Session 1: *Dominance and de Finetti’s “Book” argument*

Partition circumstances with a finite set of

*pairwise exclusive* and *mutually exhaustive* situations.

A partition with \( n \)-states \{state_1, state_2, \ldots, state_n\} is written as:

\[
\Omega = \{ \omega_1, \omega_2, \ldots, \omega_n \}.
\]

Suppose that YOU, the decision maker, can compare two acts, state by state, according to the desirability of their outcomes, \( o_{ij} \).

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \cdots & \omega_k & \cdots & \omega_n \\
Act_1 & o_{11} & o_{12} & \cdots & o_{1k} & \cdots & o_{1n} \\
Act_2 & o_{21} & o_{22} & \cdots & o_{2k} & \cdots & o_{2n} \\
\end{array}
\]

**Strict dominance**

- If YOU judge each outcome \( o_{ij} \) is strictly preferable to the outcome \( o_{2j} \) \((j = 1, \ldots, n)\),
  then you strictly prefer \( Act_1 \) over \( Act_2 \) in a pairwise choice between them.
Example 1: Suppose that you prefer more money to less. Consider the binary state decision problem where the payoffs are:

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act$_1$</td>
<td>$300$</td>
<td>$100$</td>
</tr>
<tr>
<td>Act$_2$</td>
<td>$400$</td>
<td>$200$</td>
</tr>
</tbody>
</table>

So, Act$_2$ strictly dominates Act$_1$.

- Might it be reasonable, nonetheless, to prefer Act$_1$ over Act$_2$?? For instance, what if Act$_i$ brings about state $\omega_i$? What do you choose then?

This is an instance of what is called in the insurance business “Moral Hazard.” [See the 2005 New Yorker article by Gladwell.]

UNTIL OTHERWISE NOTED, WE WILL
ASSUME THERE ARE NO MORAL HAZARDS
Example 2 (circa 1931): B. de Finetti’s betting rates and *Books*.

A bet on/against the event $E$, at odds of $r:(1-r)$, with the “pot” in a winner-take-all wager equal to the combined stakes $S > 0$ (say, bets are in $\$\ units), is specified by its payoffs, as follows.

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$E^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>bet on $E$</strong></td>
<td>(win) $(1-r)S$</td>
<td>(lose) $-rS$</td>
</tr>
<tr>
<td><strong>bet against $E$</strong></td>
<td>(lose) $-(1-r)S$</td>
<td>(win) $rS$</td>
</tr>
<tr>
<td><strong>abstain</strong></td>
<td>status quo</td>
<td>status quo</td>
</tr>
</tbody>
</table>

- By permitting what are formally negative stakes, $S < 0$, we can reverse *betting on* and *betting against*, since betting is a zero-sum game.
• We assume that the status quo (the consequence of abstaining) represents no net change in wealth. It is depicted by a 0 payoff in the units of the stake.

A collection \( E = \{ E_1, \ldots, E_n \} \)
of well defined events is presented to YOU (the bookie) and you are required to post a collection of fair odds \( R = \{ r_1, \ldots, r_n \} \)
subject to the condition that YOU are willing to accept finite combinations of bets on, or against, events in \( E \) at the corresponding rates, assuming that your shares in the combined stakes \( S = \{ S_1, \ldots, S_n \} \)
remain within YOUR financial means. Call them YOUR allowed stakes.

**Definition:** Given YOUR posted fair odds, \( R \), if there is a selection of allowed stakes such that YOUR combined bets result in a sure loss to YOU no matter what logically possible combination of events in \( E \) occurs, then you are in **book**.
• Note: Abstaining, i.e. not betting at all, strictly dominates being in a book!

What strategies available to YOU, the bookie, preclude a book against YOU? Book Theorem (aka “Dutch Book” theorem) due to de Finetti (also see F.P.Ramsey):

• YOUR fair odds \( R \) are mathematical probabilities for the events in \( E \) – that is, YOUR odds satisfy the (three) axioms of probability, if and only if no book is possible against \( R \).

Defn: When fair odds are immune to book, they are called coherent. Otherwise, they are called incoherent odds.

Example 2: Suppose that with respect to a binary partition \( \{E_1, E_2 (= E_1^c)\} \), the bookie posts fair odds of \( \{r_1 = .4 \text{ and } r_2 = .7\} \), and the bookie has a total of $10 to wager.

• Are these odds coherent?
• If not, what book(s) can be made against the bookie?
Next, we review an elementary account of mathematical probability, based on important work by A.N.Kolmogorov (1933).

