Notes for Session 7 – Basic Voting Theory

We follow up the “Impossibility” (Session 6) of pooling expert probabilities, while preserving unanimities in both unconditional and conditional probabilities. Here we explore the power of group voting as a substitute for group pooling.

Assume that the group decision problem involves $m$-many, pairwise exclusive social acts, e.g., the options might be candidates in an election, or (exclusive) bills before a legislature. And there are $n$-many citizens, or voters, or legislators in our group.

Each voter $j$ ($j = 1, \ldots, n$) has an ORDINAL ranking of the $m$-options, as summarized in the table below. The quantities $r_{-j}$ are the ranks assigned by voter $j$ to the $m$-many acts. We’ll let a rank of 1 be best, and a rank of $m$ be worst. Ties are allowed by sharing the average rank for those tied in the ranking. For example, if two acts tie for best position they share rank 1.5 ($= (1+2) / 2$), etc. Thus, for each voter, the sum of the ranks equals $m(m+1)/2$. 
**TABLE of $n$-many Voters’ Rank Order of $m$-many acts**

<table>
<thead>
<tr>
<th></th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_j$</th>
<th>$V_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{ct_1}$</td>
<td>$r_{11}$</td>
<td>$r_{12}$</td>
<td>$r_{1j}$</td>
<td>$r_{1n}$</td>
</tr>
<tr>
<td>$A_{ct_2}$</td>
<td>$r_{21}$</td>
<td>$r_{22}$</td>
<td>$r_{2j}$</td>
<td>$r_{2n}$</td>
</tr>
<tr>
<td>$A_{ct_i}$</td>
<td>$r_{i1}$</td>
<td>$r_{i2}$</td>
<td>$r_{ij}$</td>
<td>$r_{in}$</td>
</tr>
<tr>
<td>$A_{ct_m}$</td>
<td>$r_{m1}$</td>
<td>$r_{m2}$</td>
<td>$r_{mj}$</td>
<td>$r_{mn}$</td>
</tr>
</tbody>
</table>
Iterative use of majority voting in pairwise comparisons without ties.

Let the majority rule in determining pairwise comparisons. Candidates – or social options – are ordered (an “agenda”) with the first two on the list paired for the first round election. The winner in a round faces off against the next candidate on the list. Losers are eliminated. The winner of the composite election is the winner of the final pairwise comparison.

For our majority voting problems we’ll assume there are no ties, which happens if there are an odd number of voters and abstaining is not allowed.

That is, when Act\(_j\) is compared with Act\(_k\), the \(n\)-many voters vote for one or the other, with a majority of determining the outcome of this head-to-head competition.

•
Proposition: If each voter has only strict preferences – no indifferences – and there is an odd number of voters, regardless the agenda, the (Condorcet) winner of such an election is the one and only one social act* (if one exists) such that for each other social act \( i \neq act^* \), a majority of the voters prefers act* over act_i.

Proof: Class exercise! Hint: argue by induction on the location (counting from the back end of the agenda) of the position of the Condorcet winner.

However, an interesting problem arises when there is no such (Condorcet) winner.
Example: Use 3 voters and 3 options. The table of strict preferences is as follows.

<table>
<thead>
<tr>
<th></th>
<th>Voter 1</th>
<th>Voter 2</th>
<th>Voter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act₁</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Act₂</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Act₃</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note:
- a majority of voters prefers Act₁ over Act₂
- a majority prefers Act₂ over Act₃
- a majority prefers Act₃ over Act₁.

Iterative majority voting, here, does not generate a social preference ordering.
Class Exercise: In the Condorcet voting game, with these three voters and their preferences common knowledge, they face a sequence of two, pairwise votes, with the winner from the first round vote then pitted in the second round against the remaining act, which has a first round bye.

- Voting in each round is by simple majority: no abstaining, or tie-votes.

- Suppose that voter 3 is allowed to fix the agenda. Voter 3 gets to determine which two acts are voted on in the first round. Of course, the winner of that first round faces the remaining outcome in the second and final round of voting.

- Argue whether there is a Nash equilibrium strategy for the three players that leads uniquely to player 3’s favorite outcome, Act 2, being voted the winner of the game. You may eliminate weakly dominated strategies.
  - NOTE: A strategy for a voter must specify what to do under all logically possible circumstances, in all rounds of voting.

