Session 4: Classical Game Theory – von Neumann and Nash

4.1: See the attached material taken from Luce and Raiffa’s (1957) wonderful book, *Games and Decisions* – appendices 2-4 about von Neumann’s *Minimax Theorem* for Two-person, Zero-sum games. In class we’ll concentrate on the opening to Appendix 2 and the geometric argument of Appendix 3.

4.2: Limitations of the von Neumann Minimax Theorem.
   4.2.1: Non-zero sum games – multiple non-equivalent equilibria (BoS game)
   4.2.2: More than 2 players – no stability against coalitions. (Divide a dollar.)

4.3 Nash’s Theory – Preserve equilibria

4.4 Prisoner’s Dilemma – *Pareto* versus *Strict Dominance*

4.5 Strictly dominating non-Bayes decisions.
4.1 von Neumann’s Minimax Theorem for 2 person, 0 sum games.

4.2.1: Multiple, non-equivalent equilibria in 2-person, non-zero sum games.

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
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<tbody>
<tr>
<td>Bach</td>
<td>(2,1)</td>
<td>(0,0)</td>
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<tr>
<td>Stravinsky</td>
<td>(0,0)</td>
<td>(1,2)</td>
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- Equilibria pairs are: (B,B) and (S,S).
- These have different values to the players.
- Moreover, neither (B,S) nor (S,B) is in equilibrium.

4.2.2 “Divide a dollar” among 3 players, a zero-sum game.

- Each of player A, B, and C picks one of the other two as a partner and proposes a division of a dollar. If two proposals match exactly, that division is made with the dollar provided by the third party. Otherwise, there is no change of money.
4.3 Nash Equilibrium Theory: Each player simultaneously plays a best response.

4.4 Prisoner’s Dilemma

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<thead>
<tr>
<th></th>
<th>Mum</th>
<th>Rat</th>
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<tbody>
<tr>
<td>Mum</td>
<td>(2,2)</td>
<td>(0,3)</td>
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<tr>
<td>Rat</td>
<td>(3,0)</td>
<td>(1,1)</td>
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</tbody>
</table>

- What are the equilibria?
- What is dominated?
- What maximizes security?
- BUT what is Pareto?
4.5 On the strict dominance of non-Bayes decisions in finite decision problems:

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ be a finite partition of states, indexed by the set $J$.
Let $A$ be a finite class of ("pure") acts, $A = \{a_1, \ldots, a_m\}$ defined on $\Omega$, indexed by set $I$.
For $a_i \in A$, $a_i(\omega_j) = u_{ij}$, a cardinal utility of the consequence of $a_i$ when state $\omega_j$ obtains.

Let $Q$ be the class of probability distributions over $\Omega$ with a "prior" over states denoted $q$.
Similarly, let $P$ be the class of mixtures over $A$, with mixed acts denoted $p$.
The expected utility to the decision maker of mixed act $p$ with respect to prior $q$,
for $(p, q) \in P \otimes Q$, is as follows.

$$E[p,q] = \sum_j \sum_i u_{ij} p(a_i)q(\omega_j)$$

Note that as these are finite sums, the order of summation does not matter.
Theorem: Suppose that for each \( q \in Q \), act \( a \in A \) fails to maximize expected utility. That is, \( a \neq \arg \max_A E[a,q] \).

Then (i) there is a mixed alternative \( p^* \) that strictly dominates \( a \) in \( \Omega \).

\[
U(p^*(\omega_j)) > U(a(\omega_j)), \text{ for } j = 1, \ldots, n,
\]

and (ii) \( p^* \) maximizes expected utility for some \( q^* \in Q \).

Aside for those who study Statistical Decision Theory:

With this result we can apply the strict standard of deFinetti’s “incoherence” (= strict dominance) to a broad class of decisions under uncertainty, analogous to traditional Complete Class Theorems for Bayes decisions.

The standard of incoherence used in this result is stronger than the mere inadmissibility (= weak dominance) of non-Bayes decisions, as is used in those Complete Class theorems (Wald) for statistical decision problems. It follows de Finetti’s criterion of incoherence as a failure of simple dominance.

Part (ii) depends on the statistician having only finitely many (pure) options.
**Proof.** By hypothesis, assume that for each \( q \in Q \), act \( a \) fails to maximize expected utility with respect to the options in \( A \). That is, \( a \) is never a “best response” to any opinion about the states.

So, let \( f(q) = a^q \) identify an act such that \( E[a^q, q] > E[a, q] \).

