Notes for Session 7 – Basic Voting Theory and Arrow’s Theorem

We follow up the “Impossibility” (Session 6) of pooling expert probabilities, while preserving unanimities in both unconditional and conditional probabilities. Here we explore the power of group voting as a substitute for group pooling.

Assume that the group decision problem involves $m$-many, pairwise exclusive social acts, e.g., the options might be candidates in an election, or (exclusive) bills before a legislature. And there are $n$-many citizens, or voters, or legislators in our group.

Each voter $j$ ($j = 1, \ldots, n$) has an ORDINAL ranking of the $m$-options, as summarized in the table below.

The quantities $r_{\cdot j}$ are the ranks assigned by voter $j$ to the $m$-many acts.

We’ll let a rank of 1 be best, and a rank of $m$ be worst.
Ties are allowed by sharing the average rank for those tied in the ranking. For example, if two acts tie for best position they share rank 1.5 ( = (1+2) / 2), etc. Thus, for each voter, the sum of the ranks equals \( m(m+1)/2 \).

**TABLE of \( n \)-many Voters’ Rank Order of \( m \)-many acts**

<table>
<thead>
<tr>
<th></th>
<th>( V_1 )</th>
<th>( V_2 )</th>
<th>( V_j )</th>
<th>( V_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Act(_1)</strong></td>
<td>( r_{11} )</td>
<td>( r_{12} )</td>
<td>( r_{1j} )</td>
<td>( r_{1n} )</td>
</tr>
<tr>
<td><strong>Act(_2)</strong></td>
<td>( r_{21} )</td>
<td>( r_{22} )</td>
<td>( r_{2j} )</td>
<td>( r_{2n} )</td>
</tr>
<tr>
<td><strong>Act(_i)</strong></td>
<td>( r_{il} )</td>
<td>( r_{12} )</td>
<td>( r_{ij} )</td>
<td>( r_{in} )</td>
</tr>
<tr>
<td><strong>Act(_m)</strong></td>
<td>( r_{m1} )</td>
<td>( r_{m2} )</td>
<td>( r_{mj} )</td>
<td>( r_{mn} )</td>
</tr>
</tbody>
</table>
Iterative use of majority voting in pairwise comparisons without ties.

Let the majority rule in determining pairwise comparisons, with winners in earlier rounds facing off against each other in subsequent rounds. For majority voting we’ll assume there are no ties, which can be assured if there are an odd number of voters and all voters have only strict preferences.

That is, when Act\(_j\) is compared with Act\(_k\), the \(n\)-many voters express their preference for the one or the other, with a majority of expressed strict preferences determining the outcome of this head-to-head competition.

- If each voter has only strict preferences – no indifferences – and there is an odd number of voters, the \((\text{Condorcet})\) \textit{winner} of such an election is the one and only one social act* (if one exists) such that for each other social act\(_i\) \(\neq\) act*, a majority of the voters prefers act* over act\(_i\).
However, an interesting problem arises when there is no such (Condorcet) winner.

*Example:* Use 3 voters and 3 options. The table of strict preferences is as follows.

<table>
<thead>
<tr>
<th></th>
<th>Voter₁</th>
<th>Voter₂</th>
<th>Voter₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act₁</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Act₂</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Act₃</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note:
- a majority of voters prefers Act₁ over Act₂
- a majority prefers Act₂ over Act₃
- a majority prefers Act₃ over Act₁.

Iterative majority voting, here, does not generate a social preference ordering.
Class Exercise 1: In the Condorcet voting game, with these three voters and their preferences common knowledge, they face a sequence of two, pairwise votes, with the winner from the first round vote then pitted in the second round against the remaining act, which has a first round bye.

- Voting in each round is by simple majority: no abstaining, or tie-votes.

- Suppose that voter 3 is allowed to fix the agenda. Voter 3 gets to determine which two acts are voted on in the first round. Of course, the winner of that first round faces the remaining outcome in the second and final round of voting.

- Argue whether there is a Nash equilibrium strategy for the three players that leads uniquely to player 3’s favorite outcome, Act 2, being voted the winner of the game. You may eliminate weakly dominated strategies.
  - NOTE: A strategy for a voter must specify what to do under all logically possible circumstances, in all rounds of voting.

- Recall that players are free to engage in strategic voting – they may express any vote regardless their preferences – and player 3, in addition to voting, has control of the agenda.
Single Peaked Preferences and Spatial Voting

An important (1948) result, due to Duncan Black is this one.

Let acts be placed in one dimension, on a “line,” so that, with preference depicted by “height” above the line, each voter’s (ordinal) preferences are single-peaked. That is, each voter has a (unique) favorite position on the line and her/his preferences fall off monotonically from that favored position.

THEN Majority rule produces an ordering and, with an odd number of voters and no indifferences, there is a well defined Condorcet winner!

Class Exercise 2: Suppose that you are a candidate who may align with some position in the space of social acts. If there is an odd number of voters, voters’ preferences are single-peaked and no individual indifferences, where shall you position yourself? Find the position that is the Condorcet winner!

• Hint: Focus on the distribution of favored positions! Start with the game where you are one of 2 rival candidates and there is only 1 voter, then 3 voters, etc.
Borda Count Voting

A different voting scheme, not Majority Rule, is Borda Count voting. Here, each voter gives her/his rank order over the options/acts/candidates. The group’s choice is given by calculating the “sum of ranks” for each option, with the lowest sum winning.