- Recall that players are free to engage in strategic voting – they may express any vote regardless of their preferences – and player 3, in addition to voting, has control of the agenda.
Single Peaked Preferences and Spatial Voting

An important (1948) result, due to Duncan Black is this:

Let acts be placed in one dimension, on a “line,” so that, with preference depicted by “height” above the line, each voter’s (ordinal) preferences are single-peaked. That is, each voter has a unique favorite position on the line and her/his preferences fall off monotonically from that favored position.

- THEN, majority rule in pairwise comparisons produces an ordering. With an odd number of voters and no indifferences, there is a Condorcet winner!

Class Exercise: Suppose that you are a candidate who may align with some position in the space of social acts. If there is an odd number of voters, voters’ preferences are single-peaked and no individual indifferences, where shall you position yourself? Find the position that is the Condorcet winner!

Hint: Focus on the distribution of favored positions! Start with the game where you are one of 2 rival candidates and there is only 1 voter, then 3 voters, etc.

Class Exercise: Show that there may be no Condorcet winner when there are an odd number of voters, no indifferences, and preferences are single peaked but not necessarily with respect to a (1-dimensional) line!
Borda Count Voting

A different voting scheme, not Majority Rule, is Borda Count voting. Here, each voter gives her/his rank order over the options/acts/candidates. The group’s choice is given by calculating the sum of ranks for each option, with the lowest sum winning.

Class Exercise: Construct a situation where Borda Count favors, say, act₃ in a choice among 7 acts, but when act₇ is deleted, then act₂ wins!

Have we seen this phenomenon before?

- Outline a situation where, as a voter in a Borda count system, you will vote strategically, i.e., your voting behavior depends on how you think others will vote and not merely on your own preferences.

- Is there a voting scheme that is immune to strategic voting?
Can we merge INDIVIDUAL PREFERENCES into a GROUP PREFERENCE and create a “core”?
ARROW’S “Impossibility” Theorem (1950)

USE www.jstor.com to obtain: A Difficulty in the Concept of Social Welfare, Kenneth J. Arrow

Consider a (finite) set of \( m \)-many SOCIAL ACTS \( A = \{ A_1, \ldots, A_m \} \) \( (m \geq 3) \),

and \( n \)-many INDIVIDUAL PREFERENCES over these acts \( \{ \leq_1, \ldots, \leq_n \} \) \( (n \geq 2) \),

A PREFERENCE, \( \leq \), is a weak ordering of the set \( A \).

That is, \( \leq \) is a REFLEXIVE, TRANSITIVE, and COMPLETE binary relation over the set of social acts.

Arrow’s Theorem:
There does not exist a rule for creating a GROUP PREFERENCE, \( \leq_G \) that satisfies the following 4 conditions:
(C-1) The rule applies with ARBITRARY sets of ACTS and PREFERENCES.

(C-2) The rule obeys the (WEAK) PARETO AXIOM:

If $A_1 <_j A_2$ (for each $j$), then $A_1 <_G A_2$

and, if $A_1 <_j A_2$ (for each $j$), then $A_1 <_G A_2$

That is, when each person (strictly) prefers $A_2$ to $A_1$, the group does too.

(C-3) A DICTATOR is not permitted.

(C-4) The GROUP'S preference relation, $<_G$, over a particular subset $A'$ of social acts,

e.g., $<_G$ applied to the odd-numbered social acts,
depends solely on the INDIVIDUALS' PREFERENCES, $<_j$ for the acts in $A'$.

Condition (C-4) is called, INDEPENDENCE OF IRRELEVANT ALTERNATIVES.
Proof sketch – Arrow’s theorem admits rather different proofs. Here, I use ideas that connect with coalition theory, following Arrow’s own reasoning. A concise summary is found in Luce and Raiffa’s (1957) *Games and Decisions*, pp. 339-340.

**Definition:** A coalition \( V \) of individuals is *decisive* for the group’s strict preference of act \( A_2 \) over act \( A_1 \) when, for each set of individual preference orders, if \( A_1 \prec_j A_2 \) for each \( j \) in \( V \), then \( A_1 \prec_G A_2 \).