Transform the decision problem to one where the new utility \( U' \) is “regret” with respect to the act \( a \). That is, for each act

\[
a_i \in A, \quad U'(a_i(\omega_j)) = u'_{ij} = u_{ij} - U(a(\omega_j)).
\]

Hence, \( U'(a(\omega_j)) = 0 \), for \( j = 1, \ldots, n \).

Aside: When states are independent of acts, which is sufficient for dominance to be a valid principle, maximizing expected utility generates the same preference ranking of options as does maximizing expected regret-utility.
Now, consider the finite, two-person, zero-sum game in which player-1’s payoff for choosing act \( a_i \) when player-2 chooses state \( \omega_j \) is \( u'_{ij} \), and player-2’s payoff is \(-u'_{ij}\).

As before, define 
\[
E'[p,q] = \sum_i \sum_j u'_{ij} p(a_i)q(\omega_j)
\]
as the value to player-1 when the pair of mixed strategies \((p,q)\) is played.

Since, under the assumption of act/state independence, ranking acts by expected utility and ranking them by an expected regret-utility coincides,

for each \( q \in Q \) and for each pair of mixed acts \( p, p' \in P \),
\[
E'[p, q] \geq E'[p', q] \text{ if and only if } E[p, q] \geq E[p', q],
\]
since \( E'[p, q] = E[p, q] - E[a, q] \).
According the von Neumann Minimax Theorem this game has a value $V'$ where:

$$\inf_Q \sup_P E'[p,q] = \sup_P \inf_Q E'[p,q] = V'.$$

Moreover, there are strategies for each player, $(p^*, q^*) \in P \otimes Q$, that achieve the game’s value $V'$.

That is

$$\inf_Q E'[p^*, q] = \sup_P E'[p, q^*] = V'.$$

Next, observe that $E'[p^*, q^*] \geq E'[a^{q^*}, q^*] > E'[a, q^*] = 0$.

The first inequality is because $p^*$ is a best response for player-1 to player-2’s choice of $q^*$.

The second inequality holds because $a^{q^*}$ is a strictly better response to $q^*$ than is $a$.

Note that since $q^*$ is player 2’s best response to player-1’s choice of $p^*$, we have that for each $q \in Q$, $E'[p^*, q] \geq E'[p^*, q^*]$. 
Thus, 

\[ E'[p^*, \omega_j] \geq E'[p^*, q^*] \geq E'[a^q, q^*] > E'[a, \omega_j] = 0. \]

This establishes that \( p^* \) strictly dominates \( a \) across states \( \Omega \), either in \( U \) or in \( U' \) units, as the difference is never less than \( E'[a^q, q^*] > 0 \), and \( p^* \) maximizes expected utility against the “prior” \( q^* \).

The converse to the Theorem is easy! Thus, the two results may be summarized as follows. Assume act/state independence.

**Theorem:** In a finite decision problem that includes all its mixed acts, an act is never “Bayes” (i.e. maximizes expected utility for *no* “prior”) if and only if that act is dominated by some mixed-act.

Notes for Session 5 – Forecasting, wrong and right!

We begin with a general questionnaire.
Each of the following 25 assertions is factual, either true or false.
Next to each assertion, offer your personal probability that it is true,
where probability 1 is “certainly” true and probability 0 is “certainly” false.
Please use probability numbers in units of 1/20, e.g., .15, .30, .65, etc.

We will be considering how to assess expertise in probabilistic forecasting, and your answers will serve as a sample of such expert forecasts!

1. The Amazon River is longer than the Congo River.
2. The Congo River is longer than the Missouri River.
3. The Missouri River is longer than the Niger River.
4. The Niger River is longer than the Mississippi River.
5. The Mississippi River is longer than the Volga River.
6. At its deepest point the Pacific Ocean is deeper than the Atlantic Ocean.
7. At its deepest point the Atlantic Ocean is deeper than the Indian Ocean.
8. At its deepest point the Indian Ocean is deeper than the Artic Ocean.
9. At its deepest point the Artic Ocean is deeper than the Mediterranean Sea.
10. In area, the Sahara Desert is larger than the Gobi Desert.
11. In area, the Gobi Desert is larger than the Libyan Desert.
12. In area, the Libyan Desert is larger than the Kalahari Desert.
13. In area, the Kalahari Desert is larger than the Arabian Desert.
14. In area, the Arabian Desert is larger than the Painted Desert.
15. Blaise Pascal was born before Gottfried Leibnitz was born.
16. Gottfried Leibnitz was born before George Berkeley was born.
17. George Berkeley was born before David Hume was born.
18. David Hume was born before Emmanuel Kant was born.
19. Emmanuel Kant was born before Jeremy Bentham was born.
20. Jeremy Bentham was born before Georg Hegel was born.
21. The length of the Earth’s equator is greater than the length of its Meridian.
22. The Sun has greater density than does liquid water.
23. Syracuse, N.Y. has a greater average annual snowfall than Juneau, Alaska.
24. Juneau, Alaska has a greater average annual snowfall than Flagstaff, Arizona.
25. Flagstaff, Arizona has a greater average annual snowfall than Buffalo, N.Y.
Calibration curves for probabilistic forecasting

