Bernays Project: Text No. 15

Theses and remarks on the philosophical questions and on the situation of the logico-mathematical foundational research (1937)

Paul Bernays

(Thesen und Bemerkungen zu den philosophischen Fragen und zur Situation der logisch-mathematischen Grundlagenforschung, 1937.)

Translation by: *Dirk Schlimm*

Comments:

none

I. Philosophy and syntax

1. Scientific philosophy [*wissenschaftliche Philosophie*] consists of the fundamental considerations of the design resp. reorganization of the language of science [*Wissenschaftssprache*] and the considerations which refer to the possible fundamental interpretations and points of view [*Auffassungen*] of the scientific approaches [*Ansätze*].

2. The syntax, as it is developed in Carnap's book *The Logical Syntax of Language* following [*in Anlehnung an*] Hilbert's meta-mathematics, the studies [*Untersuchungen*] of the Polish logicians, and Gödel on formalized languages,

considers [*betrachtet*] the mathematical properties of formalized languages of science [*Wissenschaftssprachen*].

3. If the syntax should contain ascertainments [*Feststellungen*] it must take place in an interpreted [*gedeuteten*] language.

If a formal definition is to be used to to make precise a philosophical concept formation [*Begriffsbildung*] then either the formal definition has to be provided [*versehen*] with an interpretation or the precision is achieved indirectly [*jene Präzisierung erfolgt indirekt*] by demanding a syntactic property of the formal definition which itself has then to be determined in a way that can be interpreted [*in deutbarer Weise*].

4. That a formal language functions as a syntax-language [*Syntax-Sprache*] using, for instance, Gödel's method of arithmetization [*Arithmetisierungsmethode*], is based on the intuitive-concrete validity [*anschaulich-konkreten Gültigkeit*] of arithmetic.

II. Logic and mathematics

1. Instead of the Kantian "analytic–synthetic" distinction, which encounters fundamental problems in its general version, the introduction of a different kind of distinction is recommended between "*formally*" and "*objectively*" [*gegenständlich*] *motivated elements* of a theory, i.e., between elements (terms [*Termini*], axioms, inferences [*Schlußweisen*]) that are introduced for the sake of the elegance, the simplicity, and the rounding off [*Abrundung*] of a system, and those that are introduced with regard to the matters of fact [*Sachverhalte*] of the domain in question [*des zu behandelnden Gegenstandsgebietes*].

Remark: This distinction surely does not yield a sharp classification [*Einteilung*], since formal and objective [*gegenständliche*] motives can superpose [*? superponieren*].

2. Systematic logic forms a domain of application [*Anwendungsgebiet*] for mathematical considerations [*Betrachtung*]. The connection between logic and mathematics in the systems of logistic [*Logistik*] corresponds [*ist eine entsprechende*] to that of physics and mathematics in the systems of theoretical physics.

3. What is mathematical [*das Mathematische*] can not be found only in connection with the logical formalism of sentences [*? Satzformalismus*], rather we find mathematical relations also in intuitable objects [*? anschaulicher Gegenständlichkeit*]; in particular we meet [*treffen wir*] mathematical relationships [*Verhältnisse*] in all domains of physics and biology [*in allen Gebieten des Physikalischen und Biologischen*].— The independence of mathematics from language has been emphasized in particular by Brouwer.

4. We must acknowledge that numerical relations [*Beziehungen*] express actual facts [*? Tatsächlichkeiten*]. This becomes particularly clear by means of the syntax: e.g., if a formula A is derivable in a formalism F, then this is a fact [*Tatsache*] which as such can be exhibited [*vorweisen*] and verified [*nachgeprüft*] explicitly. On the other hand, this derivability [*Ableitbarkeit*] is represented in the language of syntax [*Syntaxsprache*] by a numerical relation.

We also have a way of verifying arithmetical statements [*Sätze*] of the form of generality [*Allgemeinheit*], e.g., the statement that every whole number can be represented as the sum of four or less quadratic numbers [*Quadratzahlen*], in a sense analogous to physical laws, only that at one time one is confronted with a computational arrangement [*? Rechenanordnung*] and at the other time with an experimental arrangement [*Anordnung*]; in both cases the particular result that is to be obtained is predicted by a law.

5. In both the logic of ordinary language [*Umgangssprache*] and symbolic logic we have formally and objectively [*gegenständlich*] motivated elements side by side. An objective [*gegenständliche*] motivation is present in so far as the logical terms [*Termini*] and principles bear reference in part to particular very general characteristics [*Charakteristika*] of actuality [*Wirklichkeit*]. Paul Hertz has pointed out this objective [*gegenständlich*] side of logic in particular. Also F. Gonseth speaks of logic as a general "théorie de l'objet".