That is, it is sufficient that each member of the decision coalition \( V \) strictly prefers \( A_2 \) over \( A_1 \) for the group to strictly prefer \( A_2 \) over \( A_1 \).

Since (by Pareto) the *Grand Coalition* is decisive for each pair of acts, there is always a *smallest* non-empty coalition that is decisive for one act over another.

So, let \( V \) be decisive for some pair, e.g., for \( A_2 \) over \( A_1 \), while no proper subset of \( V \) is decisive for any pair of acts – just delete members from the Grand Coalition until the set is not decisive. The following reasoning reduces \( V \) to a single member \( \{j\} \)!
Let $j$ be one of $V$'s members and consider the following strict preference profiles.

<table>
<thead>
<tr>
<th></th>
<th>$V-{j}$</th>
<th>$W$ (= Everybody else not in $V$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$A_3$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$A_1$</td>
<td>$A_2$</td>
</tr>
</tbody>
</table>

Thus $A_1 <_G A_2$, since $V$ is decisive on this issue. But $A_3 \leq_G A_1$ because $V-\{j\}$ is not decisive for $A_3$ over $A_1$. Therefore, by transitivity of preference, $A_3 <_G A_2$.

But since only the individual $j$ strictly prefers $A_2$ over $A_3$, while everyone else strictly prefers $A_3$ over $A_2$, $\{j\}$ is decisive for $A_2$ over $A_3$ – See the Lemma at the end.

Thus $V = \{j\}$!

Likewise, $\{j\}$ is decisive for $A_2$ over any other option different from $A_2$!

What remains in order to establish that $\{j\}$ is the dictator is to show that $\{j\}$ also is decisive for arbitrary $A_4$ over an arbitrary $A_3$. 
• Consider first whether neither of these two acts is $A_2$.

\[
\begin{array}{c|c}
\{j\} & N-\{j\} \ (= \text{Everybody else, not } j) \\
A_4 & A_3 \\
A_2 & A_4 \\
A_3 & A_2
\end{array}
\]

By Pareto, $A_2 <_G A_4$.

And $A_3 <_G A_2$, since \{j\} is decisive for $A_2$ over any other option.

Hence, $A_3 <_G A_4$ by transitivity, which makes \{j\} decisive for $A_4$ over $A_3$!

• For the remaining case, in order to show that \{j\} is decisive for the group’s preference of an arbitrary $A_4$ over $A_2$, consider this profile.

\[
\begin{array}{c|c}
\{j\} & N-\{j\} \ (= \text{Everybody else, not } j) \\
A_4 & A_3 \\
A_3 & A_2 \\
A_2 & A_4
\end{array}
\]

Then $A_3 <_G A_4$, since \{j\} is decisive for such pairs.

And $A_2 <_G A_3$, by Pareto,

Hence, $A_2 <_G A_4$ by transitivity, making \{j\} decisive for $A_4$ over $A_2$ too!
The remaining exercise is to establish the following Lemma, which is used repeatedly in the arguments above.

**Lemma**: Let $V$ be a coalition and let $U = N - V$ be the complementary coalition:

If $A_1 <_V A_2$ and $A_2 <_U A_1$ and $A_1 <_G A_2$,

then the coalition $V$ is decisive for $A_2$ over $A_1$.

**Proof hint**:  
Consider a third act $A_3$ and preferences $<_i$ for the $N$ individuals as follows.  
For each member of the coalition $V$, $A_1 <_V A_2 <_V A_3$.  
For each member of the complementary coalition $U$, $A_2 <_U A_3 <_U A_1$.  
Argue that the group’s preference then must be of the form: $A_1 <_G A_2 <_G A_3$.

Now consider different preferences $<_'$, for the $N$ individuals where:  
the preferences of the individuals in $V$ are unchanged, $A_1 <'_V A_2 <'_V A_3$ for some subset $U_1$ of the coalition $U$ their preferences are $A_3 <'_U A_1 <'_U A_2$,  
whereas the remaining members of $U$ (i.e. for individuals in $U_2 = U - U_1$ they) have the same preferences as members of $U$ above, i.e., $A_2 <'_U A_3 <'_U A_1$.  
Argue that this group’s preferences must be of the form $A_1 <'_G A_3 <'_G A_2$.  
This establishes the Lemma.
Let us press the case for SEU theory a bit harder. Suppose we require that, for each decision maker, there is a single *global Bayes model* that covers all her/his choices. That is, suppose we require that a rational decision maker has all her/his admissible options, over all choice problems, governed by a single Bayes model: \((P,U)\), where \(P\) is a personal probability over states of uncertainty, and \(U\) is a personal utility over the outcomes.

