathbb{T}_{\mathcal{M}}$ of $\mathcal{M}$ contains:

$\mathbb{T}_{\mathcal{M}}1$ The theory of $\mathcal{U}$, including ZF and the Law of Excluded Middle (LEM) for all formulas in the language of ZF, since z is Heyting.

$\mathbb{T}_{\mathcal{M}}2$ Extensionality for classes, i.e.

$(\forall z.\, z\eta X \leftrightarrow z\varepsilon Y) \rightarrow X = Y$

by topos extensionality.

$\mathbb{T}_{\mathcal{M}}3$ Replacement, i.e. for any formula ϕ:

$(\forall y \in x.\, \exists^{=1} z.\, \phi) \rightarrow \text{Ƨ}z.\, \exists y \in x.\, \phi$

by Lemma 3.3.10

$\mathbb{T}_{\mathcal{M}}4$ Class comprehension, i.e. for any formula ϕ:

$\exists X.\, \forall z.\, z\varepsilon X \leftrightarrow \phi$

where X is not free in ϕ.

Moreover, we can again define the class predicate

$$!(X) \Leftrightarrow \forall x.\, \text{Ƨ}z.\, z = \emptyset \wedge x\varepsilon X$$

or equivalently

$$!(X) \Leftrightarrow \forall x.\, S(x) \rightarrow \text{Ƨ}z.\, z \in x \wedge z\varepsilon X$$

which now holds of exactly the boolean classes, in the sense that:

$$\mathbb{T}_{\mathcal{M}}1 - \mathbb{T}_{\mathcal{M}}4 \vdash !(X) \leftrightarrow \forall x.\, x\varepsilon X \vee \neg(x\varepsilon X)$$

By Scholium 3.3.13, using that in $\mathcal{B}$ every subobject is complemented and therefore simple, we have that the following "comprehension axiom for boolean classes" holds in $\mathcal{M}$, we can call it $\mathbb{T}_{\mathcal{M}}5$:

$$\mathcal{M} \models !(X_1) \wedge \ldots \wedge !(X_n) \rightarrow (\exists X.\, !(X) \wedge \forall x.\, x\varepsilon X \leftrightarrow \phi)$$

where ϕ is a formula with no bound class variables, and all (free) class variables are in the list $X_1, \ldots, X_n$. An immediate consequence of $\mathbb{T}_{\mathcal{M}}1 - \mathbb{T}_{\mathcal{M}}5$ is that given such a formula ϕ:

$$\mathcal{M} \models \forall X_1, \ldots, X_n.\, !(X_1) \wedge \ldots \wedge !(X_n) \rightarrow \phi \vee \neg\phi$$

We shall leave it an open question what else can be said about the theory of $\mathcal{M}$, and end with a brief remark based on the sentences of this theory which have just been highlighted. For since we have assumed that z is a conservative functor, the theory of $\mathcal{M}$ is conservative over the theory of $\mathcal{U}$, which is just ZF in the case where $\mathcal{B}$ is the syntactic category of ZF and z is the Yoneda embedding of $\mathcal{B}$ into $\mathrm{Sh}(\mathcal{B})$. As long as we restrict our attention to boolean classes, we have in this case, therefore, a conservative theory over ZF which in certain respects much resembles NBG. Somewhat informally and, perhaps, conceptually, we can compare the theories $\mathbb{T}_M$ and NBG briefly as follows: First, $\mathbb{T}_M$ and NBG share the axioms of ZF, so the sets are the same, so to speak. Second, NBG allows you to form (boolean) classes that are definable in ZF, perhaps with classes as parameters. $\mathbb{T}_M$ does, too: as long as you use only boolean classes as parameters, the class formed will be boolean. However, $\mathbb{T}_M$ also allows for the formation of classes defined by formulas involving class quantification, but classes so formed will, in general, not be boolean. $\mathbb{T}_M$ does not contain LEM for such formulas, unlike NBG. An example of such a class is the class $\mathcal{N}$ of Von Neuman natural numbers satisfying

