Session 1:  *Dominance and de Finetti’s “Book” argument*

Partition circumstances with a finite set of

*pairwise exclusive* and *mutually exhaustive* situations.

A partition with \( n \)-states \( \{\text{state}_1, \text{state}_2, \ldots, \text{state}_n\} \) is written as:

\[
\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}.
\]

Suppose that YOU, the decision maker, can compare two acts, state by state, according to the desirability of their *outcomes*, \( o_{ij} \).

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \cdots & \omega_k & \cdots & \omega_n \\
A_{ct1} & o_{11} & o_{12} & \cdots & o_{1k} & \cdots & o_{1n} \\
A_{ct2} & o_{21} & o_{22} & \cdots & o_{2k} & \cdots & o_{2n} \\
\end{array}
\]

**Strict dominance**

- If YOU judge each outcome \( o_{1j} \) is strictly preferable to the outcome \( o_{2j} \) \( (j = 1, \ldots, n) \),

then you strictly prefer \( A_{ct1} \) over \( A_{ct2} \) in a pairwise choice between them.
Example 1: Suppose that you prefer more money to less. Consider the binary state decision problem where the payoffs are:

<table>
<thead>
<tr>
<th></th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act_1</td>
<td>$300</td>
<td>$100</td>
</tr>
<tr>
<td>Act_2</td>
<td>$400</td>
<td>$200</td>
</tr>
</tbody>
</table>

So, Act_2 strictly dominates Act_1.

• Might it be reasonable, nonetheless, to prefer Act_1 over Act_2? For instance, what if Act_i brings about state \( \omega_i \)? What do you choose then?

This is an instance of what is called in the insurance business “Moral Hazard.” [See the 2005 New Yorker article by Gladwell.]

Until otherwise noted, we will assume there are no moral hazards.
Example 2 (circa 1931): B. de Finetti’s betting rates and *Books*.

A bet on/against the event $E$, at odds of $r:(1-r)$, with the “pot” in a winner-take-all wager equal to the combined stakes $S > 0$ (say, bets are in $\$ units), is specified by its payoffs, as follows.

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$E^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>bet on $E$</strong></td>
<td>(win) $(1-r)S$</td>
<td>(lose) $-rS$</td>
</tr>
<tr>
<td><strong>bet against $E$</strong></td>
<td>(lose) $-(1-r)S$</td>
<td>(win) $rS$</td>
</tr>
<tr>
<td><strong>abstain</strong></td>
<td>status quo</td>
<td>status quo</td>
</tr>
</tbody>
</table>

- By permitting what are formally negative stakes, $S < 0$, we can reverse *betting on* and *betting against*, since betting is a zero-sum game.
• We assume that the *status quo* (the consequence of *abstaining*) represents no net change in wealth. It is depicted by a 0 payoff in the units of the stake.

A collection \( \mathcal{E} = \{E_1, \ldots, E_n\} \)
of well defined events is presented to YOU (the *bookie*) and you are required to post a collection of *fair odds* \( \mathcal{R} = \{r_1, \ldots, r_n\} \)
subject to the condition that YOU are willing to accept finite combinations of bets on, or against, events in \( \mathcal{E} \) at the corresponding rates, assuming that your shares in the combined stakes \( \mathcal{S} = \{S_1, \ldots, S_n\} \)
remain within YOUR financial means. Call them YOUR *allowed stakes*.

**Definition:** Given YOUR posted fair odds, \( \mathcal{R} \), if there is a selection of allowed stakes such that YOUR combined bets result in a *sure loss* to YOU no matter what logically possible combination of events in \( \mathcal{E} \) occurs, then you are in *book*. 
• Note: Abstaining, i.e. not betting at all, strictly dominates being in a book!

What strategies available to YOU, the bookie, preclude a book against YOU?
Book Theorem (aka “Dutch Book” theorem) due to de Finetti (also see F.P.Ramsey):
• YOUR fair odds $\mathcal{R}$ are mathematical probabilities for the events in $\mathcal{E}$ –
  that is, YOUR odds satisfy the (three) axioms of probability,
  \textit{if and only if} no book is possible against $\mathcal{R}$.

