Formal Epistemology: Lecture Notes

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Logical preliminaries

Let L_0 be a language containing a complete set of Boolean connectives, including *falsum* and *verum* constants $\perp$ and $\top$. The set of wff of L_0 are defined in the usual manner and $K, J, ...$denote sets of sentences. Let Cn be a function from sets of sentences to set of sentences, obeying the following Tarskian postulates:

Inclusion $A \in Cn(\{A\})$

Iteration $Cn(K) = Cn(Cn(K))$

Monotony If $K \subseteq H$, then $Cn(K) \subseteq Cn(H)$

Compactness For all X $\subseteq L_0$, Cn(X) $=\cup\{$C(Y): Y is a finite subset of X$\}$

Consistency $\perp\notin$ Cn($\emptyset$)

Deduction $A\rightarrow B\in Cn(K)$ iff B $\in Cn(K\cup\{A\})$.

We suppose also that $Cn(\emptyset)$ contains all substitution instances of classical tautologies expressible in L_0. A *theory* is any set of sentences K such that $Cn(K)=K$.

Consistency and model sets

(C.$\neg$) If $A \in M$, then $\neg A \notin M$.

(C.$\wedge$) If $A \wedge B \in M$, then $A \in M$ and $B \in M$.

(C.$\vee$) If $A \vee B \in M$, then $A \in M$ or $B \in M$.

(C. $\neg\neg$) If $\neg\neg A \in M$, then $A \in M$.

(C.$\neg\wedge$) If $\neg(A \wedge B) \in M$, then $\neg A \in M$ or $\neg B \in M$.

(C.$\neg\vee$) If $\neg(A \vee B) \in M$, then $\neg A \in M$ and $\neg B \in M$.

Consider a language with two atoms p, q. Is $Cn(p \wedge q)$ a model set for this language? What about $Cn(p \vee q)$ and $Cn(p)$?

The finite set $\{p \vee q\}$ is not a model set (why?). What about $\{p\}$? And $\{\neg p \vee q, \neg\neg p, p\}$? Hintikka argued that model sets are 'a very good formal counterpart of the informal idea of (partial) description of a possible state of affaires or a possible world.' Are model sets complete linguistic descriptions of possible worlds in the traditional sense of the term?

Consider the following rules, where μ is a model set:

If $A \in \mu$, then if $\vdash A \leftrightarrow B$, then $B \in \mu$

If $A \in \mu$, then if $\vdash A \rightarrow B$, then $B \in \mu$

are these rules satisfied by model sets?

[...] how the properties of a model sets are affected by the presence of the notions of knowledge and belief; how, in other words, the notion of model set can be generalized in such a way that the consistency of a set of statements remains tantamount to its capacity of being imbedded in a model set. What additional conditions are needed when the notions of knowledge and belief are present.

Clearly the content of [A is possible at some state of affaires w] cannot be adequately expressed by speaking only of the state of one state of affaires. The statement in question can be true only if there is a possible state of affaires where in which A would be true: but this state of affaires need not be identical with the one where the statement was made. A description of such state of affaires will be called an *alternative* to w, with respect to the agent *a*. Hence we have to impose the following conditions on the description of a model set w:

If $\Diamond A \in w$ then there is at least an alternative $w*$ to w such that $A \in w*$

If $\Box A \in w$ then for all alternatives $w*$ to w we have that $\Box A \in w*$

Let Ω be a *model system* understood as a set of model sets. Then we have:

If $\Diamond A \in w$ and w in a model system Ω, then there is at least an alternative $w*$ to w such that $A \in w*$

From now on we will tacitly assume we are working with model systems. We also need: If $\Box A \in w$ then $A \in w$. The previous account entails:

If $\Box A \in w$ then for all alternatives $w*$ to w we have that $A \in w*$

Hintikka's main goal is to develop a new notion of consistency. A set is consistent in an extended sense if it can be embedded in a model set. Since the defined modality is closed under entailment (why?) Hintikka proposes to understand it as *immunity to certain kinds of criticism* and baptizes it as *defensibility*. Valid sentences are re-baptized as *self-sustaining*.