On the other side, the fact remains that the extension [*Umkreis*] and the problems [*Problemstellung*] of logic are oriented after certain main features [*Grundzügen*] of the structure of language [*Sprachstruktur*].

*III. On the question of mathematical intuition [*Anschauung*]*

1. In Kant's doctrine of pure intuition [*Lehre von der reinen Anschauung*] the assumption of a mathematical intuition is afflicted [*behaftet*] with various questionable [*bedenklichen*] additions [*? Zusatzmomenten*]. We can leave aside all these additions [*zusätzlichen Momente*], like the claim of an obligation [*Verbindlichkeit*] of the intuition of space and time for physics or the distinction between "sensuous [*sinnlich*]" and "pure" intuition, and still acknowledge that there is an intuitive mathematical representation [*anschauliche mathematische Vorstellung*] of spacial relationships [*Verhältnissen*], on the basis of which, at least to a certain extent, we can read off properties of configurations by means

of their intuitive representation. The kind of imagination [*Phantasie*] does not need to be fundamentally different than that which a composing musician uses in the domain of notes [*Töne*] when he predetermines [*vorausbestimmt*] combinations of tones [*Klangkombinationen*] in his imagination [*Vorstellung*].

2. It is suggested not to distinguish between "arithmetical" and "geometrical" intuition according to spacial or temporal moments [*Momente des Räumlichen und Zeitlichen*], but with regard to the distinction of what is discrete and what is continuous [*dem Diskreten und dem Kontinuierlichen*]. Thereafter the representation [*Vorstellung*] of a figure that is composed of discrete parts, in which the parts themselves are considered either only in their relation to the whole figure or according to certain coarser distinctive features [*Unterscheidungsmerkmalen*] that have been specially singled out, is arithmetical. Furthermore, also the representation [*Vorstellung*] of a formal process that is performed with such a figure and that is considered only with regard to the change that it causes is arithmetical. In contrast, the representations [*Vorstellungen*] of continuous change, of continously variable [*variierbar*] magnitudes, moreover topological representations [*Vorstellungen*], like those of the shape of lines and plains [*Linien- und Flächengestalten*], are geometrical.

3. The boundaries [*Grenzen*] of what is intuitively representable [*der anschaulichen Vorstellbarkeit*] are blurred [*unscharf*]. This is the reason that has led to the systematical sharpening of the arithmetical and geometrical concepts that are obtained by intuition, as it has been done in part by the axiomatic method [*axiomatische Verfahren*], in part by the introduction of formally motivated kinds of judgments and rules of inference [*Urteils- und Schlußweisen*]. What is methodically special in this case is that the formally motivated el-

ements that were to be introduced had already been provided largely by logic, like the principle of "tertium non datur", which is synonymous [*gleichbedeutend*] with the assumption that every sentence can be negated [*der Negationsfähigkeit eines jeden Satzes*] in the sense of a strict contradictory opposite [*strikt kontradiktorischen Gegenteils*]; in addition also the objectification [*? Vergegenständlichung*] of the concepts (predicates, relations) and extensions of concepts [*Begriffsumfänge*].

Remark. [*Anmerkung*] It is noteworthy historically, that in Aristotelian logic the tertium non datur is nowhere required in the well-known 19 figures of inference [*Schlußfiguren*], because the general affirmative judgment is interpreted in such a way that it asserts the existence of objects that fall under the concept of subject [*Subjektbegriff*]. (Note the rule "ex mere negativis nihil sequitur" from this point of view.)

IV. On the problematic of the foundations [Grundlagen-Problematik]

1. The method of sharpening mathematics by abstract means [*Methode der abstrakten Verschärfung*] as it is applied [*zur Auswirkung kommt*] in analysis and set theory has, as is well known, from the very beginning found opposition from a part of the mathematicians. In its most distinctive form [*ausgeprägtesten Form*] this opposition has the goal to replace the usual method [*Verfahren*] of introducing formally motivated elements by one that is performed completely within the framework of arithmetical evidence; geometric intuitiveness [*Anschaulichkeit*] is to be eliminated [*ausgeschaltet werden*] and, on the other hand, all abstract concept formations [*Begriffsbildungen*] and rules

of inference [*Schlußweisen*] that do not possess arithmetical intuitiveness [*Anschaulichkeit*] are to be avoided.