Defn: When fair odds are immune to book, they are called \textit{coherent}. Otherwise, they
  are called \textit{incoherent} odds.

\textbf{Example 2:} Suppose that with respect to a binary partition $\{E_1, E_2 (= E_1^c)\}$, the bookie
  posts fair odds of $\{r_1 = .4$ and $r_2 = .7\}$, and the bookie has a total of $10$ to wager.
  • Are these odds coherent?
  • If not, what book(s) can be made against the bookie?
Next, we review an elementary account of mathematical probability, based on important work by A.N. Kolmogorov (1933).

A mathematical probability function $P$ assigns probability numbers to events, where an event $E$ is a set. Specifically each event $E$ is a subset of the sure event $\Omega$.

Kolmogorov's (1933) theory of probability requires that, for events $E$ and $F$

- Axiom 1 $0 \leq P(E) \leq 1$.
- Axiom 2 $P(\Omega) = 1$.
- Axiom 3 If $E \cap F = \emptyset$, then $P(E) + P(F) = P(E \cup F)$.

*Proof* that if your fair odds are coherent, they are probabilities – by the axioms!
(Axiom 1) \( 0 \leq P(E) \leq 1 \).

Suppose, to the deny the conclusion, your fair odds \( r \) for some event \( E \) are greater than 1. Then for an allowed stake \( S > 0 \) YOU will accept this "bet" on \( E \).

\[
\begin{array}{ccc}
E & E^c \\
\text{bet on } E & (1-r)S & -rS \\
\end{array}
\]

Then you lose regardless which event occurs, as each payoff is negative.

Likewise, if YOUR fair odds \( r \) on \( E \) are negative, you'll "bet" against \( E \) at allowed stake \( S > 0 \).

\[
\begin{array}{ccc}
E & E^c \\
\text{bet against } E & -(1-r)S & rS \\
\end{array}
\]

And you lose, regardless, as each payoff is negative.
(Axiom 2) \( P(\Omega) = 1 \) Using what we just showed, assume that \( 0 \leq r_\Omega \leq 1 \).

If \( r_\Omega < 1 \), then YOU find it fair to bet against \( \Omega \) at \( r < 1 \) with allowed stake \( S > 0 \).

\[
\begin{align*}
\Omega & \quad \phi = \Omega^c \\
\text{bet against } \Omega & \quad -(1-r)S
\end{align*}
\]

But then YOU lose \((1-r)S\) for sure, since \( \Omega \) is certain!

(Axiom 3) If \( A \cap B = \emptyset \), then \( P(A) + P(B) = P(A \cup B) \).

Consider 3 fair bets on \( A, B, C = (A \cup B) \),

at rates \( r_A, r_B & r_C \)

with stakes \( S_A, S_B & S_C \).

Then the simultaneous gains from these are:

\[
G_{(A&B)} = S_A + S_B - (r_AS_A + r_BS_B + r_CS_C)
\]

or \[
G_{(-A&B)} = S_B + S_C - (r_AS_A + r_BS_B + r_CS_C)
\]

or \[
G_{(-A&B)} = - (r_AS_A + r_BS_B + r_CS_C).
\]
Given the rates, these are three linear equations in the three variables $S_A$, $S_B$ and $S_C$. If the equations are linearly independent, then we may solve the rates for whatever (negative) desired gains we seek.

Hence, for the rates to be coherent, the equations must be linearly dependent, i.e. the following must obtain:

$$\begin{vmatrix} 1-r_A & -r_B & 1-r_C \\ -r_A & 1-r_B & 1-r_C \\ -r_A & -r_B & -r_C \end{vmatrix} = 0$$

Solving the determinate yields:

$$r_A + r_B = r_C$$

as required for the third axiom.
This result can be extended to conditional probability,

\[ P(A | B) \]

the conditional probability for A, given B is analyzed using "called-off" bets. The called-off bet on A (given B), results in status quo if B fails to occur.

A called-off bet on/against event A, given B, at odds of r:(1-r) with total stake S (S > 0) is specified by its payoffs, as follows.