A mathematical probability function $P$ assigns probability numbers to events, where an event $E$ is a set. Specifically each event $E$ is a subset of the sure event $\Omega$.

Kolmogorov’s (1933) theory of probability requires that, for events $E$ and $F$

- Axiom 1 $0 \leq P(E) \leq 1$.
- Axiom 2 $P(\Omega) = 1$.
- Axiom 3 If $E \cap F = \phi$, then $P(E) + P(F) = P(E \cup F)$.

*Proof* that if your fair odds are coherent, they are probabilities – by the axioms!
(Axiom 1) \(0 \leq P(E) \leq 1\).

Suppose, to deny the conclusion, your fair odds \(r\) for some event \(E\) are greater than 1. Then for an allowed stake \(S > 0\) YOU will accept this "bet" on \(E\).

\[
\begin{array}{ccc}
E & E^c \\
\text{bet on } E & (1-r)S & -rS \\
\end{array}
\]

Then you lose regardless which event occurs, as each payoff is negative.

Likewise, if YOUR fair odds \(r\) on \(E\) are negative, you'll "bet" against \(E\) at allowed stake \(S > 0\).

\[
\begin{array}{ccc}
E & E^c \\
\text{bet against } E & -(1-r)S & rS \\
\end{array}
\]

And you lose, regardless, as each payoff is negative.
(Axiom 2) \( P(\Omega) = 1 \) Using what we just showed, assume that \( 0 \leq r_\Omega \leq 1 \).

If \( r_\Omega < 1 \), then YOU find it fair to bet against \( \Omega \) at \( r < 1 \) with allowed stake \( S > 0 \).

\[
\begin{align*}
\Omega & \quad \phi = \Omega^c \\
bet against \Omega & \quad -(1-r)S
\end{align*}
\]

But then YOU lose \((1-r)S\) for sure, since \( \Omega \) is certain!

(Axiom 3) If \( A \cap B = \phi \), then \( P(A) + P(B) = P(A \cup B) \).

Consider 3 fair bets on \( A, B, C = (A \cup B) \), at rates \( r_A, r_B & r_C \) with stakes \( S_A, S_B & S_C \).

Then the simultaneous gains from these are:

\[
\begin{align*}
G_{(A \& -B)} &= S_A + S_C - (r_A S_A + r_B S_B + r_C S_C) \\
or \\
G_{(-A \& B)} &= S_B + S_C - (r_A S_A + r_B S_B + r_C S_C) \\
or \\
G_{(-A \& -B)} &= - (r_A S_A + r_B S_B + r_C S_C).
\end{align*}
\]
Given the rates, these are three linear equations in the three variables $S_A$, $S_B$ and $S_C$. If the equations are linearly independent, then we may solve the rates for whatever (negative) desired gains we seek.

Hence, for the rates to be coherent, the equations must be linearly dependent, i.e. the following must obtain:

\[
\begin{vmatrix}
1-r_A & -r_B & 1-r_C \\
-r_A & 1-r_B & 1-r_C \\
-r_A & -r_B & -r_C
\end{vmatrix} = 0
\]

Solving the determinate yields:

\[r_A + r_B = r_C\]

as required for the third axiom.
This result can be extended to *conditional probability*,

\[ P(A \mid B) \], the conditional probability for \( A \), given \( B \) is analyzed using "called-off" bets. The *called-off* bet on \( A \) (given \( B \)), results in status quo if \( B \) fails to occur.

A *called-off bet* on/against event \( A \), given \( B \), at odds of \( r:(1-r) \) with total stake \( S \) (\( S > 0 \)) is specified by its payoffs, as follows.

<table>
<thead>
<tr>
<th></th>
<th>( AB )</th>
<th>( A^cB )</th>
<th>( B^c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>on ( A )</td>
<td>( (1-r)S )</td>
<td>(-rS )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>against ( A )</td>
<td>(- (1-r)S )</td>
<td>( rS )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

Then coherent betting, including "called-off" bets, entails

\[ \text{Axiom 4: } P(A \mid B) \times P(B) = P(A \cap B). \]

The proof follows a similar argument for Axiom 3:
Let $r_{AB}$ be the fair odds on $A \cap B$.
Let $r_{A|B}$ be the fair (called-off) odds on $A$, given $B$.
Let $r_B$ be the fair odds on $B$.

Assume all three bets are placed with stakes of $S_{AB}$, $S_B$, and $S_{A|B}$.