2. The grounding [*Begründung*] of a substantial part of existing mathematics that was begun by Kronecker and has been carried out by Brouwer according to the goal mentioned in 1. has not converted the mathematicians to accept the standpoint of the arithmetical evidence. The reasons for this may be the following:

a) Those who are looking for intuitiveness in mathematics will feel the complete [*restlos*] elimination of geometrical intuition to be unsatisfying and artificial. In fact, the reduction of the continuous to the discrete succeeds only in an approximate sense. On the other hand, those who are striving for sharp concepts [*Begrifflichkeit*] will prefer those methods that are most beneficial [*am günstigsten*] from the systematic standpoint [*Standpunkt der Systematik*].

b) In the Brouwerian method distinctions are introduced into the language of mathematics and play a fundamental role, whose meaningfulness [*Bedeutsamkeit*] is only apparent from the standpoint of the syntax of this language. That the "tertium non datur" is invalid, as Brouwer claims, can only be stated [*konstatiert werden*] as a *syntactic* matter of fact [*Sachverhalt*], but not as one of mathematical objectiveness [*Gegenständlichkeit*] itself.

Comment: [*Bemerkung*] The Brouwerian idea to characterize the continuum as a set of choice sequences is by itself [*an sich*] independent of the rejection of the "tertium non datur". For sure, no "tertium non datur" can hold with regard to indefinite predicates of choice sequences. But one could as well choose a standpoint such that the "tertium non datur" is retained for number theoretic properties of lawful serieses [*gesetzlicher Folgen*]. In this manner one would obtain an extension of Weyl's theory of the continuum of 1918.

3. The standpoint that Hilbert adopts in his proof theory is thereby characterized that it meets both the requirements [*gerecht werden*] of the formal systematic [*formalen Systematik*] and those of arithmetical evidence. As a means to unify these goals he employs the distinction [*dient ihm die Sonderung*] between mathematics and meta-mathematics, which is modeled after [*nachgebildet ist*] the Kantian partitioning of philosophy into "critique" and "system". As is well known, the main task that Hilbert assigns to meta-mathematics as a critique of proof [*Beweiskritik*] is to show the consistency [*Widerspruchsfreiheit*] of the usual practice [*Verfahren*] of mathematics. The problem is thought to be tackled in stages.

During the accomplishment of the task considerable difficulties arise, which are in part unexpected. A basic reason for difficulties which have not yet been overcome is that the distance between a formalism of intuitive arithmetic and that of usual mathematics is greater than Hilbert presumed [*vermutet*].

In the formalism of number theory the "tertium non datur" can be eliminated in a certain sense. The proofs of the consistency of the number theoretic formalism by Gödel and Gentzen are based on this fact. But as soon as one considers number-*functions* [*Zahlfunktionen*] such an elimination is no longer possible [*ist nicht mehr die Rede*]. This results in particular from a theorem which has been proved by S. C. Kleene after the concept of a "computable" function had been made more precise and which says that there are number-functions which are definable with the symbols of the number theoretic formalism (including a symbol for "the smallest number x that has the property $\mathfrak{P}(x)$"), but which are not computable.

Comment. — The concept of a computable function was made more precise in two independent ways: using the concept "generally recursive" [*allgemein-rekursiven*] by Herbrand and Gödel and by Church's concept of a "λ-definable" function; both concepts have been shown to be co-extensional [*umfangsgleich*] by A. Church and Kleene.

4. While the task of a consistency proof for analysis is still an unsolved problem, in a different direction, namely in the domain [*Gebiet*] of stage-free [*stufenfreien*] formalisms of combinatorial logic, proofs of the consistency have succeeded. Such a stage-free [*stufenfreier*] calculus is the theory of "combinators" which has been formulated by H. B. Curry following Schönfinkel, moreover the theory of "conversions" which was founded by Church. Both these formal theories, whose close connection has been shown by J. B. Rosser, yield a far reaching [*weittragenden*] and logically satisfying formalism for definitions [*Definitionsformalismus*]. The consistency of operating with combinators (in the sense of unambiguousness) has been proved a while ago by Curry, that of the formalism of conversions newly by Church and Rosser.

The stage-free [*stufenfreien*] combinatorical formalisms also yield a new stimulation for the formation [*Gestaltung*] of systems of logistic. An integration of these domains may possibly lead to a reformation of logistic on the whole. Sure enough, an adequate approach for such an integration is not available yet.