**Matrix of \(m\)-many acts on the partition of \(n\)-many uncertain states**

<table>
<thead>
<tr>
<th>Act</th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_j)</th>
<th>(\omega_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act_1</td>
<td>(o_{11})</td>
<td>(o_{12})</td>
<td>(o_{1j})</td>
<td>(o_{1n})</td>
</tr>
<tr>
<td>Act_2</td>
<td>(o_{21})</td>
<td>(o_{21})</td>
<td>(o_{2j})</td>
<td>(o_{2n})</td>
</tr>
<tr>
<td>Act_i</td>
<td>(o_{i1})</td>
<td>(o_{i2})</td>
<td>(o_{ij})</td>
<td>(o_{in})</td>
</tr>
<tr>
<td>Act_m</td>
<td>(o_{m1})</td>
<td>(o_{m2})</td>
<td>(o_{mj})</td>
<td>(o_{mn})</td>
</tr>
</tbody>
</table>
What happens to Arrow’s result when we restrict rational ordering to those that admit a global Bayes model? That is, what happens when we restrict rational orderings of options to those that admit an SEU representation, both for individuals and for the group?

Consider two SEU decision makers, Dick and Jane, who wish to form a cooperative partnership that will make decisions, constrained by the following two principles.

- The partnership must satisfy the theory of SEU maximization.
- Pareto coordination—if each of Dick and Jane strictly prefers one option \(A_1\) to a second \(A_2\), then so too does the partnership.

- Suppose Dick and Jane each strictly prefers reward \(r^\ast\) over reward \(r^\ast\), for some pair, \((r^\ast, r^\ast)\).
Possibility/Impossibility Results for Cooperative SEU compromises.

1. If Dick and Jane share a common cardinal utility over the rewards, the compromises for the group’s preferences are given by an average of their two personal probabilities, and the common utility: Linear Probability Pool

2. If Dick and Jane share a common personal probability over the states, the compromises for the group’s preferences are given by an average of their two cardinal utilities, and the common probability: Linear Utility Pool.

3. If Dick and Jane have any difference in their personal probability and do not share the same cardinal utility over rewards there are only autocratic solutions. One of them makes all the decisions for the partnership!
Here is the reasoning behind case 3. Suppose that Dick and Jane are two SEU agents, with preferences for acts summarized by the probability and utility pairs: \((P_k, U_k), \ k = 1, 2\), corresponding to their coherent weak preferences: \(\sim_k\)

Pareto rule applied to their strict preferences creates a group, strict partial order \(\sim_g\).

What coherent preferences agree with the partial order \(\sim_g\)?

Theorem (S., K., S., 1989):

If \(P_1 \neq P_2\) and \(U_1 \neq U_2\) (as explained below), then the only coherent extensions of \(\sim_g\) are \(\sim_1\) and \(\sim_2\).

- One of them is the autocrat! There are not SEU compromises.
- This impossibility applies to each pair of SEU agents who disagree (to any degree) both on probability and utility.
Heuristic Example of the Problem

Suppose Dick and Jane have different beliefs: $P_1(E) = .1$ and $P_2(E) = .3$. Also, suppose they have different values: Each prefers $r^*$ to $r_*$, though they differ in their valuation of a third reward $r$.

Let $U_1(r) = .1$ and $U_2(r) = .4$, while $U_1(r_*) = U_2(r_*) = 0$ and $U_1(r^*) = U_2(r^*) = 1$.

Dick and Jane agree that
\[ .1 \leq P_k(E) \leq .3 \quad (k = 1, 2). \]
Likewise, they agree that
\[ .1 \leq U_k(r) \leq .4 \quad (k = 1, 2). \]

These induce common preferences, whose implications for coherent extensions of $\succsim_g$ are pictured in Figure 1.
Figure 1

designates the set of probability/utility pairs agreeing with the common preferences of Dick and Jane for the comparisons, above.
Other preferences that Dick and Jane have in common constrain extensions of $\sim_G$, as depicted in Figure 2.