$$\emptyset \varepsilon \mathcal{N} \wedge \forall x.\, x \varepsilon \mathcal{N} \rightarrow x \cup \{x\} \varepsilon \mathcal{N}$$

and the full induction scheme

$$\phi(\emptyset) \wedge (\forall x \varepsilon \mathcal{N}.\, \phi(x) \rightarrow \phi(x \cup \{x\})) \rightarrow \forall x \varepsilon \mathcal{N}.\, \phi(x)$$

for all formulas ϕ. This class exists since it is the extension of the formula:

$$\forall X.\, (\emptyset \varepsilon X) \wedge (\forall y.\, y \varepsilon X \rightarrow y \cup \{y\} \varepsilon X) \rightarrow x \varepsilon X$$

$\mathcal{N}$ is then a proper subclass of the set of Von Neuman natural numbers that exists by ZF. The non–boolean, or intuitionistic classes can obviously not in general be used for purposes of separation, since the sets are classical, although they can be used for purposes of replacement (by $\mathbb{T}_M 3$).

A Appendix

A.1 Ideals and inclusions

As pointed out in section 2, the fact that every small topos occurs as the small objects in a class category was already proved in [2]. We take this opportunity to recall some of the central elements of the construction in [2], and to point out some connections with the ideals construction presented in section 2. In particular, we aim to justify Remark 2.2.8 by showing that for every BIST–model of the form $\downarrow\mathcal{U}$ in some $\mathrm{Idl}(\mathcal{E})$ (where $\mathcal{U}$ contains $\mathcal{E}$) there is an elementary equivalent model of the form considered in [2] and vice versa. In the process, we will use the ideals construction (i.e. the construction of $\mathrm{Idl}(\mathcal{E})$ from a small topos $\mathcal{E}$) to give an alternative proof of the fact ([2]) that every small topos is equivalent to a small topos with a system of inclusions on it. This is Scholium A.1.9 below.

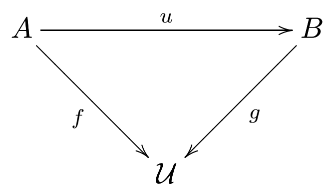
Definition A.1.1 A *system of inclusions* on a class category $\mathcal{C}$ is a subcategory of distinguished monomorphisms of $\mathcal{C}$ written $A \hookrightarrow B$, such that:

- The inclusions partially order the objects of $\mathcal{C}$, and there are binary joins written $A \cup B$.
- Every subobject $R \rightarrowtail B$ is represented by a unique inclusion $A \hookrightarrow B$.
- Inclusions are preserved by a choice of product and power object functors.

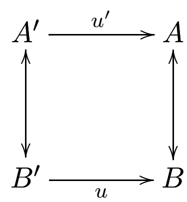
If $\mathcal{C}$ is a small class category with universal object $\mathcal{U}$, then we can consider the *category of subobjects*, $\mathcal{C}_{\mathcal{U}}$, the objects of which are subobjects of $\mathcal{U}$, that is equivalence classes of monomorphisms of $\mathcal{C}$, and the morphisms of which are corresponding equivalence classes of morphisms of $\mathcal{C}$. Specifically, the

- **objects** are equivalence classes of monomorphisms with target $\mathcal{U}$, where $f : A \rightarrowtail \mathcal{U}$ is equivalent to $f' : A' \rightarrowtail \mathcal{U}$ just in case $f \cong f'$ in $\mathcal{C}/\mathcal{U}$; and the
- **morphisms** $u : [f] \longrightarrow [g]$ are equivalence classes of triples $\langle u, f, g\rangle$ of morphisms in $\mathcal{C}$, where the source of u is the source of f, the target of u is the target of g, and the target of both f and g is $\mathcal{U}$, as in the

following (not necessarily commuting!) triangle:



Two triples $\langle u, f, g\rangle$ and $\langle u', f', g'\rangle$ are equivalent just in case $f \cong f'$ and $g \cong g'$ in $\mathcal{C}/\mathcal{U}$ and the isomorphisms yield a commuting square:



