Formal Epistemology: Lecture Notes

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Changing view as a decision problem II

The classical framework of *optimization* used in standard choice theory recommends choosing, among the feasible options, a *best* alternative. So, if S is the feasible set and R is a weak preference relation over S, *optimization* recommends focusing on the following set of maximal or best elements of S:

$$G(S, R) = \{Y \in S\text{: for all } Z \in S,\ ZRY\}$$

Maximizing

The general discipline of maximization differs from the special case of optimization in taking an alternative as choosable when it is not known to be worse that any other. [...] The basic contrast between maximization and optimization arises from the possibility that the preference ranking may be incomplete. Amartya Sen: Rationality and Freedom, p. 767.

To define a maximal set we can use the asymmetric part $P(R)$ of a binary acyclic relation R as follows:

$$M(S, R) = \{Y \in S\text{: for no } Z \in S,\ ZP(R)Y\}$$

In order to study the contraction of a theory K with a sentence A the AGM trio [**?**] has proposed to focus on the set $(K \perp A)$ of maximal subsets of K that fail to entail A as the feasible set from which one make choices. Then the idea is to utilize a *selection function* γ that when applied to $(K \perp A)$ selects a non-empty set of $(K \perp A)$.

EXAMPLES OF REMAINDER SETS

1 If A entails some element of a set of sentences S then $(K \cup \{A\}) \bot S = K \bot B$ and $K \bot (S \cup \{A\}) = K \bot S$.

2 $K \bot (A \wedge B) \subseteq K \bot A \cup K \bot B$

3 K is a theory. If $X \in K \bot A$, then $X \in K \bot B$, for all $B \in K$ minus X.

4 K is a theory and let A and B be elements of K. Then $K \bot (A \wedge B) = K \bot A \cup K \bot B$

In particular *partial meet contraction* focuses on selection functions that are *relational*, i.e. selection functions for which there is a binary relation $\leq$ such that:

$$\gamma(K\perp A, \leq) = \{Y \in K\perp A\text{: for all } Z \in K\perp A, Z \leq Y\}$$

then $K \div A =_{def} \bigcap \gamma(K\perp A, \leq)$.

So, in terms of the notation for optimization used above we have:

$$\text{(AGM) } K \div A = \bigcap G((K \perp A), R)$$

There is nevertheless an obvious option that can be applied as well when the relation R is not complete:

$$\text{(LC) } K \div A = \bigcap M((K \perp A), R)$$

This part of the lecture will focus on studying this second alternative which is rather useful when one is trying to make decisions in the face of unresolved conflict (for example, when there is more than one dimension of epistemic value and the different dimensions conflict).

In order to do this we need to present some background about the discipline of rational choice per se.

Choice functions

Choice can be analyzed in a rather abstract framework by appealing to standard techniques used in the pure theory of consumer choice. We can start with a universal set X which remains fixed throughout the analysis. Let then $\mathcal{S}$ be a distinguished non-empty collection of non-empty subsets of X. The pair $(X, \mathcal{S})$ will be called the *choice space* of the agent.

Now we can introduce a new notion that will be useful below. A *choice function* on a choice space $(X, \mathcal{S})$ is a function C defined on $\mathcal{S}$ that assigns a non-empty subset (*choice set*) $C(S)$ of S to each and every S in $\mathcal{S}$.

A weak preference relation R on X *rationalizes* (or is *a rationalization* of) a choice function C on $(X, \mathcal{S})$ if and only if, for every $S \in \mathcal{S}$, $C(S)$ consists on the greatest points of S:

(R) There is a relation R, such that for all $S \in \mathcal{S}$, $C(S) = G(S, R)$

On the other hand if a choice function C on $(X, \mathcal{S})$ has a preference relation R satisfying (R), it is customarily said that C is a *rational choice function*. Obviously this terminology reflects the dominant view according to which rational choice is choice in accordance with the *optimization* of an underlying weak preference relation.