<table>
<thead>
<tr>
<th></th>
<th>AB</th>
<th>A^cB</th>
<th>B^c</th>
</tr>
</thead>
<tbody>
<tr>
<td>on A</td>
<td>(1-r)S</td>
<td>-rS</td>
<td>0</td>
</tr>
<tr>
<td>against A</td>
<td>-(1-r)S</td>
<td>rS</td>
<td>0</td>
</tr>
</tbody>
</table>

Then coherent betting, including "called-off" bets, entails

\[
\text{Axiom 4: } P(A|B) \times P(B) = P(A \cap B).
\]

The proof follows a similar argument for Axiom 3:
Let $r_{AB}$ be the fair odds on $A \cap B$.
Let $r_{A|B}$ be the fair (called-off) odds on $A$, given $B$.
Let $r_B$ be the fair odds on $B$.

Assume all three bets are placed with stakes of $S_{AB}$, $S_B$, and $S_{A|B}$.

Denote by $G_{(AB)}$, $G_{(A^cB)}$, and $G_{(B^c)}$ the payoffs from the three bets on the condition that, respectively, event $A \cap B$, $A^cB$, or $B^c$ happens.

Then these are 3 linear equations in the three stakes.

They are linearly dependent iff

\[
\begin{vmatrix}
1-r_{AB} & 1-r_{A|B} & 1-r_B \\
-r_{AB} & -r_{A|B} & 1-r_B \\
-r_{AB} & 0 & -r_B \\
\end{vmatrix} = 0
\]

if and only if $r_{AB} = r_{A|B} \times r_B$. 

Problem set 1: On de Finetti’s game of making Book against an incoherent bookie.

Consider the partition \( \Omega = \{1, 2, 3, 4, 5, 6\} \) formed by the outcome of rolling a six-sided die. The bookie is asked to give fair betting rates \( r \) for the following collection of five events:

\[ \mathcal{E} = \{ \{1\}, \{6\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \}. \]

Suppose the bookie gives fair betting rates for these events as follows:

\[ \mathcal{R} = \{ r\{1\} = 1/6; \quad r(\{3,6\}) = r\{6\} = 1/3; \quad r(\{1,2,3\}) = r(\{1,2,4\}) \} = 1/2. \]

1.1 Using de Finetti’s coherence theorem, show that a Book cannot be made.

1.2 Next, the bookie is required to give fair betting rates also to the event \( \{4,6\} \), making a total of six events for which the bookie posts fair betting rates. Suppose that the bookie posts the rate \( r\{4,6\} = 2/5 \).

Against these six betting rates, either show that a Book cannot be made, or else give a strategy that makes a Book against the bookie.
**Hint:** To make a *Book* against the bookie, you need to arrange bets so that the bookie is both selling a bet “low” and buying a bet “high” on the same event. To do this, consider the following two cases.

Let $A$ and $B$ be disjoint events, $A \cap B = \emptyset$, and let $C = A \cup B$.

1.2.1 If the bookie has posted fair odds $r_A$ and $r_B$ respectively on $A$ and $B$, construct a bet on $C$ at the fair odds $r_C = r_A + r_B$.

   **Note well that $C$ may not belong to the set $\mathcal{E}$**.

1.2.2 If the bookie has posted fair odds $r_A$ and $r_C$ respectively on $A$ and $C$, construct a bet on $B$ at the fair odds $r_B = r_C - r_A$.

   Again, note well that $B$ may not belong to the set $\mathcal{E}$.
• De Finetti’s *Fundamental Theorem* (applied to sets of events).

Suppose coherent betting odds are given for each event $E$ in a set $\mathcal{E}$ defined with respect to some basic partition $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$.

Let $F$ be another event defined on $\Omega$ but not necessarily in $\mathcal{E}$.

Define: $\mathcal{Z} = \{E \in \mathcal{E}: E \subseteq F\}$  
$\mathcal{F} = \{E \in \mathcal{E}: F \subseteq E\}$

Let $P(F) = \sup_{E \in \mathcal{Z}} P(E)$ and $\bar{P}(F) = \inf_{E \in \mathcal{F}} P(E)$

• Then, the betting odds for $F$ that remain coherent with those already assigned to events in $\mathcal{E}$ are the values from $P(F)$ to $\bar{P}(F)$.