Class Exercise 3: Construct a situation where Borda Count favors, say, act\(_3\) in a choice among 7 acts, but when act\(_7\) is deleted, then act\(_2\) wins!

Have we seen this phenomenon before?

• Outline a situation where, as a voter in a Borda count system, you will vote strategically, i.e., your voting behavior depends how you think others will vote and not merely on your own preferences.

• Is there a voting scheme that is immune to strategic voting?
Can we merge INDIVIDUAL PREFERENCES into a GROUP PREFERENCE and create a “core”?

ARROW'S “Impossibility” Theorem (1950)

USE www.jstor.com to obtain: A Difficulty in the Concept of Social Welfare, Kenneth J. Arrow

Consider a (finite) set of $m$-many SOCIAL ACTS $A = \{ A_1, \ldots, A_m \} \ (m \geq 3)$, and $n$-many INDIVIDUAL PREFERENCES over these acts $\{ \preceq_1, \ldots, \preceq_n \} \ (n \geq 2)$,

A PREFERENCE, $\preceq$, is a weak ordering of the set $A$.

That is, $\preceq$ is a REFLEXIVE, TRANSITIVE, and COMPLETE binary relation over the set of social acts.

Arrow’s Theorem:
There does not exist a rule for creating a GROUP PREFERENCE, $\preceq_G$ that satisfies the following 4 conditions:
(C-1) The rule applies with ARBITRARY sets of ACTS and PREFERENCES.

(C-2) The rule obeys the (WEAK) PARETO AXIOM:

If $A_1 \preceq_j A_2$ (for each $j$), then $A_1 \preceq_G A_2$
and, if $A_1 \prec_j A_2$ (for each $j$), then $A_1 \prec_G A_2$

That is, when each person (strictly) prefers $A_2$ to $A_1$, the group does too.

(C-3) A DICTATOR is not permitted.

(C-4) The GROUP'S preference relation, $\preceq_G$, over a particular subset $A'$ of social acts,
e.g., $\preceq_G$ applied to the odd-numbered social acts,
depends solely on the INDIVIDUALS’ PREFERENCES, $\prec_j$ for the acts in $A'$.

Condition (C-4) is called, INDEPENDENCE OF IRRELEVANT ALTERNATIVES.
Proof sketch – Arrow’s theorem admits rather different proofs. Here, I use ideas that connect with coalition theory, following Arrow’s own reasoning. A concise summary is found in Luce and Raiffa’s (1957) *Games and Decisions*, pp. 339-340.

**Definition:** A coalition \( V \) of individuals is *decisive* for the group’s strict preference of \( A_2 \) over \( A_1 \) when, if \( A_1 \prec_j A_2 \) (for each \( j \) in \( V \)) then \( A_1 \prec_G A_2 \).

Since (by Pareto) the Grand Coalition is decisive for each pair of acts, there is always a smallest non-empty coalition that is decisive for one act over another.

So, let \( V \) be decisive for some pair, e.g., for \( A_2 \) over \( A_1 \), while no proper subset of \( V \) is decisive for any pair of acts – just delete members from the Grand Coalition until the set is not decisive. The following reasoning reduces \( V \) to a single member \( \{j\} \)!

Let \( j \) be one of \( V \)’s members and consider the following strict preference profiles.

\[
\begin{array}{ccc}
\{j\} & V-\{j\} & W \\
A_2 & A_3 & A_1 \\
A_1 & A_2 & A_3 \\
A_3 & A_1 & A_2 \\
\end{array}
\]

Thus \( A_1 \prec_G A_2 \), since \( V \) is decisive on this issue. But \( A_3 \prec_G A_1 \) because \( V-\{j\} \) is not decisive for \( A_3 \) over \( A_1 \). Therefore, by transitivity of preference, \( A_3 \prec_G A_2 \).
But since only the individual j strictly prefers A₂ over A₃, while everyone else strictly prefers A₃ over A₂, {} is decisive for A₂ over A₃ – See the “exercise” at the end.

Thus V = {j}!

Likewise, {j} is decisive for A₂ over any other option different from A₂!

What remains in order to establish that {j} is the dictator is to show that {j} also is decisive for arbitrary A₄ over an arbitrary A₃.

• Consider first whether neither of these two acts is A₂.

{J}  N-{J} (= Everybody else, not j.)
A₄    A₃
A₂    A₄
A₃    A₂

By Pareto, A₂ <₆ A₄.
And  A₃ <₆ A₂, since {j} is decisive for A₂ over any other option.
Hence, A₃ <₆ A₄ by transitivity, which makes {j} decisive for A₄ over A₃!
• For the remaining case, to show that \{j\} is decisive for arbitrary A_4 over A_2, consider this profile.

\[
\begin{array}{cc}
\{j\} & N-\{j\} \ (= \text{Everybody else, not } j.) \\
A_4 & A_3 \\
A_3 & A_2 \\
A_2 & A_4 \\
\end{array}
\]

Then \[ A_3 \simless G A_4, \] since \{j\} is decisive for such pairs.

And \[ A_2 \simless G A_3, \] by Pareto,

Hence, \[ A_2 \simless G A_4 \] by transitivity, making \{j\} decisive for A_4 over A_2 too!

The remaining “exercise,” is to show that when U is a coalition and V = N-U is the complementary coalition:

If \[ A_1 \simless_U A_2 \] and \[ A_2 \simless_V A_1 \] and \[ A_1 \simless G A_2, \]

Then U is decisive for A_2 over A_1.