So far we have talked about partial descriptions of states of affaires via model sets. We can now introduce formally a set of possible worlds W. Even when these possible worlds are primitives in the construction their description will require maximal and consistent sets. Then we can define a truth set for a sentence A as the set of states where A is true. We will denote this truth set as $|A|$. The relation between model sets and possible worlds will be discussed below.

Relational semantics for knowledge

A **relational frame** is a pair $\langle W, R \rangle$ where R is an alternativeness relation on W (i.e., $R \subseteq W \times W$). A **relational model** based on a frame $\mathbb{F}$ is a pair $\langle \mathbb{F}, V \rangle$ where $V : \mathsf{At} \to 2^W$ is a valuation function. Formulas from $\mathcal{L}$ are interpreted at states from W.

Formally, truth in a relational model $\mathbb{M} = \langle KW, R, V \rangle$ is defined inductively as follows. Let $w \in W$ and $\phi \in \mathcal{L}$,

1. $\mathbb{M}, w \models p$ iff $w \in V(p)$

2. $\mathbb{M}, w \models \neg\phi$ iff $\mathbb{M}, w \not\models \phi$

3. $\mathbb{M}, w \models \phi \wedge \psi$ iff $\mathbb{M}, w \models \phi$ and $\mathbb{M}, w \models \psi$

4. $\mathbb{M}, w \models \Box\phi$ iff for each $v \in W$, if wRv then $\mathbb{M}, v \models \phi$

5. $\mathbb{M}, w \models \Diamond\phi$ iff there is a $v \in W$ such that wRv and $\mathbb{M}, v \models \phi$

Scott-Montague-style semantics

Definition 1 *A pair* $\langle W, E\rangle$ *is a called a* **neighborhood system**, *or a* **epistemic frame**, *if* W *a non-empty set and* E *is an epistemic function.*

Definition 2 *Let* $\mathbb{F} = \langle W, N\rangle$ *be an epistemic frame. A* **model** *based on* $\mathbb{F}$ *is a tuple* $\langle W, N, V\rangle$ *where* $KV : \mathsf{At} \to 2^W$ *is a valuation function.*

Let $\mathbb{M} = \langle W, E, V \rangle$ be a model and $w \in W$.

- $\mathbb{M}, w \models p$ iff $w \in V(p)$
- $\mathbb{M}, w \models \neg\phi$ iff $\mathbb{M}, w \not\models \phi$
- $\mathbb{M}, w \models \phi \wedge \psi$ iff $\mathbb{M}, w \models \phi$ and $\mathbb{M}, w \models \psi$
- $\mathbb{M}, w \models \Box\phi$ iff $(\phi)^{\mathbb{M}} \in N(w)$
- $\mathbb{M}, w \models \Diamond\phi$ iff $W - (\phi)^{\mathbb{M}} \not\in E(w)$

Non-valid axiom schemes.

1. $\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$

2. $\Box\top$

3. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

4. From $\phi \rightarrow \psi$ infer $\Box\phi \rightarrow \Box\psi$

5. $\Box\phi \rightarrow \phi$

HIntikka and possible worlds semantics

Let $\mathbb{M} = \langle S, W, R, l, V \rangle$ be a Hintikka model, where S is a state of affaires, W a set of possible worlds, R a relation on S and L a labeling function $l : S \rightarrow 2^W$ such that $w \in l(s)$ if and only if for all sentences $A \in S$, $w \models A$.

Define now: $s \models^* A$ if and only if for all w, such that $w \in l(s), w \models A$

$s \models^* \Box A$ if and only if for all s' such that sRs', $s' \models^* A$.

How we can construct an epistemic function for this relational model? Which are the main differences between this model and the straight relational models presented above?