Denote by $G_{(AB)}$, $G_{(A^cB)}$, and $G_{(B^c)}$ the payoffs from the three bets on the condition that, respectively, event $A \cap B$, $A^cB$, or $B^c$ happens.
Then these are 3 linear equations in the three stakes.

They are linearly dependent iff

$$\begin{vmatrix}
1-r_{AB} & 1-r_{A|B} & 1-r_B \\
-r_{AB} & -r_{A|B} & 1-r_B \\
-r_{AB} & 0 & -r_B
\end{vmatrix} = 0$$

if and only if $r_{AB} = r_{A|B} \times r_B$. 
Problem set 1: On de Finetti’s game of making Book against an incoherent bookie.

Consider the partition $\Omega = \{1, 2, 3, 4, 5, 6\}$ formed by the outcome of rolling a six-sided die. The bookie is asked to give fair betting rates $r$ for the following collection of five events:

$$\mathcal{E} = \{ \{1\}, \{6\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \}.$$

Suppose the bookie gives fair betting rates for these events as follows:

$$\mathcal{R} = \{ r\{1\} = 1/6; \ r(\{3,6\}) = r\{6\} = 1/3; \ r(\{1,2,3\}) = r(\{1,2,4\}) = 1/2. \}$$

1.1 Using de Finetti’s coherence theorem, show that a Book cannot be made.

1.2 Next, the bookie is required to give fair betting rates also to the event $\{4,6\}$, making a total of six events for which the bookie posts fair betting rates. Suppose that the bookie posts the rate $r(\{4,6\}) = 2/5$.

Against these six betting rates, either show that a Book cannot be made, or else give a strategy that makes a Book against the bookie.
**Hint:** To make a *Book* against the bookie, you need to arrange bets so that the bookie is both selling a bet “low” and buying a bet “high” on the same event. To do this, consider the following two cases.

Let $A$ and $B$ be disjoint events, $A \cap B = \emptyset$, and let $C = A \cup B$.

1.2.1 If the bookie has posted fair odds $r_A$ and $r_B$ respectively on $A$ and $B$, construct a bet on $C$ at the fair odds $r_C = r_A + r_B$.

   **Note well that $C$ may not belong to the set $\mathcal{E}$.

1.2.2 If the bookie has posted fair odds $r_A$ and $r_C$ respectively on $A$ and $C$, construct a bet on $B$ at the fair odds $r_B = r_C - r_A$.

   Again, note well that $B$ may not belong to the set $\mathcal{E}$.
• De Finetti’s *Fundamental Theorem* (applied to sets of events).

Suppose coherent betting odds are given for each event $E$ in a set $\mathcal{E}$ defined with respect to some basic partition $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$.

Let $F$ be another event defined on $\Omega$ but not necessarily in $\mathcal{E}$.

Define:

\[ \mathcal{J} = \{ E \in \mathcal{E} : E \subseteq F \} \]
\[ \overline{\mathcal{J}} = \{ E \in \mathcal{E} : F \subseteq E \} \]

Let \[ P(F) = \sup_{E \in \mathcal{J}} P(E) \quad \text{and} \quad \overline{P}(F) = \inf_{E \in \overline{\mathcal{J}}} P(E) \]

• Then, the betting odds for $F$ that remain coherent with those already assigned to events in $\mathcal{E}$ are the values from $P(F)$ to $\overline{P}(F)$.

Outside this interval, the enlarged set of betting odd is incoherent.

*Note*: De Finetti’s coherence criterion does not require the rational agent to identify betting odds beyond those for which the Fundamental Theorem constrains them.
Specifically, the rational agent is not required by *coherence* to have probabilities defined on an algebra of events, let alone on a power-set of events. It is sufficient to have probabilities defined *as-needed* for the arbitrary set $\mathcal{E}$, as might arise in a particular decision problem.

- See, e.g., F. Lad, 1996 for interesting applications of this result.

**Problem 1.3** –
\[ \Omega = \{1, 2, 3, 4, 5, 6\} \] the outcome of rolling an ordinary die, as before.
\[ \mathcal{E} \] is the set of these four events
\[ \mathcal{E} = \{ \{1\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \} \]

Suppose YOU give betting odds for these four events that agree with the judgment that the die is “fair.”
\[ P(\{1\}) = 1/6; \quad P(\{3,6\}) = 1/3; \quad P(\{1,2,3\}) = P(\{1,2,4\}) = 1/2. \]

The *Fundamental Theorem* identifies those events, and the values for which precise betting odds are required by coherence.

