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Considerations regarding the paradox of Thoralf Skolem (1957)

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(Betrachtungen zum Paradoxon von Thoralf Skolem, 1957.)

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Comments:

none

About 35 years ago from now, in occasion of a congress in Helsingfors, Thoralf Skolem pointed out a paradoxical consequence of a theorem by Löwenheim, for which he had presented a simplified proof two years earlier using the logical normal form [*logischen Normalform*] named after him. This well-known theorem by Löwenheim says that for every mathematical theory axiomatized in the framework of elementary predicate logic — i.e., without bound variables for predicates [*Prädikatenvariablen*] — there exists a model in which the individuals [*Individuen*] are natural numbers, provided that it has a model that satisfies it at all. The theorem can be extended to the case in which one or more axiom *schemata* occur in the axiom system besides the proper [*eigentliche*] axioms. In the axiom schema an arbitrary predicate that can be constructed using the formative [*formativen*] rules of the axiom system, resp. a set or function that is arbitrary in the same sense, occurs as parameter.

Now Skolem realized that this theorem can be applied to axiomatic set theory, provided it has been sharpened [*verschärft*] by a more precise concept of definite property [*definiten Eigenschaft*] over the original formulation of Zermelo. That it is possible to sharpen it in such a way [*Verschärfung*], whereby the axiom system can be represented as a calculus [*kaklülmäßig*] by axioms and schemata, had been realized shortly before by Skolem and, in a different way, by A. Fraenkel. By the way, J. v. Neumann even succeeded to set up a system of finitely many axioms (without schemata) for set theory.

Thereby the possibility of such models for set theory arose in which sets are represented by natural numbers. This possibility is very paradoxical, because the cardinal numbers of the occurring sets rise to such sense dizzying heights [*schwindelhafte Höhen*] according to the theorems of set theory, so that the infinity of the number sequence (the countably infinite) is exceeded by large.

That this does not constitute a proper contradiction follows, as is well-known, from the fact that the enumerations [*Abzählungen*] which work as such in the axiomatic framework do not yet exhaust all possible enumerations. The concept of set is restricted by the axiomatic fixation [*Festlegung*] in such a way that one can speak of "set" only relative to a particular framework, if one generally insists on the demand of axiomatic accurateness [*? Präzisierung*]. This relativization is extended to a series of other concepts that are closely connected to the concept of set, in particular the concept of the uniquely invertible [*? umkehrbar eindeutig*] mapping between two totalities [*Gesamtheiten*] and thereby also the concept of cardinality [*Mächtigkeit*] (generalized concept of number [*Anzahl*]), and especially that of enumerability [*Abzählbarkeit*].

At first the impression arises that the detected [*vorgefundene*] paradox shows above all that differences of magnitudes are apparent and especially that the properly uncountable is an illusion. At the same time the thought is excited whether an operative construction [*Aufbau*] of mathematics and in particular of analysis is preferable to an axiomatic formulation in the light of the ascertained [*festgestellten*] relativity.

An operative understanding of mathematics is championed [*verfochten*] by many. It is characteristic for it that it does not regard the object of mathematics as something that is given in advance and that should be made accessible to our cognition [*? Erkennen*] by formations of concepts [*Begriffsbildungen*] and axiomatic descriptions, but that the mathematical operations themselves and the objects [*Gegenständlichkeiten*] that are brought about [*zustande kommen*] in them are regarded as the theme [*Thema*] of mathematics. Mathematics should create its objects by itself to some extent [*gewissermaßen*]. Thereby the character of arithmetic is prescribed [*vorgezeichnet*] eo ipso, since the structures of the operative creation are not fundamentally more general than those of the number sequence.

Herein lies a strength of this standpoint on the one hand, and a weakness on the other. It possesses a strength insofar arithmetical (constructive, combinatorical) thinking has the methodical advantage [*ausgezeichnet*] of being elementary and intuitive. However, it is dubitable if we get by with it for mathematics and if a ,so to speak, monistic conception of mathematics in the sense of the operative view can do full justice to its content — even to only as much as is already there—.

This idea is especially reinforced [*bestärkt*] when we consider the enterprises [*Unternehmungen*] of an operative build-up [*Aufbau*] of analysis as they have been pursued in newer times following different programmatic points of view. All these kinds of build-ups have in common that we are hindered by distinctions which are of no relevance for the geometrical idea of the continuum and are not necessary for the consistent functioning of the concepts [*Begriffsbildungen*]. The usual procedure of classical analysis proves to be vastly superior in this respect; and if the treatment of analysis had historically begun with an operative procedure, the detection of the possibility of the so much simpler classical methods would have been an eminent discovery, hardly less as it meant a de facto eminent progress in a different direction, namely compared [*gegenüber*] to the vagueness [*Unschärfe*] of the former operations [*Operierens*] in analysis.