1. We say $\mathcal{F}$ is **closed under intersections** if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\bigcap_{i\in I} X_i \in \mathcal{F}$. For any cardinal κ, we say that $\mathcal{F}$ is **closed under $\leq \kappa$-intersections** if for each collections of sets $\{X_i\}_{i\in I}$ from $\mathcal{F}$ with $|I| \leq \kappa$, $\cap_{i\in I} X_i \in \mathcal{F}$.

2. We say $\mathcal{F}$ is **closed under unions** if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i\in I} X_i \in \mathcal{F}$. For any cardinal κ, we say that $\mathcal{F}$

is **closed under $\leq \kappa$-unions** if for each collections of sets $\{X_i\}_{i\in I}$ from $\mathcal{F}$ with $|I| \leq \kappa$, $\cup_{i\in I} X_i \in \mathcal{F}$.

3. We say that $\mathcal{F}$ is **closed under complements** if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.

4. We say $\mathcal{F}$ is **supplemented**, or **closed under supersets** provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

5. We say $\mathcal{F}$ **contains the unit** provided $W \in \mathcal{F}$; and $\mathcal{F}$ **contains the empty set** if $\emptyset \in \mathcal{F}$.

6. Call the set $\bigcap_{X\in\mathcal{F}} X$ the **core of** $\mathcal{F}$. We say that $\mathcal{F}$ **contains its core** provided $\cap_{X\in\mathcal{F}} X \in \mathcal{F}$.

7. We say $\mathcal{F}$ is **consistent** if $\emptyset \notin \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.

Definition 3 *1. $\mathcal{F}$ is a* **filter** *if $\mathcal{F}$ contains the unit, closed under finite intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.*

2. $\mathcal{F}$ is a **topology** *if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.*

3. $\mathcal{F}$ is **augmented** *if $\mathcal{F}$ contains its core and is supplemented.*

Lemma 4 *If $\mathcal{F}$ is augmented, then $\mathcal{F}$ is closed under arbitrary intersections.*

Of course, the converse is false.

Lemma 5 *There are consistent filters that are not augmented.*

It should be now clear that in our previous analysis we focused only on *augmented* epistemic frames. In fact, it is easy to see that:

Lemma 6 *If $\mathcal{F}$ is augmented, then $X \in \mathcal{F}$ if and only if $\bigcap \mathcal{F} \subseteq X$.*

Definition 7 *Suppose that* $R \subseteq W \times W$ *is a relation. The pair* $\langle W, R\rangle$ *is called a* **relational frame**, *or a* **relational structure**.

Given a relation R on a set W (i.e., $R \subseteq W \times W$) we can define the following functions:

1. $R^{\rightarrow} : W \rightarrow 2^W$ defined as follows. For each $w \in W$, let $R^{\rightarrow}(w) = \{v \mid wRv\}$.

2. $R^{\leftarrow} : 2^W \rightarrow 2^W$ defined as follows. For each $X \subseteq W$, $R^{\leftarrow}(X) = \{w \mid \exists v \in X \text{ such that } wRv$

Definition 8 *Given a relation R on a set W and a state $w \in W$ a set $X \subseteq W$ is* **known at** *w if $R^{\rightarrow}(w) \subseteq X$. Let $\mathcal{N}_w$ be the set of sets that are known at w. That is,*

$$\mathcal{N}_w = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

Lemma 9 *Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w$ is augmented.*

Observation 10 *Let W be a set and $R \subseteq W \times W$.*

1. *If R is reflexive, then for each $w \in W$, $w \in \bigcap \mathcal{N}_w$*
2. *If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.*

Proof Suppose that R is reflexive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. Then since R is reflexive, wRw and hence $w \in R^{\rightarrow}(w)$. Therefore by the definition of $\mathcal{N}_w$, $w \in X$. Since X was an arbitrary element of $\mathcal{N}_w$, $w \in X$ for each $X \in \mathcal{N}_w$. Hence $w \in \bigcap \mathcal{N}_w$.