- **What are the events that have coherent betting odds fixed by $\mathcal{E}$?**
Directions for playing the *Matter/Anti-matter Tetrahedron Game*
(after a similar game created by M.Stone, *circa* 1976)
This game requires a tetrahedral die, marked with faces:

- electron (e⁻);
- positron (e⁺);
- muon (μ⁺); and
- anti-muon (μ⁻).

The game is played by rolling the tetrahedron and *recording* each outcome – the downward showing face – according to this rule:

- The *record of outcomes* **contracts** whenever opposite particles **collide**. Otherwise the record **expands**.

For example, let the current *record* be the sequence:

...... μ⁺ e⁻ e⁻ μ⁻

and suppose the next roll is a *muon* (μ⁺). Then these collide and the *record contracts* to

...... μ⁺ e⁻ e⁻.

If the succeeding roll is a *positron* (e⁺), again the *record contracts* to

...... μ⁺ e⁻.

However, if the succeeding roll is, instead, an anti-muon (μ⁻) the *record expands* to

...... μ⁺ e⁻ e⁻ μ⁻

and, thus, returns to the state it was in two rolls previous.
In teams of 3, you will play the game by creating a record corresponding to about 10 minutes of (about 50) rolls: call this the 10 minute record.

- After these 10 minutes, you will perform one final roll.

Then you will each (individually) bet 50 cents on, or against, the proposition that:

The majority of the others’ final rolls were collisions with their 10 minute records.

Winners will share the pot equally.

Of course, we will have a market for buying and selling your bets, prior to settling-up. As with real markets, these exchanges will be driven by insider-information.
Additional Notes on Problem 1.3
The set of events with coherent betting odds fixed by events in $\mathcal{E}$ does not form an algebra. Only 22 of 64 events have precise previsions.
For instance, by the Fundamental Theorem,
\[
P\left(\{6\}\right) = 0 < \bar{P}\left(\{6\}\right) = 1/3;
\]
likewise
\[
P\left(\{4\}\right) = 0 < \bar{P}\left(\{4\}\right) = 1/3;
\]
however,
\[
P(\{4,6\}) = 1/3.
\]

• Moreover, the smallest algebra containing these 4 events is the power set of all 64 events on $\Omega$.

De Finetti’s Fundamental Theorem applies with called-off bets, given an event $F$.

The Fundamental Theorem, in its full generality for prices on (bounded) random variables $X$ uses this representation of payoffs
\[
I_F \beta [X - P_F(X)]
\]
where:  $I_F$ is the indicator function for the conditioning event $F$
$\beta$ is the generalized “stake,” with sign reflectging “buy” vs “sell”
and $P_F(X)$ is the decision maker’s “fair price” for buying/selling units of $X$. 

Session 2: *Expected Utility*

In our discussion of betting from Session 1, we required the bookie to accept (as *fair*) the combination of two gambles, when each gamble, on its own, is judged *fair*.

That is, the bookie posted “fair odds” for betting on events and, subject only to a concern about the *magnitude* of the stakes involved in the bets, we required the bookie to accept arbitrary combinations of fair bets.

But, is that always your attitude towards a combinations of two fair bets? For example, might you have a different view about an even money bet on a coin landing heads at $40 stake, compared with an even money $2 bet on the same event.

<table>
<thead>
<tr>
<th>Coin lands heads</th>
<th>Coin lands tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even-money $20 bet on heads</td>
<td>win $20</td>
</tr>
<tr>
<td>Even-money $1 bet on heads</td>
<td>win $1</td>
</tr>
</tbody>
</table>
The von Neumann – Morgenstern theory of cardinal utility (1944)

The goal is a theory of qualitative preference over options that admits a quantitative representation, one that captures a notion of strength of preference.

That is, the goal is a theory of cardinal (rather than merely ordinal) preference.

A simple lottery $L$ is probability distribution over a finite set of rewards
\[ \text{Rewards} = \{r_1, \ldots, r_n\}. \]

\[ \text{Write a lottery } L \text{ as a sequence} \quad < p_1, p_2, \ldots, p_n > \]
\[ \text{where} \quad p_j \geq 0 \quad \text{and} \quad \sum_j p_j = 1 \quad (j = 1, \ldots, n). \]

The quantity $p_j$ is the chance of winning reward $r_j$. 
Matrix of $m$-many lotteries on the set of $n$-many rewards.