The morphisms of $\mathcal{C}_{\mathcal{U}}$ that are also morphisms of $\mathrm{Sub}(\mathcal{U})$ are then a natural candidate for a system of inclusions on $\mathcal{C}_{\mathcal{U}}$. A choice, for each object A of $\mathcal{C}$, of a monomorphism $A \rightarrowtail \mathcal{U}$ allows us to define an equivalence $\Upsilon : \mathcal{C} \to \mathcal{C}_{\mathcal{U}}$, which sends an object to the subobject determined by the chosen morphism for that object, and a morphism, $u : A \longrightarrow B$ say, to the equivalence class $[\langle u, a, b\rangle]$, where a and b are the chosen morphisms of A and B respectively (it is straightforward to check that Υ is full, faithful, and essentially surjective). $\mathcal{C}_{\mathcal{U}}$ is therefore a class category, and a canonical choice of class category structure can be specified as follows to make our candidate system of inclusions into a real one (i.e. to make sure that inclusions are preserved under taking products and power objects): As usual when we talk about a universal object (or universe) $\mathcal{U}$, we are assuming a choice of a monomorphism $\iota : P\mathcal{U} \rightarrowtail \mathcal{U}$, relative to which we can define an ordered pair map $op : \mathcal{U} \times \mathcal{U} \rightarrowtail \mathcal{U}$ (see Proposition 2.2.5) . The universal object of $\mathcal{C}_{\mathcal{U}}$ is the equivalence class of $id : \mathcal{U} \longrightarrow \mathcal{U}$. For $R \rhd\!\!\longrightarrow \mathcal{U}$ and $S \rhd\!\!\longrightarrow \mathcal{U}$, we choose representatives $a : A \rightarrowtail \mathcal{U}$ and $b : B \rightarrowtail \mathcal{U}$ and define the product to be the composite $op \circ a \times b : A \times B \rightarrowtail \mathcal{U} \times \mathcal{U} \rightarrowtail \mathcal{U}$, and the power object (of R) to be the composite $\iota \circ Pa : PA \rightarrowtail P\mathcal{U} \rightarrowtail \mathcal{U}$. It is now straightforward to verify that if two morphisms u and v are inclusions then so are $u \times v$ and Pu. We have, then that any small class category $\mathcal{C}$ with universal object $\mathcal{U}$ is equivalent to a small class category $\mathcal{C}_{\mathcal{U}}$ with universal object and a system of inclusions [2]. We state this for reference:

Proposition A.1.2 *Any small class category $\mathcal{C}$ with universal object $\mathcal{U}$ is equivalent to a small class category $\mathcal{C}_{\mathcal{U}}$ with universal object and a system of inclusions.*

Definition A.1.3 Let $\mathcal{E}$ be a small topos with a system of inclusions on it. The *category of inclusion ideals*, $\mathrm{Incl}(\mathcal{E})$, consists of:

Objects Sets of objects of $\mathcal{E}$ which are downward closed with respect to inclusions, as well as closed under binary unions. I.e. order ideals with respect to the inclusion ordering. We write such an object as $\mathbf{A}$, letting $|\mathbf{A}|$ denote its underlying set of objects.

Morphisms A morphism $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{B}$ consists of an order preserving function $f : |\mathbf{A}| \to |\mathbf{B}|$ and a $|\mathbf{A}|$–indexed family of covers $\{f_A : A \longrightarrow\!\!\!\triangleright\, f(A)\}_{A \in |\mathbf{A}|}$ such that whenever $A' \hookrightarrow A$ in $\mathbf{A}$, the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{f_A} & f(A) \\ \uparrow & & \uparrow \\ A' & \xrightarrow[f_{A'}]{} & f(A') \end{array}$$

Composition and identities are defined in the obvious manner (see [2] for more details and properties).