Figure of preferences which separate the family of agreeing probability/utility pairs.

designates the set of probability/utility pairs agreeing with the common preferences of Dick and Jane for A7 over A6, $e = 0.01$. 

Notes for Session 8, Group Preference? Arrow’s and other Impossibility Theorems, Summer School 2011  T.Seidenfeld 13
If we superimpose the two figures, we obtain Figure 3

Figure of preferences which separate the family of agreeing probability/utility pairs.

designates the set of probability/utility pairs agreeing with the common preferences of Dick and Jane, e = .01.
Figure 3 shows that the set of coherent extensions of the partial order $\sim_g$ is disconnected.

The set of coherent extensions is not convex.

Thus, the assumption that rationality requires a global SEU model is inconsistent with a simple Pareto rule, unless there is perfect agreement on either probabilities or utilities.

How can we relax the SEU model without abandoning the requirement that an admissible option carries a local Bayes model – on pain of violating strict dominance?
Let us go back to the beginning of our investigations and remind ourselves what are the minimal conditions of individual rationality, and they try to lift that up to the standards for a cooperative group, at least where binding agreements are possible. Here, think of the approach Plato uses in *The Republic*, applied to Rationality rather than to Justice.

Setting aside problems with Moral Hazard (act-state dependence), we did not find reason to overturn strict dominance, and we relied on it, e.g., in de Finetti’s “book” argument.

- What, generally, can we settle with a condition of strict dominance?

What follows, next, is a result that is more general than de Finetti’s “book.”
Recall the background conditions we used even with the betting arguments.

- Act-state independence: no cases of “moral hazards” are considered – so strict dominance is valid.
- State-independent utility: no cases where the value of a prize depends upon the state in which it is received.
- There is a defined cardinal utility for prizes – only uncertainties in beliefs are addressed by “book” – group decision problems: team play.
- The focus is on normal form (aka “strategic form”) games/decisions.

This is where the decision maker can commit, in advance, to all contingency planning. There is, however, no assumption that normal and extensive form decisions are equivalent.
**General framework:** Given a (closed) set $O$ of feasible options, a *choice function* $C$ identifies the set $A$ of *acceptable options* $C[O] = A$, for a non-empty subset $A \subseteq O$.

Note: There may be no acceptable option if the option set is not closed, e.g., there is no “best” option from the continuum of utility values in $[0, 1)$.

Also, by considering only a choice function, and not requiring an ordering over the options in $O$, we do not presume rationality entails an ordering.

**Definition:** Option $o \in O$ *has a local Bayes model* $P$ if

$$o \text{ maximizes the } P\text{-expected utility over the options in } O.$$ 

**Theorem 1** (Pearce, 1984 for finite state spaces): If an option $o \in O$ fails to have a local Bayes model then it is uniformly, strictly dominated by a finite mixture of options already available from $O$.

**Recall:** This result is a corollary of von Neumann’s Minimax Theorem!
Consider a decision problem on 2 states $\{\omega_1, \omega_2\}$, with 3 options $\{f, g, h\}$, and a convex set of probabilities $\mathcal{Q} = \{ P: 0.25 \leq P(\omega_2) \leq 0.75 \}$.

**Example 1**
Thus, strict dominance in binary comparisons, when options may be combined into mixed strategies, entails existence of a local Bayes model. In other words, each admissible option in a decision problem must be supported by some Bayes model or other, as a best response, on pain of simple dominance by a mixed options, otherwise.

**Important Example**: Iterative elimination of dominated strategies. Consider the following $2 \times 3$ game.

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$M$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>(2,1)</td>
<td>(1,4)</td>
<td>(0,3)</td>
</tr>
<tr>
<td>$B$</td>
<td>(1,8)</td>
<td>(0,2)</td>
<td>(1,3)</td>
</tr>
</tbody>
</table>

Though there is no strict dominance between Row-player’s two options, from Column-player’s perspective, $R$ is never a best reply. For example, the mixed strategy $[.25L \oplus .75M]$ strictly dominates $R$.