<table>
<thead>
<tr>
<th></th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_j$</th>
<th>$r_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>$p_{11}$</td>
<td>$p_{12}$</td>
<td>$p_{1j}$</td>
<td>$p_{1n}$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$p_{21}$</td>
<td>$p_{22}$</td>
<td>$p_{2j}$</td>
<td>$p_{2n}$</td>
</tr>
<tr>
<td>$L_m$</td>
<td>$p_{m1}$</td>
<td>$p_{m2}$</td>
<td>$p_{mj}$</td>
<td>$p_{mn}$</td>
</tr>
</tbody>
</table>
Von Neumann - Morgenstern theory introduces one operation for the combination of two lotteries into a third lottery.

The convex combination of two lotteries is denoted by “⊕”.
Fix a quantity $x$, $0 \leq x \leq 1$.
\[ xL_1 \oplus (1-x)L_2 = L_3 \]
where
\[ p_{3j} = xp_{1j} + (1-x)p_{2j} \quad (j = 1, ..., n). \]

You may think of “⊕” as involving a compound chance where, first a coin (biased $x$ for "heads") is flipped and,

- if it lands heads then lottery $L_1$ is run
- if it lands tails then lottery $L_2$ is run.
Von Neumann - Morgenstern preference axioms.

The theory is given by 3 axioms for a binary preference relation over lotteries.

**Axiom-1** Preference \( \sim \) is a weak order:

- Preference \( \sim \) is reflexive \( \forall L \ L \sim L \)
- Preference \( \sim \) is transitive \( \forall \{L_1, L_2, L_3\} \text{ if } L_1 \sim L_2 \& L_2 \sim L_3 \text{ then } L_1 \sim L_3 \)

and lotteries are comparable under \( \sim \) \( \forall \{L_1, L_2\} \text{ either } L_1 \sim L_2 \text{ or } L_2 \sim L_1. \)

Define the asymmetric and symmetric parts of \( \sim \) in the usual way:

\[ L_1 \sim L_2 \text{ if } L_1 \sim L_2 \text{ but not } L_2 \not\sim L_1 \]

\[ L_1 \not\sim L_2 \text{ if } L_1 \not\sim L_2 \text{ and } L_2 \not\sim L_1 \]

**Axiom-2** Independence – \( \oplus \) with a common lottery doesn't affect preference

\[ \forall \{L_1, L_2, L_3\}, 0 < x \leq 1, \ L_1 \not\sim L_2 \text{ if and only if } xL_1 \oplus (1-x)L_3 \not\sim xL_2 \oplus (1-x)L_3. \]
Axiom-3 (Archimedes)  This is a technical condition to allow the use of real numbers to provide magnitudes for cardinal utilities.

If \( L_1 \sim L_2 \sim L_3 \),
then \( \exists \ 0 < x, y < 1 \), \( xL_1 \oplus (1-x)L_3 \sim L_2 \sim yL_1 \oplus (1-y)L_3 \).
Or then \( \exists \ 0 < z < 1 \), \( zL_1 \oplus (1-z)L_3 \sim L_2 \).

Von Neumann - Morgenstern Theorem

These three axioms are necessary and sufficient for the existence of a unique cardinal utility function \( U(\bullet) \) on rewards,

\[
U(r_j) = u_j \quad (j = 1, \ldots, n) \text{ such that:}
\]

\[
L_1 \sim L_2 \text{ if and only if } \sum_j p_{1j}u_j \leq \sum_j p_{2j}u_j
\]

This is the Expected Utility property of cardinal utilities.
U is unique up to a positive linear transformation. That is, for a > 0 and b an arbitrary real number, the utility U', defined by U' = aU + b is equivalent to U.

- Note: **Strength of preference** is captured by the fact that the quantity

\[
\frac{[U(L_1) - U(L_2)]}{[U(L_3) - U(L_4)]}
\]

is invariant over equivalent cardinal utility functions.

That is,

\[
\frac{[U'(L_1) - U'(L_2)]}{[U'(L_3) - U'(L_4)]}
\]

is the same quantity whenever U' = aU + b.
A geometric presentation of von Neumann-Morgenstern Cardinal Utility theory.

Assume that $r_1 \prec r_2 \prec r_3$. Relative to a lottery $L$, note the region of lotteries that must be strictly preferred to $L$ and the region of lotteries that must be strictly dispreferred to $L$.

- **Stochastic dominance**: Moving probability from less to more preferred rewards.
Line of indifference between $r_2$ and $L_2 = u_2r_3 \oplus (1-u_2)r_1$.