Proposition A.1.4 *For any small topos $\mathcal{E}$ with a system of inclusions, the category of inclusion ideals* $\mathrm{Incl}(\mathcal{E})$ *is a class category with universal object.*

Proof In [2], we briefly indicate the relevant definitions:

- (Terminal object) $\downarrow 1$, i.e. $\{A \in \mathcal{E} \mid A \hookrightarrow 1\}$.
- (Binary products) $\mathbf{A} \times \mathbf{B} := \{C \hookrightarrow A \times B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$.
- (Equalizers) The equalizer of $\mathbf{f}, \mathbf{g} : \mathbf{A} \rightrightarrows \mathbf{B}$ is the (evident inclusion of the) ideal $\{A \in \mathbf{A} \mid \mathbf{f}_A = \mathbf{g}_A\}$.
- (Binary coproducts) $\mathbf{A} + \mathbf{B} = \{A + B \mid A \in \mathbf{A}, B \in \mathbf{B}\}$.
- (Regular epimorphisms) $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{B}$ is a regular epimorphism if $f : |\mathbf{A}| \longrightarrow |\mathbf{B}|$ is surjective.

- (Dual images) The dual image of a subideal $\mathbf{A} \subseteq \mathbf{C}$ along a morphism $\mathbf{f} : \mathbf{C} \longrightarrow \mathbf{D}$ is defined as

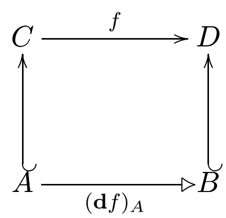
  $\mathbf{f}_*(\mathbf{A}) := \{D \in \mathbf{D} \mid \text{for all } C \in \mathbf{C}, f(C) \hookrightarrow D \Rightarrow C \in \mathbf{A}\}$

- (Power objects) $P\mathbf{A} := \{C \hookrightarrow PA \mid A \in \mathbf{A}\}$

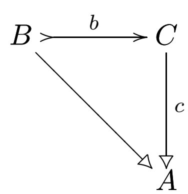
- (Universal object) $\mathbf{U} := \{A \mid A \in \mathcal{E}\}$ ⊣

Recall that if $\mathcal{C}$ is a class category, then the full subcategory of small objects is a topos. If $\mathcal{C}$ has a system of inclusions, then so, of course, does the subcategory of small objects.

Definition A.1.5 Let $\mathcal{C}$ be a small class category with a system of inclusions, and denote by $\mathcal{E}$ the full subcategory of small objects. The *derivative functor* $\mathbf{d} : \mathcal{C} \to \mathrm{Incl}(\mathcal{E})$ is defined on objects $C \in \mathcal{C}$ by $C \mapsto \{A \in \mathcal{E} | A \hookrightarrow C\}$, and on morphisms $f : C \longrightarrow D$ by factoring, as indicated in the following diagram:



We say that a class category $\mathcal{C}$ has *small generators* if the small objects generate $\mathcal{C}$ (see e.g. [7, A1.2.4]). We say that $\mathcal{C}$ has *small covers* if for every small object A in $\mathcal{C}$ and cover $c : C \longrightarrow A$, there exists small object B and monomorphism $b : B \rightarrowtail C$ such that $c \circ b$ is a cover.



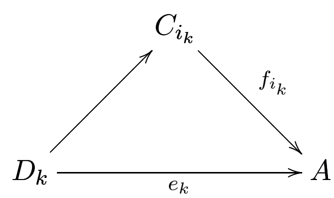
Proposition A.1.6 *Let $\mathcal{C}$ be a small class category with a system of inclusions, and suppose that $\mathcal{C}$ has small generators and small covers. Then the derivative functor is logical and conservative.*

Proof [2] ⊣

Lemma A.1.7 *For any small topos $\mathcal{E}$,* $\mathrm{Idl}(\mathcal{E})$ *has small generators and small covers.*

Proof The small objects in $\mathrm{Idl}(\mathcal{E})$ are the representables, which generate $\mathrm{Sh}(\mathcal{E})$.