Figure 1

Calibration curves for:
(A) underconfident
(B) calibrated
(C) overconfident
(D) mixed-case forecasters.

Frequency of Correct Responses

Forecast Probability
• What do you think will be your calibration curve for the survey with which we began?

• For which probability forecasts will you be best calibrated? Worst calibrated?

How will your forecasts be affected by feedback?

I will now reveal the truth values of each prime numbered question:
Q2 is __, Q3 is __, Q5 is __, Q7 is __, Q11 is __, Q13 is __, Q17 is __, Q19 is __, Q23 is __.

• Reset your probability forecasts for the remaining 16 questions given this news.

• Do you think your revised 16 forecasts will have better calibration than the first forecasts you gave for these 16 forecasts? Why??

• Now I will reveal the truth values for the remaining assertions. Please check your calibration.
• Suppose that, before final calibration scores are awarded, you are asked to include forecasts for the following 15 additional assertions.

26. George Washington’s famous white horse was dark brown in color.

27. George W. Bush has his picture on the US $1 bill.

28. Las Vegas is the capital of the USA.

29. Hollywood is the capital of the USA.

30. The Pittsburgh Pirates, who have won about 45% of their games this season, have the best won/loss record in Major League Baseball.

31. It will snow at least 1 meter in Pittsburgh today (June 8, 2011).

32. ...

33. ...

34. ...

35. Dick Cheney is the most popular American living.

• IS CALIBRATION RELEVANT TO EXPERTISE IN FORECASTING?
Regret-like Scoring Rules for Probabilistic Forecasting.

In all of the following, multiple forecasts get the sum of the individual scores.

1. “0-1” loss for non-probabilistic forecasts. (T/F exams)

You are penalized 1 utile if a forecast differs from the realized value of the random variable, 0 if the forecast agrees with the realized value.

In what follows, $X$ is a random quantity, i.e. $X(\omega)$ is a real number for each $\omega \in \Omega$. For forecasting events, note that the event $E$ can be identified with its indicator function,

$$
\chi_E = \begin{cases} 
1 & \text{if } E \text{ obtains} \\
0 & \text{if } E \text{ fails to obtain.}
\end{cases}
$$

**Problem:**

- As an expected utility maximizer, if your personal probability for event $E$ is $P(E)$ what should you offer as your forecast $Q(E)$ of $E$ subject to 0-1 loss?
- What should you announce as your forecast for the simple random variable $X$?
2. “Mean Deviation” loss.
For random quantity $X$, you are penalized $|X – P(X)|$ for your forecast $P(X)$.

*Problem:* As an expected utility maximizer, what is your forecast for $X$ subject to mean-deviation loss?

3. “Squared-error” loss.
For random quantity $X$, your penalty is $[X – P(X)]^2$ utiles for your forecast $P(X)$.

*Problem:* As an expected utility maximizer, what is your best forecast for event $E$ subject to squared error loss?
Coherence and Forecasting

- de Finetti’s “fair prices” – not a loss function.

For random quantity $X$, you receive $c_X[X - P(X)]$ utiles for your forecast $P(X)$.

In your role as merchant, your fair prices are incoherent\textsubscript{1} if the customer can choose the $c$’s so that you suffer a sure loss.

In your role as “forecaster,” your forecasts are incoherent\textsubscript{2} if there is a rival set of forecasts that assure you a smaller penalty. Consider squared error loss.

\textbf{Example}: The two previsions: $P(A) = .6$ and $P(A^c) = .7$ are incoherent\textsubscript{1}

A \textit{Book} is achieved against these previsions with the gambler’s strategy $c_A = c_{A^c} = 1$.

Then the net payoff to the bookie is -0.3 regardless which state $\omega$ obtains.
In order to see that these are also incoherent\(_2\) forecasts, review the Figure, below.