Note: If $r_2 \sim L_2$ then $r_2 \sim xr_2 \oplus (1-x)L_2$
Parallel lines of indifference

Note: Reasoning by similar triangles shows that indifference lines are parallel.
The Allais Problem

Choice set #1:

Option 1: Receive $3.5M (million dollars) for sure.

Option 2: Receive $0 with chance .01; receive $3.5M with chance .90, and receive $5M with chance .09.

Choice set #2:

Option 3: Receive $0 with chance .90 and $3.5M with chance .10.

Option 4: Receive $0 with chance .91 and $5M with chance .09.
If I’ve picked the dollar amounts right, you have an inclination to pick Option 1 rather than Option 2, and Option 4 rather than Option 3.

However, these choices cannot reflect “rational” (strict) preferences using a von Neumann-Morgenstern utility for money.

To see this, imagine that these options are generated with a fair 100-slot roulette wheel where you win the dollar amounts, below, according to which slot results on the spin of the wheel:

<table>
<thead>
<tr>
<th></th>
<th>#1-90</th>
<th>#91-99</th>
<th>#100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option 1</td>
<td>$3.5M</td>
<td>$3.5M</td>
<td>$3.5M</td>
</tr>
<tr>
<td>Option 2</td>
<td>$3.5M</td>
<td>$5 M</td>
<td>$0</td>
</tr>
<tr>
<td>Option 3</td>
<td>$0</td>
<td>$3.5M</td>
<td>$3.5M</td>
</tr>
<tr>
<td>Option 4</td>
<td>$0</td>
<td>$5 M</td>
<td>$0</td>
</tr>
</tbody>
</table>
Define these three lotteries on the 3 rewards \{\$0, \$3.5M, \$5M\}:

Lottery \(L_A\) by probs \{0, 1., 0\} for the 3 rewards \{\$0, \$3.5M, \$5M\}

Lottery \(L_B\) by probs \{.1, 0, .9\} for the 3 rewards \{\$0, \$3.5M, \$5M\}

Lottery \(L_C\) by probs \{1., 0, 0\} for the 3 rewards \{\$0, \$3.5M, \$5M\}.

Then,

Option 1  =  .9\(L_A\) \(\oplus\) .1\(L_A\)

Option 2  =  .9\(L_A\) \(\oplus\) .1\(L_B\)

And

Option 3  =  .9\(L_C\) \(\oplus\) .1\(L_A\)

Option 4  =  .9\(L_C\) \(\oplus\) .1\(L_B\)

So,  

Option 1 \(\sim\) Option 2 \hspace{5pt} if and only if \hspace{5pt} Option 3 \(\sim\) Option 4.
Class Problems Session 2.
These 3 problems deal with the framework of lotteries over 3 rewards, given on slide 8.

2.1 Regarding slide 9, argue that the coefficient $u_2$ in the lottery
$L_2 = u_2 r_3 \oplus (1 - u_2) r_1$ serves as the utility $U(r_2)$ for reward $r_2$.
What are the associated utilities for $r_1$ and $r_3$?

2.2 Graph the Allais Problem with the simplex for lotteries involving 3 rewards.

2.3 Using the simplex, formulate modifications of the von Neumann-Morgenstern Theory that will accommodate Allais-styled choices?
Session 3: Security based Reasoning

In yesterday’s first session, we discussed de Finetti’s approach to a personal probability, as an interpretation of coherent fair odds – odds immune to a book.

However, because the bookie is required to accept a finite combination of fair bets, with net payoff the sum of the individual payoffs, at least within a range of allowed stakes, those stakes behave as though outcomes can be added while preserving relative values of gains and losses. That is, de Finetti requires that the bookie’s fair odds are the same \((r = r')\) for each of these two bets, assuming that the stakes are within the allowed range for betting at all.

<table>
<thead>
<tr>
<th></th>
<th>Rain this week</th>
<th>No rain this week</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fair-odds $20 bet on rain @ (r)</td>
<td>win ((1-r)$20)</td>
<td>lose (r$20)</td>
</tr>
<tr>
<td>Fair odds $1 bet on rain @ (r'): (1-(r'))</td>
<td>win ((1-r')$1)</td>
<td>lose (r'$1)</td>
</tr>
</tbody>
</table>

Then, \(U($20 \text{ gain}) = 10 \times U($2 \text{ gain})\) and the stakes are in units of cardinal utiles.
However, unlike with von Neumann – Morngenstern lotteries, the chances for the outcomes of simple bets are not specified.