Suppose that R is transitive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. We must show $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$. That is, we must show $R^{\rightarrow}(w) \subseteq \{v \mid X \in \mathcal{N}_v\}$. Let $x \in R^{\rightarrow}(w)$. Then wRx. To complete the proof we need only show $X \in \mathcal{N}_x$. That is, we must show $R^{\rightarrow}(x) \subseteq X$. Since R is transitive, $R^{\rightarrow}(x) \subseteq R^{\rightarrow}(w)$ (why?). Hence since $R^{\rightarrow}(w) \subseteq X$, $R^{\rightarrow}(x) \subseteq X$.

$\square$

From this point of view, we can think of relational frames and (augmented) epistemic frames as two different ways of presenting the same information. That is, we are after a mathematical structures that can represent for each state, the set of known propositions at each state. It should be clear that with epistemic frames, there is more freedom in which collection of sets can be known at a particular state. A natural question to ask is under what circumstances do an epistemic frame and a relational frame represent the same information.

Definition 11 *Let* $\langle W, E\rangle$ *be an epistemic frame and* $\langle V, R\rangle$ *be a relational frame. We say that* $\langle W, E\rangle$ *and* $\langle V, R\rangle$ *are* **equivalent** *if there is a function* $f : 2^W \to 2^V$ *such that for each* $X \subseteq W$*,* $X \in E(w)$ *iff* $f[X] \in \mathcal{N}_w$*.*

Theorem 12 *Let* $\langle W, R \rangle$ *be a relational frame. Then there is an equivalent augmented neighborhood frame.*

Proof The proof is trivial given the previous Lemma. For each $w \in W$, let $E(w) = \mathcal{N}_w$.

$\square$

Theorem 13 *Let* $\langle W, E \rangle$ *be an augmented epistemic frame. Then there is an equivalent relational frame.*

PC Any axiomatization of propositional calculus

E $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

M $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

N $\Box\top$

$$K \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

$$RE \quad \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

$$Nec \quad \frac{\phi}{\Box\phi}$$

$$MP \quad \frac{\phi \quad \phi \to \psi}{\psi}$$

Soundness and Completeness

Let F be a collection of neighborhood frames. A formula $\phi \in \mathcal{L}$ is **valid in** F, or F-**valid** if for each $\mathbb{F} \in$ F, $\mathbb{F} \models \phi$. We say that a logic $\mathbf{L}$ is **sound** with respect to F, provided $\mathbf{L} \subseteq$ F. That is for each formula $\phi \in \mathcal{L}$, $\vdash_{\mathbf{L}} \phi$ implies ϕ is valid in F.

Given a set of formulas Γ, a formula ϕ and a collection of frames F, we say Γ **semantically entails** ϕ with respect to F, denoted $\Gamma \models_{\mathsf{F}} \phi$, if for each $\mathbb{F} \in \mathsf{F}$, if $\mathbb{F} \models \Gamma$ then $\mathbb{F} \models \phi$. Here $\mathbb{F} \models \Gamma$ means for each $\phi \in \Gamma$, $\mathbb{F} \models \phi$. Finally we write $\models_{\mathsf{F}} \phi$ if for each $\mathbb{F} \in \mathsf{F}$, $\mathbb{F} \models \phi$. A logic $\mathbf{L}$ is **weakly complete** with respect to a class of frames F, if $\models_{\mathsf{F}} \phi$ implies $\vdash_{\mathbf{L}} \phi$. A logic $\mathbf{L}$ is **strongly complete** with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \phi$ implies $\Gamma \vdash_{\mathbf{L}} \phi$.