Outside this interval, the enlarged set of betting odd is incoherent.

*Note*: De Finetti’s coherence criterion does *not* require the rational agent to identify betting odds beyond those for which the Fundamental Theorem constrains them.
Specifically, the rational agent is not required by *coherence* to have probabilities defined on an algebra of events, let alone on a power-set of events. It is sufficient to have probabilities defined *as-needed* for the arbitrary set $\mathcal{E}$, as might arise in a particular decision problem.

- See, e.g., F. Lad, 1996 for interesting applications of this result.

*Problem 1.3* –

$\Omega = \{1, 2, 3, 4, 5, 6\}$ the outcome of rolling an ordinary die, as before.

$\mathcal{E}$ is the set of these four events

$\mathcal{E} = \{ \{1\}, \{3,6\}, \{1,2,3\}, \{1,2,4\} \}$

Suppose YOU give betting odds for these four events that agree with the judgment that the die is “fair.”

$P(\{1\}) = 1/6; \ P(\{3,6\}) = 1/3; \ P(\{1,2,3\}) = P(\{1,2,4\}) = 1/2.$

The *Fundamental Theorem* identifies those events, and the values for which precise betting odds are required by coherence.

- What are the events that have coherent betting odds fixed by $\mathcal{E}$?
Directions for playing the *Matter/Anti-matter Tetrahedron Game*  
(after a similar game created by M. Stone, *circa* 1976)

This game requires a tetrahedral die, marked with faces:

- electron ($e^-$);
- positron ($e^+$);
- muon ($\mu^+$); and
- anti-muon ($\mu^-$).

The game is played by rolling the tetrahedron and recording each outcome – the downward showing face – according to this rule:

- **The record of outcomes *contracts* whenever opposite particles *collide*. Otherwise the record *expands*.**

For example, let the current *record* be the sequence:

```
..... $\mu^+$ e- e- $\mu^-$
```

and suppose the next roll is a *muon* ($\mu^+$). Then these collide and the *record contracts* to

```
..... $\mu^+$ e- e-.
```

If the succeeding roll is a *positron* ($e^+$), again the *record contracts* to

```
..... $\mu^+$ e-.
```

However, if the succeeding roll is, instead, an anti-muon ($\mu^-$) the *record expands* to

```
..... $\mu^+$ e- e- $\mu^-$
```

and, thus, returns to the state it was in two rolls previous.
In teams of 3, you will play the game by creating a record corresponding to about 10 minutes of (about 50) rolls: call this the **10 minute record**.

- After these 10 minutes, you will perform one **final roll**.

Then you will each (individually) bet 50 cents on, or against, the proposition that:

The *majority* of the others’ **final rolls** were **collisions** with their **10 minute records**.

**Winners will share the pot equally.**

Of course, we will have a market for buying and selling your bets, prior to settling-up. As with real markets, these exchanges will be driven by insider-information.
Additional Notes on Problem 1.3

The set of events with coherent betting odds fixed by events in $\mathcal{E}$ does not form an algebra. Only 22 of 64 events have precise previsions. For instance, by the Fundamental Theorem,

$$\underline{P}(\{6\}) = 0 < \overline{P}(\{6\}) = 1/3;$$

likewise

$$\underline{P}(\{4\}) = 0 < \overline{P}(\{4\}) = 1/3;$$

however,

$$P(\{4,6\}) = 1/3.$$ 

• Moreover, the smallest algebra containing these 4 events is the power set of all 64 events on $\Omega$.

De Finetti’s *Fundamental Theorem* applies with called-off bets, given an event $F$.

The *Fundamental Theorem*, in its full generality for prices on (bounded) random variables $X$ uses this representation of payoffs

$$I_F \beta [X - P_F(X)]$$

where: $I_F$ is the indicator function for the conditioning event $F$

$\beta$ is the generalized “stake,” with sign reflecting “buy” vs “sell”

and $P_F(X)$ is the decision maker’s “fair price” for buying/selling units of $X$. 