Let an ideal C in $\mathrm{Idl}(\mathcal{E})$ be given, and write it as a ideal diagram of colimits, $C = \varinjlim_I(yC_i)$. Let a small object yA and a cover $\phi : \varinjlim_I(yC_i) \longrightarrow\!\!\!\triangleright yA$ be given. For each $i \in I$, denote by c_i the colimit cocone monomorphism $c_i : yC_i \rightarrowtail \varinjlim_I(yC_i)$, and by $yf_i : yC_i \longrightarrow yA$ the composite $\phi \circ c_i$. Since ϕ is an epimorphism in $\mathrm{Sh}(\mathcal{E})$, ϕ is locally surjective (see [9]), so we may chose a finite epimorphic family $(e_k)_{k \leq n}$ with target A in $\mathcal{E}$ such that for all $e_k : D_k \longrightarrow A$, e_k is in the image of $\phi_{D_k} : \varinjlim_I(yC_i(D_k)) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(D_k, A)$. We may choose, therefore, an $i_k \in I$ such that e_k factors through f_{i_k}:



Since I is filtered, we may choose C_i such that for all $k \leq n$, $C_{i_k} \rightarrowtail C_i$. $f_i : C_i \longrightarrow\!\!\!\triangleright A$ is a cover, then, and is preserved as such by Yoneda. ⊣

Proposition A.1.8 *For any small topos $\mathcal{E}$ and any class category $\downarrow\!\mathcal{U}$ in* $\mathrm{Idl}(\mathcal{E})$ *such that $\downarrow\!\mathcal{U}$ contains $\mathcal{E}$, there exists a small topos $\mathcal{E}'$ with a system of inclusions and a logical, conservative functor $\downarrow\!\mathcal{U} \to$* $\mathrm{Incl}(\mathcal{E}')$.

Proof By Proposition A.1.2 $\downarrow\!\mathcal{U}$ is equivalent to a class category $\downarrow\!\mathcal{U}_{\mathcal{U}}$ with a system of inclusions, which in turn embeds logically and conservatively into $\mathrm{Incl}(\mathcal{E}')$ by Proposition A.1.6, where $\mathcal{E}'$ is the category of small objects in $\downarrow\!\mathcal{U}_{\mathcal{U}}$. ⊣

Scholium A.1.9 *Every small topos $\mathcal{E}$ is equivalent to a small topos $\mathcal{E}'$ with a system of inclusions.*

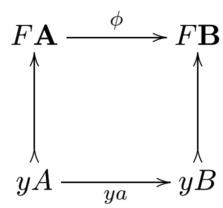
Proposition A.1.10 *Let $\mathcal{E}$ be a small topos with a system of inclusions. Then there exists an object $\mathcal{U}$ in* $\mathrm{Idl}(\mathcal{E})$ *such that $\downarrow\!\mathcal{U}$ is equivalent to* $\mathrm{Incl}(\mathcal{E})$.

Proof An object $\mathbf{A}$ in $\mathrm{Incl}(\mathcal{E})$ can be considered as an ideal diagram in $\mathcal{E}$ (e.g. as the inclusion functor $\mathbf{A} \hookrightarrow \mathcal{E}$). Define a functor $F : \mathrm{Incl}(\mathcal{E}) \to \mathrm{Idl}(\mathcal{E})$ by sending an object $\mathbf{A}$ to $\varinjlim_{A \in \mathbf{A}} yA$, and by sending a morphism $\mathbf{f} : \mathbf{A}$

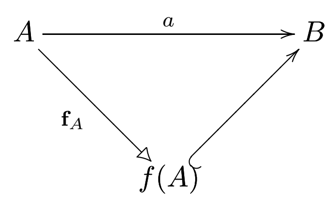
$\longrightarrow \mathbf{B}$ to the unique morphism $\varinjlim_{A\in\mathbf{A}} yA \longrightarrow \varinjlim_{B\in\mathbf{B}} yB$ corresponding to the cocone over the diagram $y\mathbf{A}$ with vertex $\varinjlim_{B\in\mathbf{B}} yB$ obtained from $\mathbf{f}$. Set $\mathcal{U} := F\mathbf{U}$, where $\mathbf{U} = \{A \mid A \in \mathcal{E}\}$.