The sense of an appropriate formation of concepts [*Begriffsbildung*] for analysis apparently [*allem Anschein nach*] lies in a suitable [*geeigneten*] compromise. We can make that plausible to us by the following. The diverging moments [*? widerstrebenden Momente*] for the concept [*Begriffsbildung*] that has to be chosen are, on the one hand, those that are intended by the homogeneity of the idea of the continuum and, on the other hand, those of the requirement [*Erfordernisses*] of the conceptual distinction for the determination of the measure of magnitudes [*Maßbestimmung der Größen*]. From an arithmetical point of view, every element of the number sequence is an individual [*Individuum*] with its very specific [*ganz besondere*] properties; from a geometric point of view we have here only the succession [*Aufeinanderfolge*] of repeating similar things [*Gle-*

ichartigen]. The task of formulating a theory of the continuum is not simply descriptive, but a reconciliation of two diverging tendencies. In the operative treatment one of them is given too much weight, so that homogeneity comes too short [*? zu kurz kommt*].

The investigations about the effectiveness and the fine structure in the formation of number serieses [*Zahlenfolgen*] and sets of numbers [*Zahlenmengen*] have their unquestionable importance for their specific direction of the question [*Fragestellung*]. But the insights that have been gained here do not contain a definite clue [*Hinweis*] that the usual procedure of analysis should be replaced by the more arithmetical methods.

The method on which the proceeding in classical analysis is based upon consists, in its logical means, in the application of a contentful "second order" logic, in which the general concepts [*Allgemeinbegriffe*] like "proposition" [*Aussage*], "set", "series" [*Folge*], "function" etc. are used in an unbound [*ungebundenen*] way that is not further specified. This second order logic shows its strength not only in its application to the theory of the continuum, but that it generally allows for the identification [*Kennzeichnung*] of mathematical structures, that may even be uncountable, by explicit definitions. Namely, to what is usually called "implicit definition" of mathematical objects there corresponds an explicit definition of a whole structure [*Strukturganzen*] wherein those objects occur as dependent [*unselbständige*] components. The model theoretic concepts of satisfiability [*Erfüllbarkeit*] and categoricity [*Kategorizität*] also find here their unproblematic application.

For sure [*Freilich*], second order logic is reproached for having a certain impreciseness [*Unschärfe*] in the concepts and it is the aim [*Absicht*] of the new sharper [*verschärften*] form of the axiomatic approach [*der Axiomatik*] to repair this defect [*diesem Mangel abzuhelfen*]. Logistic [*Logistik*] and axiomatic set theory have developed the methods for it. The phenomena of the relativity of the higher general concepts [*höhere Allgemeinbegriffe*] discussed above is evidence that this has not succeeded in a completely adequate way to make the concepts precise [*Präzisierung*].

Let us bring this to mind [*vergegenwärtigen*] again with an example. The property that an ordering [*Ordnung*] is without gaps [*Lückenlosigkeit*] is expressed in second order logic [*Logik der zweiten Stufe*] by the condition that every proper initial segment of the ordering [*echte Anfangsstück der Ordnung*] which has no last element possesses an immediately succeeding one. The general concept of set [*allgemeine Mengenbegriff*] appears here by means of the proper initial segment. If this now made precise by giving certain instructions on how to obtain sets, the manifold of the initial segments under consideration is narrowed and thereby the condition is weakened. This means that some orderings are admitted [*zugelassen*] as being gapless that can no longer count as being such if the concept of set is sufficiently expanded (i.e., if further processes are admitted for the formation of sets).

The difficulty considered here which is related to the task of making a theory formally precise [*? Präzisierung*] not only occurs in the characterization of uncountable structures, but especially also in the characterization of the structure of the sequence of numbers [*Zahlenreihe*]. We can explain, in Dedekind's sense, that a set M has the structure of the sequence of numbers with regard to a mapping φ (from M to itself) if φ is uniquely invertible [*? umkehrbar eindeutig*] and there is an element a of M which is not mapped to by φ [*nicht als φ-Bild auftritt*], and which has the property that no proper subset of M exists that contains a as well as $\varphi(c)$ for every element c. Here again to stipulate a narrower concept of subset can have the consequence that the above condition is satisfied by models to which we would not attribute the structure of the number sequence based on the unrestricted condition. This state of affairs [*Sachverhalt*] results likewise [*gleichermaßen*] if we use an axiom system [*Axiomensystem*] to identify [*Kennzeichnung*] the number sequence [*Zahlenreihe*] instead of the explicit definition of structure [*Strukturdefinition*]. In the usual [*gebräuchlichen*] form of such an axiom system one has the axiom of complete induction [*vollständigen Induction*] in which the general concept [*Allgemeinbegriff*] of proposition [*Aussage*] (of predicate) occurs. If the axiomatics is formally sharpened [*formalen Verschärfung*] this axiom is replaced by a formal inference principle [*Schlußprinzip*] in which the extension [*Umkreis*] of the allowed predicates is formally delimited [*abgegrenzt*] by a substitution rule [*? Einsetzungsregel*]. This constriction [*Einengung*] also allows for the possibility of models for number theory that satisfy all statements provable within the formal framework, but that deviate from the structure of the number sequence [*Zahlenreihe*] when they are considered in isolation [*? losgelöst von diesen*]. Again it was Skolem who pointed out this state of affairs of the "non-characterizability" [*Nichtcharakterisierbarkeit*] of the number sequence [*Zahlenreihe*] by a formalized axiom system using drastic examples.