If the forecast previsions are not coherent\(_1\), they lie outside the probability simplex. Project these incoherent\(_1\) forecasts into the simplex. As in the Example, (.60, .70) projects onto the coherent\(_1\) previsions depicted by the point (.45, .55). By elementary properties of Euclidean projection, the resulting coherent\(_1\) forecasts are closer to each endpoint of the simplex. Thus, the projected forecasts have a dominating Brier score regardless which state obtains. This establishes that the initial forecasts are incoherent\(_2\). Since no coherent\(_1\) forecast set can be so dominated, we have coherence\(_1\) of the previsions if and only coherence\(_2\) of the corresponding forecasts.
Figure
Notes for Session 6 – Combining Expert Opinions. Can it be done?!

The challenge for this session is to determine whether there are defensible rules for combining a set of $n$-many “expert” probability distributions into one common probability distribution.

We suppose that each of our $n$-many experts has an opinion about some common domain of interest, represented by the partition into relevant states:

$$\Omega = \{\omega_1, \ldots, \omega_k\},$$

$\text{Expert}_i$’s opinion is probability distribution

$$P_i = <p_{i1}, \ldots, p_{ik}> \text{ over } \Omega, \ i = 1, \ldots, n.$$
Can we combine these \( n \)-many probabilities, \( P_i \), into a single probability \( P_G \) that reflects the group’s combined wisdom?

**Linear Pooling:**

Assign each expert a non-negative “weight” \( w_i \geq 0 \) to reflect her/his relative expertise in the group, and standardize these so that \( \sum_i w_i = 1 \)

Form \( P_G = \sum_i w_i P_i \), the \( w_i \)-weighted average of their separate opinions.

Defn: \( P_G \) is called a *Linear Pool* of the expert opinions.

- Geometric Interpretation of the Linear Pool using the simplex of probabilities on \( \Omega \).

  The Linear Pool puts \( P_G \) inside the *hull* (= closed, convex set) of the \( n \)-many points \( P_i \) (\( i = 1, \ldots, n \)).
What are some of the nice features of a Linear Pool?

• Preservation of unanimity of probabilistic opinions

\[
\text{If } c_1 \leq P_i(E) \leq c_2 \text{ (i = 1, \ldots, n) then } c_1 \leq P_G(E) \leq c_2.
\]

Suppose there is a common utility U for outcomes across the group, that is, suppose the group is a Team.

If each expert judges that Act_1 is better than Act_2 by the standards of SEU, then so too the Team will make the same Pareto judgment – using the shared utility U and pooled opinion P_G.

• The Linear Pool is computationally convenient in the following sense of being a local computation.

Once the \( w_i \) (i = 1, \ldots, n) are fixed \( P_G(E) \) depends solely on the \( n \)-values \( P_i(E) \). In other words, \( P_G(E) \) does not depend upon how the \( n \)-many experts divide up their probabilities on \( E^c \).
Is there a problem with the Linear Pool?

Learning and Pooling.

Let us use conditional probability as the rule for updating new information.

- \( P_i(\bullet \mid F) \) is the revised opinion for \( P_i \) when new information \( F \) is added.

1. Consider allowing the experts all to learn the same new information \( F \) before pooling their opinions with weights \( w_i \).

So, by this method of first updating and then pooling we obtain

\[
P^1_G(\bullet \mid F) = \sum_i w_i P_i(\bullet \mid F).
\]

2. However, we might first pool the expert opinions and then update \( P_G \) with the same information \( F \), to yield

\[
P^2_G(\bullet \mid F) = P_G(\bullet \cap F) \div P_G(F)
= \sum_i w_i P_i(\bullet \cap F) \div \sum_i w_i P_i(F).
\]

Alas, generally,

\[ P^1_G(\bullet \mid F) \neq P^2_G(\bullet \mid F)! \]
Consider the 3-dimensional simplex of probabilities on two events and the saddle-shaped surface of independence between two events.

We see that, generally, linear pooling two probability distributions that make the events E and F independent will make them dependent! This method of pooling creates some strange decisions for the group.
For example, if $n = 2$ and both experts think that $E$ and $F$ are independent events, then each will refuse to pay anything to learn about $F$ before betting on $E$. However, if a linear opinion pool is formed first, that opinion will make $E$ and $F$ dependent events, and there will be value in first learning $F$ before wagering on $E$.