F is faithful: If $F\mathbf{f} = F\mathbf{g} : \varinjlim_{A\in\mathbf{A}} yA \longrightarrow \varinjlim_{B\in\mathbf{B}} yB$, then for each $A \in \mathbf{A}$, $f(A) = g(A)$.

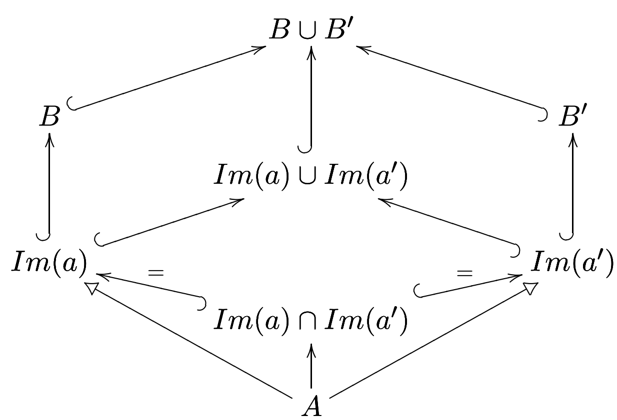
F is full: Let $\phi : \varinjlim_{A\in\mathbf{A}} yA \longrightarrow \varinjlim_{B\in\mathbf{B}} yB$ be given. We obtain a morphism $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{B}$ by factoring as follows: For each $A \in \mathbf{A}$, there exists a $B \in \mathbf{B}$ and a morphism $a : A \longrightarrow B$ such that the following square commutes:



where the monomorphisms are the colimit cocone arrows. There is a unique reg.epi–inclusion factorization of a which gives us $\mathbf{f}_A$:



Note that $\mathbf{f}_A$ is independent of the choice of the morphism a, which can be seen by an easy diagram chase: Let $a : A \longrightarrow B$ and $a' : A \longrightarrow B'$ be two morphisms which factor $yA \rightarrowtail F(\mathbf{A}) \longrightarrow F(\mathbf{B})$ through the base of $F\mathbf{B}$. This implies that the outer diagram in the following commutes (recall from [2] that the central diamond is a pullback):



F is essentially surjective on $\downarrow\mathcal{U}$: Let an ideal with monomorphism $\phi : \varinjlim_I (yA_i) \rightarrowtail \varinjlim_{B \in \mathbf{U}} (yB)$ be given. Denote by f_i the colimit cocone monomorphism $yA_i \rightarrowtail \varinjlim_I (yA_i)$. By the argument for fullness, there exists, for each $i \in I$, a unique "least" object $\delta(A_i)$ in $\mathcal{E}$ such that $\phi \circ f_i$ factors through $y\delta(A_i)$. Note that $A_i \cong \delta(A_i)$. $\{B \hookrightarrow \delta(A_i) | i \in I\}$ is then an ideal, which we may denote $\mathbf{B}$, and the isomorphisms $yA_i \longrightarrow y\delta(A_i)$ gives us an isomorphism $\varinjlim_I (yA_i) \longrightarrow F(\mathbf{B})$. $\dashv$

In sum, then, we have the following:

Corollary A.1.11 *For every small topos $\mathcal{E}$, and every universe $\mathcal{U}$ in* $\mathrm{Idl}(\mathcal{E})$ *such that $\downarrow\mathcal{U}$ contains $\mathcal{E}$, there exists a topos $\mathcal{E}'$ with a system of inclusions such that $\downarrow\mathcal{U}$ is equivalent to* $\mathrm{Incl}(\mathcal{E}')$ *(and so $\mathcal{E} \simeq \mathcal{E}'$). Moreover, for every small topos $\mathcal{E}$with a system of inclusions there exists a object $\mathcal{U}$ in* $\mathrm{Idl}(\mathcal{E})$ *such that $\downarrow\mathcal{U}$ is equivalent to* $\mathrm{Incl}(\mathcal{E})$.