We will follow here the terminology of Suzumura according to which this notion of rationality is called G-rationality. The obvious alternative in terms of maximization will be called M-rationality. We will focus in the following subsections on characterizations of types of M-rationality.

Definition 1 *A choice function* C *is* quasi-transitive *M-rational if and only if it is M-rational with a quasi-transitive (reflexive and transitive) rationalization.*

The quasi-transitivity of R requires that the relation is both reflexive and that $P(R)$ is transitive.

(Finite domain) $\mathcal{S}$ consists of all finite nonempty subsets of X.

(Finitely additive domain) For any S_1, S_2 in $\mathcal{S}$, $S_1 \cup S_2$ in $\mathcal{S}$.

(General domain) $\mathcal{S}$ is a specified nonempty collection of nonempty subsets of X.

We will assume from now on the Finite Domains constraint. As an important auxiliary step we will define here the *base relation* R^C for C as follows:

Definition 2 $R^C = \{(x, y) \in X \times X\colon\ x \in C(\{x, y\})\ \}$

(Chernoff Property) For all $S, T \in X$, if $S \subseteq T$, then $S \cap C(T) \subseteq C(S)$.

(Superset Axiom) For all S, T in $\mathcal{S}$: If $S \subseteq T$ and $C(T) \subseteq C(S)$, then $C(S) = C(T)$.

(M-Condorcet property) For all $S \in \mathcal{S}$: $M(S, R^C) \subseteq C(S)$

Observation 3 *C is M-rational if and only if R^C is a M-rationalization of C.*

(Path Independence-α) For all $S, T \in \mathcal{S}$,

$$C(S \cup T) = C(C(S) \cup C(T)).$$

(Path Independence-β) For all $S, T \in \mathcal{S}$,

$$C(S \cup T) = C(C(S) \cup T).$$

Theorem 4 *(Suzumura-Th. 2.4) A choice function C on a space $(X, \mathcal{S})$ satisfying the Finite Domain condition satisfies Path Independence-α if and only if it satisfies Path Independence-β.*

Theorem 5 *(Suzumura-Th. 2.5) A choice function C on $(X, \mathcal{S})$ satisfies path independence if and only if it satisfies Chernoff axiom and the Superset axiom.*

Theorem 6 *A choice function C on $(X, \mathcal{S})$ satisfying the Finite Domain condition is quasi-transitive M-rational if and only if satisfies Chernoff's axiom, the Superset axiom and the M-Condorcet property.*

Proof To prove the theorem it is enough to show that path independence (PI) and M-Condorcet are necessary and sufficient for quasi-transitive M-rationality. For sufficiency it is enough to show (in the presence of the previous observation) that R^C is quasi transitive and that for all finite subsets S of $\mathcal{S}$, $C(S) = M(S, R^C)$.

Assume by contradiction that $x \in C(S)$ but $x \notin M(S, R^C)$. If so there is $y \in S$ such that $(y, x) \in P(R^C)$. This means that $x \notin C(\{x, y\})$ and $y \in C(\{x, y\})$. Now we can use the following instance of path independence:

$$C(S) = C(C(\{x, y\}) \cup C(S - \{x, y\})$$

from which we can conclude that $x \notin C(S)$. Contradiction. So, this proof plus the M-Condorcet property yields that $C(S) = M(S, R^C)$

as we wanted. For transitivity of the asymmetric part of R^C assume that $(x, y) \in P(R^C)$ and $(y, z) \in P(R^C)$. This entails that $\{x\} = C(\{x, y\})$ and $\{y\} = C(\{y, z\})$. But then in virtue of Path Independence (β) we have that $C(\{x, z\}) = C(C(\{x, y\}) \cup \{z\}) = C(\{x\} \cup \{y, z\}) = C(\{x\} \cup C(\{y, z\})) = C(\{x, y\}) = \{x\}$.