On the whole, after what has been said so far the success of attempting to make a theory sharper and more precise using axioms [*verschärften axiomatischen Präzisierung*] might appear highly questionable. But, thereby the circumstance is not taken into consideration that there are frameworks [*Rahmen-*

systeme] for which classical mathematics does not have reasons to transgress them – as has been shown by the axiomatic and logistic [*logistische*] analysis of mathematical theories. The domain of sets and functions, e.g., as it is provided by the axioms of set theory [*Axiomatik der Mengenlehre*], is closed in such a way [*besitzt eine solche Geschlossenheit*] that the formal axiomatic restriction is hardly palpable [*fühlbar*] when forming concepts [*Begriffsbildungen*] and proofs [*Beweisführungen*].

Furthermore [*Es kommt noch der Umstand hinzu*], the set theoretic theorems are not affected by the relativity that holds for the general concepts [*Allgemeinbegriffe*]. This relativism surely does not mean that the continuum is shown to be uncountable [*überabzählbar*] in *one* framework of set theory and countable in *another*. The discrepancy consists rather only in the fact that the totality [*Gesamtheit*] of things that are represented in a set theoretic system, e.g., the set of subsets of the number sequence, can be countable in a more comprehensive [*umfassenderen*] system; but then it does not act there as a representation of that set of subsets, and thus it is impossible to map the numbers unambiguously [*eindeutig*] to the sets of numbers. In such a way [*Solchermaßen*] the cardinality theorems [*Mächtigkeitssätze*] of Cantor's set theory are invariant against [*invariant gegenüber*] the axiomatic framework despite the relativity of the concepts of sets [*Mengenbegriffe*].

For sure it must be conceded that this relativity brings the circumstance [*Umstand*] more forcefully into our consciousness that the higher cardinalities [*Mächtigkeiten*] in set theory are only intended, so to speak, but not properly constructed [*eigentlich aufgebaut*]. In this sense the graduations [*Abstufungen*] of cardinalities are in a certain way unreal [*Uneigentlichkeit*].

If one becomes aware of this state of affairs [*Sachverhalt*] it is often explained that everything in mathematics is countable in "actuality" [*in "Wirklichkeit"*]. But this formulation is misleading in so far as it does not take into account [*Rechnung trägt*] the fundamental fact which is expressed both in operative mathematics and in the consideration [*Betrachtung*] of formal axiom systems, namely that mathematical thinking in principle transcends [*hinausgeht*] every countable system. The framework for the mathematical formation of concepts [*mathematische Begriffsbildung*] is the open contentual [*inhaltliche*] second number class [*Zahlenklasse*] both when proceeding constructively and within a theory of stages [*? Stufentheorie*], if these are not restricted in an arbitrary fashion, or also in the sequence of the ascending systems of axiomatic set theory. It represents something that is in the proper sense uncountable, and sure enough it cannot be addressed [*angesprochen*] as a particular mathematical structure.

We are reminded here of the fact that also the number sequence is presented to us originally as an open domain [*Bereich*] compared to which the number sequence that we address [*ansprechen*] as a structure is somehow unreal [*eine Art der Uneigentlichkeit hat*]. The difference with respect to the second number class is that the openness of the number sequence is only due to the incompletedness [*Unabgeschlossenheit*] of the iterations of a single process, whereas the openness of the second number class is due to the incompletedness [*Unabgeschlossenheit*] of the formations of concepts [*Begriffsbildungen*].

That the unreal character [*Uneigentlichkeit*] of particular uncountable structures is much more noticeable [*soviel merklicher*] than the unreal character that lies in the conception of the number sequence as a structure is due to the fact that our concept of a formal theory intends exactly the same kind of infinity as that of the number sequence.