A.2 Slicing and Logic

Several proofs refer to Theorem 1.3.8 in order to justify ignoring cases involving formulas with parameters or in order to simplify proofs of conditionals. We briefly recall the mechanism underlying these proof techniques.

Consider a Heyting category $\mathcal{C}$ and an object Z of $\mathcal{C}$. The pullback functor (as described in e.g. [7]) $Z^* : \mathcal{C} \to \mathcal{C}/Z$ is Heyting. It therefore provides us with a truth–preserving translation from the language of $\mathcal{C}$, $\mathcal{L}_{\mathcal{C}}$, to the language of $\mathcal{C}/Z$, $\mathcal{L}_{\mathcal{C}/Z}$: For any formula–in–context $\vec{x} \mid \phi$ in $\mathcal{L}_{\mathcal{C}}$, the translation $\vec{x} \mid \psi$ in $\mathcal{L}_{\mathcal{C}/Z}$ is obtained by replacing each type (relation/function) symbol A in $\vec{x} \mid \phi$ by the type (relation/function) symbol denoting $Z^*[\![A]\!]_{\mathcal{C}}$ in the language of $\mathcal{C}/Z$. We refer to the translation of ϕ by $Z^*\phi$. Since $Z^* : \mathcal{C} \to \mathcal{C}/Z$ is Heyting, $[\![\vec{x} \mid Z^*\phi]\!]_{\mathcal{C}/Z} = Z^*[\![\vec{x} \mid \phi]\!]$. Now, the following is straightforward to prove:

Proposition A.2.1 *Let $\mathcal{C}$ be a Heyting category.*

1. *Let a subobject $Q \rightarrowtriangle Z$ be given. Then*

   $\mathcal{C} \models \forall x{:}Z.\, Q(x)$ *iff* $\mathcal{C}/Z \models (Z^*Q)(a)$,

   where $a : 1 \longrightarrow Z^*Z$ is the arrow $\Delta : Z \longrightarrow Z \times Z$ in $\mathcal{C}$.

2. *Let P and Q be subobjects of* 1. *Then*

   $\mathcal{C} \models P \to Q$ *iff* $\mathcal{C}/P \models P^*Q$.

Notice that the subobject $(Z^*Q)(a) \rightarrowtriangle 1$ in $\mathcal{C}/Z$ is precisely the subobject $Q \rightarrowtriangle Z$ in $\mathcal{C}$ considered as a subobject of 1 in $\mathcal{C}/Z$.

Together with Theorem 1.3.8, then, Proposition A.2.1 allows us to greatly simplify certain proofs.

Example A.2.2 Corollary 3.2.8(3) states that for any class category $\mathcal{C}$ and any formula ϕ in $\mathcal{C}$,

$$\mathcal{C} \models \forall u{:}PA.\,(\forall y\epsilon_A u.\,!\phi) \rightarrow !(\exists y\epsilon_A u.\,\phi)$$

Assume for the sake of avoiding too much tedious repetition (or, alternatively, product related notation) that there are no unbound variables in ϕ in the statement above. We loose no generality, since we would get rid of additional parameters using the same method by which we get rid of the "parameter" $u{:}PA$. Then the full statement to be proved is:

$$\mathcal{C} \models \forall u{:}PA.\,(\forall y{:}A.\,(y\epsilon_A u) \rightarrow (\exists v{:}P1.\,\forall x{:}1.\,x\epsilon_1 v \leftrightarrow \phi))$$
$$\rightarrow (\exists v{:}P1.\,\forall x{:}1.\,(x\epsilon_1 v) \leftrightarrow (\exists y{:}A.\,y\epsilon_A u \wedge \phi))$$