Now we need to focus on the necessity of Path Independence and M-Condorcet for C to be quasi-transitive M-rational. It is not difficult to see that if C is quasi-transitive M-rational it should satisfy Chernoff, and this, in turn, guarantees that:

For all $S, T \in \mathcal{S}$, $C(S \cup T) \subseteq C(C(S) \cup C(T))$

For the converse assume by contradiction that $x \in C(C(S) \cup C(T))$ but that $x \notin C(S \cup T)$. Without loss of generality we can assume that $x \in C(S)$. We therefore have that:

(I) For all $z \in S$: $(z, x) \notin P(R^C)$.

Moreover we also know that there is no $z \in \{C(S) \cup C(T)\}$ such that $(z, x) \in P(R^C)$. In particular there is no $z \in C(T)$ such that $(z, x) \in P(R^C)$.

Now, since we assumed that $x \notin C(S \cup T)$ we also know that there is $y \in \{S \cup T\}$ such that $(y, x) \in P(R^C)$. Moreover y cannot belong to S in virtue of (I). And since we also established that there is no $z \in C(T)$ such that $(z, x) \in P(R^C)$, we have that y must belong to $T - C(T)$. Therefore there must be $r \in T$ such that $(r, y) \in P(R^C)$. And, again, we know that r must belong to $T - C(T)$. Repeating this procedure we can generate an infinite sequence $y, r, ...$ in a finite set T, which gives us the desired contradiction.

$\square$

(Sen's γ) For all $S, T \in \mathcal{S}$, such that $S \cup T \in \mathcal{S}$, $C(S) \cap C(T) \subseteq C(T \cup S)$.

(Aizerman Property) For all $S, T \in \mathcal{S}$, if $S \subseteq T$, and $C(T) \subseteq S$, then $C(S) \subseteq C(T)$.

(Arrow's Axiom) For all $S, T \in \mathcal{S}$, such that $S \cup T \in \mathcal{S}$, if $C(S \cup T) \cap S \neq \emptyset$, then $C(S) \subseteq C(T \cup S)$.

Theorem 7 *A choice function C on a space $(X, \mathcal{S})$ satisfying the condition of Finite Domain is quasi-transitive M-rational if and only if satisfies Chernoff's axiom, the Superset axiom and γ.*

Proof The proof depends on a classic result first established by Sen. Consider a choice function C on a finite space $(X, \mathcal{S})$, i.e. a space $(X, \mathcal{S})$ where $\mathcal{S}$ consists of and only of all finite nonempty subsets of X.

(Weak Revealed Base Preference) $xR^{+}y$ if and only if $x \in C(\{x, y\})$.

(Binariness of Choice) $C(S) = G(S, R^+)$

The original choice function $C(S)$ is *binary* if and only if the revealed preference relation R^+ generated by the choice function, if used as the basis of choice, will, in turn, regenerate the choice function itself. Amartya Sen proved that a choice function is binary if and only if it satisfies Properties α (Chernoff) and Property γ.

Now, if we have Chernoff, the Superset axiom and the M-Condorcet property for a choice function defined over a space satisfying Finite Domain, we also have that for all finite subset S of $\mathcal{S}$, $C(S) = M(S, R^C)$. This flows from the first part of the proof of the previous theorem. Now, since the Finite Domain condition is satisfied, the revealed preference relation $R^+ = R^C$ should be complete. In this case we have that $M(S, R^C) = G(S, R^+)$. Therefore we have, as desired, that $C(S) = G(S, R^+)$.

So, any choice function satisfying Chernoff, the Superset axiom and the M-Condorcet property, is binary in Sen's sense, and therefore it obeys γ in virtue of his result.

Moreover, if we start with a binary choice function $C(S)$ obeying both Chernoff and γ we also know that $C(S) = G(S, R^+)$. Since R^+ is complete (by the Finite Domain condition) we have that $C(S) = G(S, R^+) = M(S, R^C)$. So, Chernoff and γ entail that a function $C(S)$ can be expressed as $C(S) = M(S, R^C)$ while Chernoff and Superset suffice to establish that R^C should be quasi transitive. These two facts give us M-Condorcet.