Let $M_{\mathbf{L}}$ be the set of $\mathbf{L}$-maximally consistent sets of formulas. Recall that *Lindenbaum's Lemma* (see [**?**] and [**?**] for an extended discussion) states that given any consistent collection of formulas Γ' there is a maximally consistent set of formulas that contains Γ'. Given a formula $\phi \in \mathcal{L}$, let $|\phi|_{\mathbf{L}}$ be the **proof set** of ϕ in $\mathbf{L}$. Formally, $|\phi|_{\mathbf{L}} = \{\Delta \mid \Delta \in M_{\mathbf{L}} \text{ and } \phi \in \Delta\}$. We first note that proof sets share a number of properties in common with truth sets.

Lemma 14 *1.* $|\phi \wedge \psi|_{\mathbf{L}} = |\phi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$

2. $|\neg\phi|_{\mathbf{L}} = M_{\mathbf{L}} - |\phi|_{\mathbf{L}}$

3. $|\phi \vee \psi|_{\mathbf{L}} = |\phi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$

4. If $|\phi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ *then* $\vdash_{\mathbf{L}} \phi \rightarrow \psi$

5. $\vdash_{\mathbf{L}} \phi \leftrightarrow \psi$ *iff* $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$

Lemma 15 *For each* $\phi \in \mathcal{L}$*,* $\psi \in \bigcap |\phi|_{\mathbf{L}}$ *iff* $\vdash_{\mathbf{L}} \phi \rightarrow \psi$*.*

Given any model $\mathbb{M} = \langle W, N, V \rangle$ and a set $X \subseteq W$, we say that X is **definable** in $\mathbb{M}$ if there is a formula $\phi \in \mathcal{L}$ such that $(\phi)^{\mathbb{M}} = X$. Let $\mathcal{D}_{\mathbb{M}}$ be the set of all sets that are definable in $\mathbb{M}$.

Define the **canonical valuation**, $V_{\mathbf{L}} : \mathsf{At} \rightarrow 2^{M_{\mathbf{L}}}$ as follows. Let $p \in \mathsf{At}$, then $V_{\mathbf{L}}(p) = |p|_{\mathbf{L}} = \{\Gamma \mid \Gamma \in M_{\mathbf{L}}$ and $p \in \Gamma\}$.

Definition 16 $\mathbb{M} = \langle W, N, V \rangle$ *is a* **canonical for** $\mathbf{L}$ *if*

1. $W = M_{\mathbf{L}}$

2. *for each* $\Gamma \in W$ *and each formula* ϕ, $|\phi|_{\mathbf{L}} \in N(\Gamma)$ *iff* $\Box\phi \in \Gamma$

3. $V = V_{\mathbf{L}}$

Lemma 17 (Truth Lemma) *For any consistent logic* $\mathbf{L}$ *and any consistent formula* ϕ*, if* $\mathbb{M}$ *is canonical for* $\mathbf{L}$*,*

$$(\phi)^{\mathbb{M}} = |\phi|_{\mathbf{L}}$$

Proof The boolean connectives are as usual and left for the reader. We focus on the modal case. Let $\mathbb{M} = \langle W, N, V \rangle$ be canonical for $\mathbf{L}$. Suppose that $\Gamma \in (\Box\phi)^{\mathbb{M}}$, then by definition $(\phi)^{\mathbb{M}} \in N(\Gamma)$. By the induction hypothesis, $(\phi)^{\mathbb{M}} = |\phi|_{\mathbf{L}}$, hence $|\phi|_{\mathbf{L}} \in N(\Gamma)$. By part 2 of Definition 16, $\Box\phi \in \Gamma$. Hence

$\Gamma \in |\Box\phi|_{\mathbf{L}}$. Conversely, suppose that $\Gamma \in |\Box\phi|_{\mathbf{L}}$. Then by definition of a truth set, $\Box\phi \in \Gamma$. Hence by part 2 of Definition 16, $|\phi|_{\mathbf{L}} \in N(\Gamma)$. By the induction hypothesis, $|\phi|_{\mathbf{L}} = (\phi)_{\mathbf{L}}$, hence $(\phi)^{\mathbb{M}} \in N(\Gamma)$. Hence $\Gamma \in (\Box\phi)^{\mathbb{M}}$.