By Proposition A.2.1 this is equivalent to:

$$\mathcal{C}/PA \models PA^*((\forall y{:}A.\,(y\epsilon_A u) \rightarrow (\exists v{:}P1.\,\forall x{:}1.\,x\epsilon_1 v \leftrightarrow \phi))$$
$$\rightarrow (\exists v{:}P1.\,\forall x{:}1.\,(x\epsilon_1 v) \leftrightarrow (\exists y{:}A.\,y\epsilon_A u \wedge \phi))[\frac{a{:}PA^*PA}{u{:}PA^*PA}])$$

where $a : 1 \longrightarrow PA^*PA$ is the arrow $\Delta : PA \longrightarrow PA \times PA$ in $\mathcal{C}$. Using the fact that the pullback functor is Heyting and the fact that it preserves the class structure, we write the statement above, using the language of $\mathcal{C}/PA$ to the right of the double turnstile, as:

$$\mathcal{C}/PA \models (\forall y{:}PA^*A.\,(y\epsilon_{PA^*A} a) \rightarrow (\exists v{:}P1.\,\forall x{:}1.\,x\epsilon_1 v \leftrightarrow (PA^*(\phi))[\frac{a}{u}]))$$
$$\rightarrow (\exists v{:}P1.\,\forall x{:}1.\,(x\epsilon_1 v) \leftrightarrow (\exists y{:}PA^*A.\,y\epsilon_{PA^*A} a \wedge (PA^*(\phi))[\frac{a}{u}]))$$

which we can write as:

$$\mathcal{C}/PA \models (\forall y{:}A'.\,(y\epsilon_{A'} a) \rightarrow (\exists v{:}P1.\,\forall x{:}1.\,x\epsilon_1 v \leftrightarrow \phi'))$$
$$\rightarrow (\exists v{:}P1.\,\forall x{:}1.\,(x\epsilon_1 v) \leftrightarrow (\exists y{:}A'.\,y\epsilon_{A'} a \wedge \phi'))$$

Choosing a monomorphism $m : S \rightarrowtail 1$ from the subobject

$$[\![\,|\,\forall y{:}A'.\,(y\epsilon_{A'} a) \rightarrow (\exists v{:}P1.\,\forall x{:}1.\,x\epsilon_1 v \leftrightarrow \phi')]\!]$$

we apply Proposition A.2.1 (second part) and see that our original statement holds if and only if

$$(\mathcal{C}/PA)/S \models S^*(\exists v{:}P1.\, \forall x{:}1.\, (x\epsilon_1 v) \leftrightarrow (\exists y{:}A'.\, y\epsilon_{A'} a \wedge \phi'))$$

which we write as (now in the language of $(\mathcal{C}/PA)/S$ on the right hand side of the double turnstile):

$$(\mathcal{C}/PA)/S \models \exists v{:}P1.\, \forall x{:}1.\, (x\epsilon_1 v) \leftrightarrow (\exists y{:}A''.\, y\epsilon_{A''} a' \wedge \phi'')$$

We have that

$$(\mathcal{C}/PA)/S \models \forall y{:}A''.\, (y\epsilon_{A''} a') \rightarrow (\exists v{:}P1.\, \forall x{:}1.\, (x\epsilon_1 v) \leftrightarrow \phi'')$$

for the class category $(\mathcal{C}/PA)/S$, which we write as $\mathcal{C}''$. Now, if B is the small object such that

$$\begin{array}{ccc} B & \longrightarrow & \epsilon_{A''} \\ \downarrow & \lrcorner & \downarrow \\ A'' \times 1 & \xrightarrow[Id \times a']{} & A'' \times P(A'') \end{array}$$

then this tells us that the subobject

$$\begin{array}{c} [\![\phi'']\!] \\ \downarrow \\ B \end{array}$$

is simple. And Proposition 3.2.6 tells us that simple subobjects are closed under bounded quantification, so the subobject

$$\begin{array}{c} [\![\exists y{:}A''.\, y\epsilon_{A''} a' \wedge \phi'']\!] \\ \downarrow \\ 1 \end{array}$$

is simple, whence

$$(\mathcal{C}/PA)/S \models \exists v{:}P1.\, \forall x{:}1.\, x\epsilon_1 v \leftrightarrow (\exists y{:}A''.\, y\epsilon_{A''} a' \wedge \phi'')$$

and we are done.

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