So, quasi-transitive M-rationality is characterized by Chernoff, Superset and γ.

$\square$

Now we can go back to our main problem which is the characterization of:

$$\text{(LC) } K \div A = \bigcap M((K \perp A), <_c)$$

where $<_c$ is a notion of categorical preference including all shared pairs across a set of potentially conflicting value dimensions. The relation is therefore quasi-transitive if all the original value dimensions were represented by weak orderings.

(÷ 1) $K \div A = Cn(K \div A)$ [closure]

(÷ 2) $K \div A \subseteq K$ [inclusion]

(÷ 3) If $A \notin K$ or $A \in Cn(\emptyset)$, then $K \subseteq K \div A$ [vacuity]

(÷ 4) If $A \notin Cn(\emptyset)$, then $A \notin K \div A$ [success]

(÷ 5) $K \subseteq Cn((K \div A) \cup \{A\})$ [recovery]

($\dot{-}$ 6) If $Cn(A) = Cn(B)$, then $K \dot{-} A = K \dot{-} B$ [extensionality]

(÷ 7) $K \div A \cap K \div B \subseteq K \div (A \wedge B)$ [conjunctive overlap]

Lemma 8 *If* $\div$ *is defined via (C) for some choice function* $M((K \perp A), <_c)$*, then if* M *satisfies Chernoff (Sen's* α*)* $\div$ *satisfies postulates* (÷1) *to* (÷7)*. Moreover if* $\div$ *satisfies postulates* (÷1) *to* (÷7) *and the choice function that determines* $\div$ *is complete, then* M *satisfies Chernoff.*

(÷ 8c) If $B \in K \div (A \wedge B)$, then $K \div (A \wedge B) \subseteq K \div A$.

Lemma 9 *If ÷ is defined via (C) for some choice function* $M((K \bot A), <_c)$, *then if* M *satisfies Sen's* ϵ *(Superset) ÷ satisfies postulates* (÷1) *to* (÷6) *and* (÷8c). *Moreover if ÷ satisfies postulates* (÷1) *to* (÷6) *and* (÷8c) *and the choice function that determines ÷ is complete, then* M *satisfies Sen's* ϵ.

($\div$ 8) If $A \notin K \div (A \wedge B)$, then $K \div (A \wedge B) \subseteq K \div A$.

This postulate naturally corresponds to the so-called Arrow Axiom (or Sen's property β^{+}). But it is easy to see that Arrow's Axiom does not hold for the maximizing choice functions we are considering here (why?).

The idea that rationality presupposes an optimization process has been attacked by many scholars trying to develop *bounded* accounts of rationality. The terminology was coined by Herb Simon in a series of fascinating writings on this and related topics. It is interesting to notice, nevertheless, that some *bounded* methods of choice, like Simon's *satisficing* can also be seen as forms of maximization.

The businessman who is willing to settle for $x =$ \$ 1.00 million without concerns about raising it to $y =$ \$ 1.01 million, regards both x and y as acceptable, but this does not mean that he sees as 'equally good'. With respect to *his welfare function* the businessman might place y over x.

On the other hand, given the bounded nature of *his choice behavior* he is ready to settle for either x or y. So, given his goals neither x is placed over y nor viceversa. Nor there is a decision to accept the options as equally good *given the agent's goals*. This has lead Sen to conclude that:

So, in terms of *the goal function* (as opposed to his *welfare function*) there is a 'tentative incompleteness' here and both x and y can be seen as 'maximal' in terms of the *operational* goals. Thus interpreted satisficing corresponds entirely to maximizing behavior.

But, as we discussed earlier, the use of this *as if* preference is *interpretatively* quite different. Thus the substantive gap between satisficing and optimizing remains (closable only in a *purely formal* way), whereas the gap between satisficing and maximizing is both formally and substantively absent.

The insistence on transitively relational accounts of optimizing contraction in the AGM tradition can therefore be seen as a tacit dismissal of indeterminacy and therefore also of the use of bounded methods of choice of the sort that Simon has in mind.