$\Box$

Theorem 18 *The logic* $\mathbf{E}$ *is sound and strongly complete with respect to the class of all neighborhood frames.*

Proof The proof is standard and so will only be sketched. Soundness is straightforward (and in fact already shown in earlier exercises). As for strong completeness, we will show that every consistent set of formulas can be satisfied in some model. Before proving this, we briefly explain why this implies strong completeness. The proof is by contrapositions. Suppose that it is not the case that $\Gamma \vdash_{\mathbf{L}} \phi$. Then $\Gamma \cup \{\neg\phi\}$ is consistent. Since any such set has a model, there $\Gamma \cup \{\neg\phi\}$ is true in some model. But then Γ cannot semantically entail ϕ. Thus if $\Gamma \not\vdash_{\mathbf{L}} \phi$ then $\Gamma \not\models_{\mathsf{F}} \phi$ (where F is the class of all neighborhood frames).

Let Γ be a consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Then consider the model $\mathbb{M}^{min}_{\mathbf{E}}$. By the Truth Lemma (Lemma 17), $\mathbb{M}^{min}_{\mathbf{L}}, \Gamma' \models \Gamma'$. Thus

Γ is true is some model, namely the minimal canonical model.

$\square$

What are canonical models like? Notice that for one fixed system we will have a large class of canonicals. The smallest canonical is one where the neighborhood of every world w is constituted by $\{|\phi|_{\mathbf{L}}\colon \Box\phi \in w\}$.

The largest canonical for a classical modal system will contain in addition all the non-prrof sets for the given logical system.

Lemma 19 *If* $\mathbb{M} = \langle W, N, V\rangle$ *is the smallest canonical for a normal system of modal (propositional) logic then its augmentation is also a canonical model for the system.*

Proof Let Σ be a normal system and suppose that $\mathbb{M}^!$ is the augmentation of $\mathbb{M}$. To prove that $\mathbb{M}^!$ is a canonical for Σ we should prove that for every w in $\mathbb{M}^!$: $\Box A \in w$ iff $|A|_\Sigma \in N^!_w$

We know that $|A|_\Sigma \in N^!_w$ means that $\bigcap N_w \subseteq |A|_\Sigma$ (why?), which, in turn means that:

$$\bigcap\{|A|_{\Sigma} : \Box A \in w\} \subseteq |A|_{\Sigma}$$

(why?). Then we can show (try to provide a proof) that: $w' \in \bigcap\{|A|_{\Sigma} : \Box A \in w\}$ iff $\{|A| : \Box A \in w\} \subseteq w'$. So, we wish to show that $\Box A \in w$ if and only if A belongs to every Σ-maximal set of sentences w' such that $\{|A| : \Box A \in w\} \subseteq w'$.

Suppose that $A \in w'$ for every Σ-maximal set w' such that $\{|A| : \Box A \in w\} \subseteq w'$; i.e. that

A belongs to every Σ-maximal extension of the set $\{|A| : \Box A \in w\}$. By a corollary to Lindenbaum's lemma we have that A is Σ-deducible from this set of sentences:

$$\{|A| : \Box A \in w\} \vdash_{\Sigma} A$$

The compactness of $\vdash_{\Sigma}$ gives us in turn that there is a finite set of sentences $A_1, ..., A_n$ ($n \geq 0$) in the set $\{|A| : \Box A \in w\}$ such that:

$$A_1, ..., A_n \vdash_{\Sigma} A$$

Since the logic is normal we can then infer the boxed version of this entailment. Notice that all items in the boxed version of the entailment are members of w, so we have that so we have (by monotony of $\vdash_{\Sigma}$):

$$w \vdash_{\Sigma} \Box A$$

and we are done.

