Methods for proving consistency proofs and their limitations
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The methods that were used to prove the consistency of formalized theories from the finite standpoint can be surveyed according to the following classification [Einteilung].

1. Method of valuation [Wertung]. It has obtained its fundamental development in the Hilbertian procedure of trying [Ausprobieren] the valuation. Using this procedure Ackermann and v. Neumann proved the consistency of number theory — admittedly under the restriction that the application of the inference from $n$ to $(n+1)$ is only allowed on formulas with free variables.

2. Method of integration [Ausintegrieren]. This can only be applied to such domains [Gebiete] that are completely controlled mathematically [mathematisch vollkommen beherrscht]. It allows for these not only the question of consistency
to be answered in a completely positive sense, but also those of completeness and decision [Entscheidbarkeit]. Such domains are in particular:

a) the calculus of one place functions \([\text{einstellige Funktionenkalkul}]\), which has been handled concludingly [abschließend] by Löwenheim Skolem, and Behmann.

b) Subformalisms [Teilformalismen] of number theory. Herbrand and Presburger have applied the method to such. It has been shown that Peano’s axioms do not suffice to develop number theory based on the function calculus [Funktionenkalkul] of “first order” (with axioms for equality). Only the addition of the recursive equations [Rekursionsgleichungen] for addition and multiplication brings full number theory about [kommt zustande]\(^1\).

3. Method of elimination. The idea for it can be found already in Russell and Whitehead, in particular applied to the concept “that, which” [\(\text{derjenige, welcher}\)] . However, the actual implementation [Durchführung] of the thought [Gedankens] is laborious [mühsam]. A fundamental simplification is effected by an approach by Hilbert, which follows the introduction of the “\(\epsilon\)-symbol”.

Firstly, — as has been shown by Ackermann — this yields, in a simpler way, again the result of the method of valuation.

Moreover, a new proof for a theorem can be reached from here, that has been discovered and proved for the first time by Herbrand, and which consists

\(^1\)The situation differs if the standpoint of class logic [Klassenlogik] is taken as basic [zugrunde legt] at the outset, like Dedekind; however, this contains stronger assumptions than are needed for number theory.
in a reversal of Löwenheim’s famous theorem about the satisfiability in the countable realm [im Abzählbaren]. It also yields a general procedure for the treatment of questions about consistency.

The present narrowness [vorliegende Begrenztheit] of the results presents itself as fundamental, because of the new theorem on the limits of decision procedures [Entscheidbarkeit] for formal systems by Gödel in connection with the conjecture by v. Neumann that followed it, despite the manifold [mannigfachen] of insights that have been obtained.