**Exercise**: Identify the set of strategies using $L$ and $M$ that dominate $R$. 
Thus, Column player eliminates option R, and Row player knows this.

Then, having eliminated the state $R$, option $T$ strictly dominates option $B$ for Row player.

Then, Column player knows that Row player has chosen state $T$, so that now $M$ is the sole winner for Column player.

The combination (T, M) is the sole survivor of iterated elimination of strictly dominated strategies.

And, it is THE Nash equilibrium of this game – the only rationalizable pair.
Coherent Choice Functions Under Uncertainty
joint work with M.J. Schervish and J.B. Kadane

By coherent choice, we mean
a decision rule that restricts the admissible options in a problem to
those that maximize expected utility for some probability in a set \( \mathcal{P} \).

Specifically, here we develop an account:

- Where coherent choice from a set of options does not reduce to
  a set of binary comparisons between the options available.
- Each two sets of probabilities yield different coherent choices.
- Coherent choice is axiomatized by constraints on choice functions
  that parallel the familiar axioms for coherent (binary) preferences.
- Coherent choice generally does not reduce to pairwise comparisons
  between options, taken two-at-a-time.
Given a (closed) set $O$ of feasible options, a *choice function* $C$ identifies the set $A$ of *acceptable options* $C[O] = A$, for a non-empty subset $A \subseteq O$.

With local coherence, only $\{f, g\}$ are admissible from the triple $\{f, g, h\}$; however, all pairs are admissible in pairwise choices.

![Diagram](image)

The set of (Bayes) mixed strategies $\alpha f \oplus (1-\alpha)g$ is **pink**

- Note well that $m$ is among the coherent acts from $\{f, g, m\}$ *if and only if* $P(\omega_2) = .5$ belongs to the set $\mathcal{P}$. 

Session 9, Group Decision Making After Arrow, Summer School 2011  T.Seidenfeld
This observation about the admissibility of a mixed option generalizes.

- Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be a partition and $\mathcal{P}$ a set of probabilities on $\Omega$. We consider horse lotteries on $\Omega$ using only the two utility rewards $\{0,1\}$.
- Let $p = (p_1, \ldots, p_n)$ be one probability distribution on $\Omega$ and denote by $p$ the smallest nonzero coordinate of $p$.
- Define the constant horse lottery act $a = p1 \oplus (1-p)0$.

- For each $j = 1, \ldots, n$, define the act $h_j$ by
  \[
  h_j(\omega_i) = \begin{cases} 
  1 & \text{if } i = j \text{ and } p_j = 0, \\
  a & \text{if } i \neq j \text{ and } p_j = 0, \\
  (p/p_j)1 + (1-p/p_j)0 & \text{if } i = j \text{ and } p_j > 0, \\
  0 & \text{if } i \neq j \text{ and } p_j > 0.
  \end{cases}
  \]

Consider the option set $O_p = \{a, h_1, \ldots, h_n\}$.

**Theorem 2**: $p \in \mathcal{P}$ if and only if $C(O_p) = O_p$ -- all of $O_p$ is admissible.
**Corollary:**

- Each (arbitrary) set of probabilities has its own distinct coherent choice function.
- For each two different sets of probabilities there is a (finite) decision problem where they have distinct coherent choices.

**Application:** We can represent the set of probability distributions that make two events independent, since convexity of the set is not required in our approach.

Coherent choice functions are characterized by axioms on admissible sets that parallel familiar axioms for coherent preferences over *horse-lotteries*.

**Horse Lotteries:** *Functions from states to von N-M lotteries.*

\[ h: \mathcal{Q} \to vN-M \text{ lotteries} \]
Coherent Preference \( \prec \)

Axiom 1 \( \prec \) is a weak order

Axiom 2 \( \prec \) obeys Independence
\[
 o_1 \prec o_2 \iff xo_1 \oplus (1-x)o_3 \prec xo_2 \oplus (1-x)o_3
\]

Axiom 3: Archimedean postulate to assure a real-valued representation, for utility and personal probability.

Axiom 4: To insure existence of a state-independent utility requires that the decision maker’s preference for constant horse lotteries reproduces under each non-null state in the form of called-off horse lotteries.