- Does it matter if the bookie is betting without knowing chances for outcomes?

Matrix of $m$-many acts on the partition of $n$-many uncertain states

$\begin{array}{cccc}
\omega_1 & \omega_2 & \omega_j & \omega_n \\
\hline
o_{11} & o_{12} & o_{1j} & o_{1n} \\
\hline
o_{21} & o_{22} & o_{2j} & o_{2n} \\
\hline
o_{i1} & o_{i2} & o_{ij} & o_{in} \\
\hline
o_{m1} & o_{m2} & o_{mj} & o_{mn} \\
\end{array}$
Subjective Expected Utility thesis: A decision maker chooses as-if she/he has a personal probability $P(\bullet)$ over states of uncertainty, and a cardinal utility $U(\bullet)$ over outcomes, and maximizes subjective expected utility.

Act$_1$ is dispreferred to Act$_2$ if and only if $\sum_j P(\omega_j) U(o_{1j}) \leq \sum_j P(\omega_j) U(o_{2j})$

Note: When acts and states are probabilistically independent, i.e.,

whenever $P(\omega_j) = P(\omega_j | Act_i) \quad i = 1, \ldots, m \quad j = 1, \ldots, n$

then strict dominance is a valid decision rule.

That is, when there is no moral hazard, and Act$_2$ strictly dominates Act$_1$,

then the Subjective Expected Utility of Act$_2$ is greater than of Act$_1$. 
Ellsberg’s (1961) Paradox for SEU theory.

We use only 2 rewards, $0 and $1,000 in the following decision problems.

Background: There is an urn containing 90 balls, one of which will be drawn at random, i.e., the probability is 1/90 of drawing a particular ball from the urn.

- 30 of the 90 balls are colored **RED**.
- Of the remaining 60 each is either **GREEN** or **BLUE**, with no restrictions.

Evaluate each of two pairs of options.

Act\(_1\): Receive $1,000 if the ball drawn is **RED**, $0 otherwise.
Act\(_2\): Receive $1,000 if the ball drawn is **GREEN**, $0 otherwise.

Act\(_3\): Receive $1000 if the ball drawn is **RED** or **BLUE**, $0 if **GREEN**.
Act\(_4\): Receive $1000 if the ball drawn is **GREEN** or **BLUE**, $0 if **RED**.
Consider the following graph of the expected utilities of these four acts as a function of $x = \text{personal probability of GREEN}$.
Note that, in terms of subjective expected utilities over different values of the unknown proportion of GREEN balls \(0 \leq x \leq 1\)

- Act\(_1\) maximizes minimum value \(1/3\) compared with Act\(_2\) \((0)\).
- and Act\(_4\) maximizes minimum value \(2/3\) compared with Act\(_3\) \((1/3)\).

**Principle of MaxiMin** in decisions under uncertainty:

Among the options available, choose that act whose minimum value is maximum.

So, in the Ellsberg Paradox problem, if we measure security in terms of expected utility with respect to different personal probabilities about the contents of the urn, then, by Maximin, the decision maker ranks Act\(_1\) over Act\(_2\) and Act\(_4\) over Act\(_3\).

Also, as with the Allais Paradox, we can see how these choices are in conflict with the first two postulates of von Neumann – Morgenstern’s theory.
<table>
<thead>
<tr>
<th></th>
<th>Red</th>
<th>Green</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act₁</td>
<td>$1,000</td>
<td>$0</td>
<td>$0</td>
</tr>
<tr>
<td>Act₂</td>
<td>$0</td>
<td>$1,000</td>
<td>$0</td>
</tr>
<tr>
<td>Act₃</td>
<td>$1,000</td>
<td>$0</td>
<td>$1,000</td>
</tr>
<tr>
<td>Act₄</td>
<td>$0</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
</tbody>
</table>

Note: The first pair of acts are identical with the second pair of acts for the first two columns, and differ solely in the common payoff for the third column.
With respect to our Axiomatic EU Theory,

Maximin

- Satisfies the “Ordering” postulate.
  This is because each option is assigned a real number on its own – its worst case value – and options are compared by these numbers.

But Maximin

- Violates the “Independence” postulate, as we see in the following example.

Consider a binary decision problem with three options:

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option$_f$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Option$_g$</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Option$_h$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Consider a decision problem on 2 states $\{\omega_1, \omega_2\}$, with 3 options $\{f, g, h\}$, and a convex set of probabilities $\mathcal{P} = \{ P : 0.25 \leq P(\omega_2) \leq 0.75 \}$.