$\square$

General Neighborhood Frames

Definition 20 *A* **general neighborhood frame** *is a tuple* $\mathbb{F}^g = \langle W, N, \mathcal{A} \rangle$*, where* W *is a non-empty set of states,* N *is a neighborhood function, and* $\mathcal{A}$ *is a collection of subsets of* W *closed under intersections, complements, and the* m_N *operator* ($m_N(X) = \{w : X \in N_w\}$*.*

We say a valuation V : At $\rightarrow 2^W$ is admissible for a general frame $\langle W, N, \mathcal{A}\rangle$ if for each $p \in$ At, $V(p) \in \mathcal{A}$.

Definition 21 *Suppose that* $\mathbb{F}^g = \langle W, N, \mathcal{A}\rangle$ *is a general neighborhood frame. A general model based on* $\mathbb{F}^g$ *is a tuple* $\mathbb{M}^g = \langle W, N, \mathcal{A}, V\rangle$ *where* V *is an admissible valuation.*

Lemma 22 *Let $\mathbb{M}^g$ be an general neighborhood model. Then for each $\phi \in \mathcal{L}$, $(\phi)^{\mathbb{M}^g} \in \mathcal{A}$.*

Given a logic $\mathbf{L}$, it is easy to show that the set $\mathcal{A}_{\mathbf{L}} = \{|\phi|_{\mathbf{L}} \mid \phi \in \mathcal{L}\}$ is a boolean algebra and closed under the m_N operator. A general frame is called a $\mathbf{L}$-frame, if $\mathbf{L}$ is valid on that frame. We show that for each modal logic $\mathbf{L}$ the canonical frame is a $\mathbf{L}$-frame.

Lemma 23 *Let $\mathbf{L}$ be any logic extending $\mathbf{E}$. Then the general canonical frame $\mathbb{F}^g_{\mathbf{L}} \models \mathbf{L}$.*

Proof Let $\phi \in \mathbf{L}$ and V an arbitrary admissible valuation. We must show that $\mathbb{M}^g = \langle M_{\mathbf{L}}, N, V \rangle$ validates ϕ. Since V is admissible, for each propositional letter p_i occurring in ϕ, $V(p_i) \in \mathcal{A}_{\mathbf{L}}$. Hence for each (there are only finitely many), p_i, $V(p_i) = |\psi_i|_{\mathbf{L}}$ for some formula ψ_i. Let ϕ' be ϕ where each p_i is replaced with ψ_i. We prove by induction of ϕ that $(\phi)^{\mathbb{M}^g} = (\phi')^{\mathbb{M}^g_{\mathbf{L}}}$.

The base case is when $\phi = p$. Then $\phi' = \psi$ for some $\psi \in \mathcal{L}$ where $V(p) = |\psi|_{\mathbf{L}} \in \mathcal{A}_{\mathbf{L}}$. Then $\Gamma \in (p)^{\mathbb{M}^g}$ iff $\Gamma \in V(p) = |\psi|_{\mathbf{L}}$ iff $\Gamma \in (p)^{\mathbb{M}^g_{\mathbf{L}}}$. The boolean connectives are straightforward. Suppose that ϕ is of the form $\Box\gamma$ and $(\gamma)^{\mathbb{M}^g} = (\gamma')^{\mathbb{M}^g_{\mathbf{L}}}$. Note that $\phi' = \Box\gamma'$. Hence $\Gamma \in (\phi)^{\mathbb{M}^g}$ iff $(\gamma)^{\mathbb{M}^g} \in N(\Gamma)$ iff $(\gamma')^{\mathbb{M}^g_{\mathbf{L}}} \in N(\Gamma)$ iff $\Gamma \in (\phi')^{\mathbb{M}^g}$.

$\Box$

Basic Topological Notions

Definition 24 *A* **topological space** *is a neighborhood frame* $\langle W, \mathcal{T} \rangle$ *where* W *is a nonempty set and*

1. $W \in \mathcal{T}$, $\emptyset \in W$

2. $\mathcal{T}$ *is closed under finite intersections*

3. $\mathcal{T}$ *is closed under arbitrary unions.*

The collection $\mathcal{T}$ is called a **topology**. Elements $O \in \mathcal{T}$ are called **opens**. A set C such that $W - C \in \mathcal{T}$ is called **closed**. Given a topology $\langle W, \mathcal{T} \rangle$, let $\mathcal{T}_C$ be the collection of closed subsets of W, i.e., $\mathcal{T}_C = \{C \mid W - C \in \mathcal{T}\}$. The following observation is an easy consequence of the above definition.