Consider two doctors who are unsure both about your allergic state and about the weather in China, but who agree these are independent events. Do you mind if, instead of checking your medical history for information about your drug allergies, instead they spend the insurance money learning about the weather in China and using that information to decide on your treatment?
Example for challenging Linear pooling:

Consider a decision problem among three options – three treatment plans \{T_1, T_2, T_3\} defined over 4 states \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\) with determinate utility outcomes given in the following table. That is, the numbers in the table are the utility outcomes for the options (rows) in the respective states (columns).

<table>
<thead>
<tr>
<th></th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_3)</th>
<th>(\omega_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_1)</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>(T_2)</td>
<td>1.00</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>(T_3)</td>
<td>0.99</td>
<td>-0.01</td>
<td>-0.01</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Let a convex set \(P\) of probabilities be generated by two extreme points, distributions \(P_1\) and \(P_2\), defined by the following table. Distribution \(P_3\) is the .50-.50 (convex) mixture of \(P_1\) and \(P_2\).

<table>
<thead>
<tr>
<th></th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_3)</th>
<th>(\omega_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1)</td>
<td>0.08</td>
<td>0.32</td>
<td>0.12</td>
<td>0.48</td>
</tr>
<tr>
<td>(P_2)</td>
<td>0.48</td>
<td>0.12</td>
<td>0.32</td>
<td>0.08</td>
</tr>
<tr>
<td>(P_3)</td>
<td>0.28</td>
<td>0.22</td>
<td>0.22</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Note that (for \(i = 1, 2, 3\)) under probability \(P_i\), only option \(T_i\) is Bayes-admissible from the option set of \{\(T_1\), \(T_2\), \(T_3\)\}. 
Without convexity – that is, using only the set comprised by the two (extreme) distributions \( \{ P_1, P_2 \} \) – option T3 is the sole \textit{inadmissible} option from among the three options \( \{ T_1, T_2, T_3 \} \).

Now, interpret these states as the cross product of two binary partitions:
- a binary medical event – \( A \) (patient allegric); \( A^c \) (patient not-allergic),
- a binary meteorological partition – \( S \) (sunny) and \( S^c \) (cloudy).

Specifically: \( \omega_1 = A \& S \quad \omega_2 = A \& S^c \quad \omega_3 = A^c \& S \quad \omega_4 = A^c \& S^c \)

Under \( P_1 \), the two events are independent with \( P_1(A) = .4 \) and \( P_1(S) = .2 \).
Likewise, under \( P_2 \), the events are independent, \( P_2(A) = .6 \) and \( P_2(S) = .8 \).

But under \( P_3 \) the \( A \) and \( S \) are positively correlated: \( .56 = P_3(A \mid S) > P_3(A) = .5 \), as happens for each non-trivial mixture of \( P_1 \) and \( P_2 \).
The three options have the following interpretations:

- $T_1$ and $T_2$ are ordinary medical options, with outcomes that depend solely upon the patient’s allergic state.
- $T_3$ is an option that makes the allocation of medical treatment a function of the meteorological state, with a “fee” of 0.01 utile assessed for that input.

That is, $T_3$ is the option

“$T_1$ if cloudy and $T_2$ if sunny, while paying a fee of 0.01.”

Suppose $P_1$ represents the opinion of medical expert 1, and $P_2$ that of medical expert 2.

- Without convexity of the probabilities, $T_3$ is inadmissible. That supports the shared agreement between the two medical experts that $T_3$ is unacceptable from the choice of three \{$T_1$, $T_2$, $T_3$\}. It supports the pre-systematic understanding that under $T_3$ you pay to use medically irrelevant inputs about the weather to determine a medical treatment.

- However, with convexity of the set generated by $p_1$ and $p_2$, then $T_3$ is admissible.
Externally Bayesian Pooling Rules.

There is a family of pooling rules that is invariant over the order of pooling and updating by conditioning. These are *Externally Bayesian Pooling rule.*

It is a “logarithmic pool”: \[ P_G \propto \prod_i P_i^{w_i} \]

It is a linear pool in the logarithms of the expert opinions.

- What is problematic about this pooling rule?

Example with three states and two experts.

\[ \Omega = \{\omega_1, \omega_2, \omega_3\} \quad P_1 = <.3, .5, .2>, \quad P_2 = <.3, .2, .5>, \quad \text{and} \quad w_1 = w_2. \]

**Exercise:** Show that with the logarithmic pooling rule, \( P_G(\omega_1) \neq .3 \), which is a violation of unanimity for pooling of the unconditional probabilities.