Coherent Choice Functions

C-Axiom 1a Sen’s Property \( \alpha \):
An inadmissible option remains so upon addition of other options.

C-Axiom 1b Aizerman condition, almost:
Deleting inadmissible options does not promote other inadmissible options.

- These two axioms determine a strict partial order \( O_1 \prec O_2 \) on option sets: \( O_1 \) contains no admissible options from among the choice of \( O_1 \cup O_2 \).

- The remaining 3 pairs of C-Axioms are expressed with the convenience of the partial order \( \ll \).
C-Axiom 2a  « obeys Independence  \( O_1 \ll O_2 \) iff \( xO_1 \oplus (1-x)O_3 \ll xO_2 \oplus (1-x)O_3 \).  

Let \( H(O) \) be the option set formed by taking the convex hull of options in \( O \).  

C-Axiom 2b Mixtures  If \( o \in O \) and \( o \not\in C[H(O)] \), then \( o \not\in C[O] \).  

This axiom asserts that unacceptable options from a mixed set remain so even before mixing – see Pearce’s Theorem for motivation!  

C-Axiom 3: The Archimedean condition requires a technical adjustment. The canonical form used by, e.g. von Neumann-Morgenstern theory or Anscombe-Aumann theory is too restrictive in this setting. (See section II.4 of our 1995.) The reformulated version of the Archimedean condition is as a continuity principle compatible with strict preference as a strict partial order. It reads as follows.  

Let \( A_n \) and \( B_n \) \((n = 1, \ldots)\) be sets of options converging pointwise, respectively, to the option sets \( A \) and \( B \). Let \( N \) be an option set.  

C-Axiom 3a: If, for each \( n \), \( B_n \ll A_n \) and \( A \ll N \), then \( B \ll N \).  

C-Axiom 3b: If, for each \( n \), \( B_n \ll A_n \) and \( N \ll B \), then \( N \ll A \).
C-Axiom 4 – State-neutrality/dominance is captured by the following relations.

Consider horse lotteries $h_1$ and $h_2$, involving the two rewards 0 and 1 exclusively, with $h_i(\omega_j) = \beta_{ij}I \oplus (1-\beta_{ij})\theta$; $i = 1, 2$; $j = 1, \ldots, n$.

**Definition:** $h_2$ weakly dominates $h_1$ if $\beta_{2j} \geq \beta_{1j}$ for $j = 1, \ldots, n$.

**C-Axiom 4:** Assume that $o_2$ weakly dominates $o_1$, and that $a$ is an option different from each of these two.

- **4a:** If $o_2 \in O$ and $a \notin C[\{o_1\} \cup O]$ then $a \notin C[O]$.
- **4b:** If $o_1 \in O$ and $a \notin C[O]$ then $a \notin C[\{o_2\} \cup O \setminus o_1]$.

- In words, Axiom 4a requires that when a weakly dominated option is removed from the set of options, other inadmissible options remain inadmissible. So, by Axiom 1, when an option is replaced in the option set by one that it weakly dominates, other admissible options remain admissible.

- Axiom 4b says that when an option is replaced by one that weakly dominates it, (other) inadmissible options remain inadmissible. Trivially by Axiom 1, merely adding a weakly dominating option cannot promote an inadmissible option into one that is admissible.
Representation Theorem

• The 4-pairs of axioms are necessary for a choice function to be coherent.

• The axioms suffice for representing a choice function with the coherence rule for admissibility applied to a set of
  Probability/Almost-state-independent utility pairs.

• We offer a sufficient condition for a single, state-independent utility on rewards.
Summary

We develop an account:

• Where coherent choice from a set of options does not reduce to a set of binary comparisons between the options available.
• Each two sets of probabilities yield different coherent choices.
• Coherent choice is axiomatized by constraints on choice functions that parallel the familiar axioms for coherent (binary) preferences.
• The standards of coherent choice afford a common model for individual and cooperative group choice subject to a simple Pareto principle: The set of probabilities that represent a group’s coherent choices is the union of the sets of probabilities for the individuals that comprise the group!

See [www.sipta.org](http://www.sipta.org) for discussion about how to use sets of probabilities.