Observation 25 *Let* $\langle W, \mathcal{T} \rangle$ *be a topological space. Then* $\mathcal{T}_C$ *has the following properties:*

1. $\emptyset, W \in \mathcal{T}_C$

2. $\mathcal{T}_C$ *is closed under finite unions*

3. $\mathcal{T}_C$ *is closed under arbitrary intersections*

Given a topological space $\langle W, \mathcal{T} \rangle$ and a point $w \in W$, a **neighborhood of** w is any open set that contains w. Let $\mathcal{T}_w = \{O \mid O \in \mathcal{T} \text{ and } w \in O\}$ be the collection of all neighborhoods of w.

Lemma 26 *Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.*

Definition 27 *Let* $\langle W, \mathcal{T} \rangle$ *be a topological space. A pair* $\langle W, N \rangle$ *is called a* **neighborhood system** *provided* $N : W \to \mathcal{T}$ *is defined as follows:* $N(w) = \mathcal{T}_w$.

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X \subseteq W$ any set. The largest open subset of X is called the **interior** of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the **closure** of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

It is easy to see that a set X is open if $Int(X) = X$ and closed if $Cl(X) = X$. The following Lemma will be helpful when studying the topological semantics of the next section.

Lemma 28 $\langle W, \mathcal{T} \rangle$ *is a topological space. Then*

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$

2. $Int(\emptyset) = \emptyset$, $Int(W) = W$

3. $Int(X) \subseteq X$

4. $Int(Int(X)) = Int(X)$

5. $Int(X) = W - Cl(W - X)$

Let $\langle W, N, V\rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter

- for each $w \in W$, $w \in \bigcap N(w)$

- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

We can now show that there is a topological model that is point-wise equivalent to $\mathcal{M}$. Consider the set $\mathcal{B} = \{m_N(X) \mid X \subseteq W\}$. We will show that $\mathcal{B}$ is a base. That is we must show that

1. $\cup\mathcal{B} = W$ and 2. for each $X, Y \in \mathcal{B}$ and each $x \in X \cap Y$ there is a $Z \in \mathcal{B}$ such that $x \in Z \subseteq X \cap Y$.

Suppose that $X = m_N(x_1)$ and $Y = m_N(X_2)$ and $x \in X \cap Y$. Since $N(w)$ is a filter, $m_N(X_1) \cap m_N(X_2) = m_N(X_1 \cap X_2)$.

Thus $x \in m_N(X_1 \cap X_2) \subseteq m_N(X_1) \cap m_N(X_2)$. Thus we are done if we can show that $m_N(X_1 \cap X_2) \in \mathcal{B}$. But this follows form thethird property above since $X_1 \cap X_2 \in N(w)$.

Topological Semantics

$\mathbb{M}^T, w \models \Box\phi$ iff $\exists O \in \mathcal{T}, w \in O$ such that $\forall v \in O, \mathbb{M}^T, v \models \phi$

Notice the similarity between this definition an the definition of truth of the modal operator $\langle\ ]$. The only difference is the extra clause $w \in O$. Then, the above clause can be written as

$\mathbb{M}^T, w \models \Box\phi$ iff $\exists O \in \mathcal{T}_w$ such that $\forall v \in O, \mathbb{M}^T, v \models \phi$

Although this difference is a trivial change in terminology, it demonstrates a close connection between neighborhood frames and topological frames. Finally it is easy to see that

$$(\Box\phi)^{\mathbb{M}^T} = Int((\phi)^{\mathbb{M}^T})$$