Act $h$ is never Bayes but never beaten.
Each one of options \( f \) and \( g \) has a minimum values of 0.0, whereas option \( h \) has a minimum value of 0.4. So, by standards of *Maximin* decision making, option \( h \) is strictly preferred to options \( f \) and \( g \), which are indifferent to each other.

However, the (.5, .5) convex combination of \( f \) and \( g \), \( .5f \oplus .5g \) has minimum value .5, which makes it strictly preferred to \( h \), in violation of the “Independence” postulate.

A rival “security” based decision rule is to consider *Maximin Regret*, to which we turn next.
Matrix of $m$-acts on the partition of $n$-states, with outcomes in cardinal utiles

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_j$</th>
<th>$\omega_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act_1</td>
<td>$u_{11}$</td>
<td>$u_{12}$</td>
<td>$u_{ij}$</td>
<td>$u_{1n}$</td>
</tr>
<tr>
<td>Act_2</td>
<td>$u_{21}$</td>
<td>$u_{21}$</td>
<td>$u_{2i}$</td>
<td>$u_{2n}$</td>
</tr>
<tr>
<td>Act_i</td>
<td>$u_{i1}$</td>
<td>$u_{i2}$</td>
<td>$u_{ij}$</td>
<td>$u_{in}$</td>
</tr>
<tr>
<td>Act_m</td>
<td>$u_{m1}$</td>
<td>$u_{m2}$</td>
<td>$u_{mj}$</td>
<td>$u_{mn}$</td>
</tr>
</tbody>
</table>

Let $u_{ij}^* = \max_i \{u_{ij}\}$, which is the value of the best choice possible if state $\omega_j$ obtains.
Transform the decision matrix to its regret form by subtracting $u^*_{ij}$ from $u_{ij}$,

So, $u^*_{ij} = u_{ij} - u^*_{j}$

**REGRET Matrix**

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_j$</th>
<th>$\omega_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act_1</td>
<td>$u^*_{11}$</td>
<td>$u^*_{12}$</td>
<td>$u^*_{ij}$</td>
<td>$u^*_{1n}$</td>
</tr>
<tr>
<td>Act_2</td>
<td>$u^*_{21}$</td>
<td>$u^*_{22}$</td>
<td>$u^*_{2j}$</td>
<td>$u^*_{2n}$</td>
</tr>
<tr>
<td>Act_i</td>
<td>$u^*_{i1}$</td>
<td>$u^*_{i2}$</td>
<td>$u^*_{ij}$</td>
<td>$u^*_{in}$</td>
</tr>
<tr>
<td>Act_m</td>
<td>$u^*_{m1}$</td>
<td>$u^*_{m2}$</td>
<td>$u^*_{mj}$</td>
<td>$u^*_{mn}$</td>
</tr>
</tbody>
</table>
• *Note well:* If there are no moral hazards, the SEU ranking of options in a decision problem is the same as the SEU ranking of these same options in their REGRET MATRIX format.

The reason for this is because, without moral hazards, adding a constant to a column makes no difference to the differences in Subjective Expected Utilities for rival acts.

**HOWEVER,** *Maximin REGRET* is not generally the same as *Maximin,* as we see next.
Consider a binary decision problem with utility matrix:

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option$_1$</td>
<td>-20</td>
<td>0</td>
</tr>
<tr>
<td>Option$_2$</td>
<td>-3</td>
<td>3</td>
</tr>
<tr>
<td>Option$_3$</td>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>Option$_4$</td>
<td>0</td>
<td>-20</td>
</tr>
</tbody>
</table>

The Regret Matrix is identical, since the max is 0 in each state.

Then Maximin and Maximin Regret lead to the same choice:

Option$_2$ is the unique winner from the quadruple \{O$_1$, O$_2$, O$_3$, O$_4$\}
Consider the modified decision problem where, first, Option $1$ is removed.

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option $2$</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>Option $3$</td>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>Option $4$</td>
<td>0</td>
<td>-20</td>
</tr>
</tbody>
</table>

But now the Regret Matrix is not the same:

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option $2$</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>Option $3$</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>Option $4$</td>
<td>0</td>
<td>-17</td>
</tr>
</tbody>
</table>

*Maximin Regret* now recommends Option $3$ from the triple $\{O_2, O_3, O_4\}$, in violation of the Ordering postulate.