Bernays Project: Text No. 19

Mathematical Existence and Consistency (1950b)

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(Mathematische Existenz und Widerspruchsfreiheit. In *Etudes de Philosophie des sciences en hommage à Ferdinand Gonseth*, pp. 11–25. Editions du Griffon, Neuchatel, 1950.)

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Comments:

I've combined the translation by Sigrid Goodman and Wilfried Sieg with my draft translation. Changes to the Goodman/Sieg translations are indicated as ~~deletions~~ and insertions. About halfway through the paper I decided to be less picky and not indicate purely stylistic changes (in particular, remove dangling prepositions, avoid starting sentences with connectives, etc.) — Richard Zach, 9/19/00

It is a familiar thesis in the philosophy of mathematics that existence, in the mathematical sense, means nothing but consistency. It is a thesis used to describe the specific character of mathematics. The claim is that there is no ~~philosophical existence problem~~ *philosophical question of existence* for mathematics. However, this thesis is neither as simple in content nor as ~~trivial~~ *self-evident* as it may seem, and a study of it [reflections on it] may help to shed light on several issues current in philosophical discussions.

Let us begin by describing ~~what~~ *that against which* the thesis is directed ~~against~~. Quite obviously, it opposes the view that attributes to mathematical objects an ideal existence (i.e., a manner of existence that is independent on the one hand of being thought or imagined, and on the other hand also of appearing as the ~~datum~~ *determination* [Bestimmungsstück] of something real) and which claims furthermore that the existential statements of mathematics are to be understood with reference to this ideal existence. From the very outset one fact speaks against this view; namely, that ~~here~~ without apparent necessity an assumption is introduced *here* ~~that achieves nothing with respect to methodology~~ *which does no methodological work at all.* To ~~elucidate matters~~ *make things clear*, it may be advantageous to compare this with existential claims in the natural sciences. It is well known that an extreme phenomenalistic philosophy sought to eliminate the assumption of objects that exist independently of perception even from the representation of ~~relations in~~ nature. However, even a rough orientation about our experience suffices to show that such ~~a beginning~~ *an undertaking*—apart from the manifold obstacles that confront ~~the~~ *its* implementation—is also inappropriate from the ~~viewpoint of scientific concern~~ *scientific perspective*. Perception alone does not provide us with perspicuous laws. The world of our experience would have to be ~~a totally~~ *completely* different ~~one~~ in order for a theory based on *purely perceptible* concepts ~~of the purely perceptible~~ to be successful. Hence, positing the objective existence of entities in nature [Naturgegensändlichkeiten] is by no means solely an effect of our instinctive attitude but is appropriate from the standpoint of scientific ~~methods~~ *methodology.* ~~(That still holds~~ *(This is* true *also* for ~~today's~~ *contemporary* quantum physics, even though according to it ~~there is no such thing as a completely determined fixed state~~ *states cannot be specified with complete precision*.)

Comparing this case with that of mathematical entities, we find the following obvious difference. In the theoretical and concrete use of mathematical objects an independent [losgelöste] existence of these objects plays no role (i.e., an existence independent of their ~~individual~~ *respective* appearance as ~~the data [components of determination]~~ *determination* of something otherwise objective). The assumption of objective physical entities, by contrast, has ~~an~~ explanatory value only because the objects and states in question are posited as existing at particular times and in particular locations.

What we find here concerning mathematical objects holds in general for all those things that can be called "ideal objects."[1] We are referring to such objects of ~~thought~~ *reflection* [Betrachtung] to which we cannot ascribe, at least not directly, the character of the real, or more precisely, of the independently real; e.g., species, totalities, qualities, forms, norms, relations, concepts. All mathematical objects belong to this realm.

One can hold the view—and this view has indeed been defended by some philosophers—that all statements about ideal objects are reducible, if made precise, to statements about the real [über Wirkliches]. This kind of reduction would yield, in particular, an interpretation of existence statements in mathematics. However, at this point fundamental difficulties arise. In the first place, on a somewhat closer inspection it turns out that the task of

[1] I think it best to use the word "ideal" with a footnote. The New Muret-Saenders Encyclopedic Dictionary, Langenscheidt, has the entry: ideell: 3. math. (Zahi, Punkt) ideal. – Possible alternatives: conceptual, notional, ideate, ideative, ideational. ['ideal' is standard, no footnote needed —RZ]

reduction is by no means uniquely determined, since several conceptions of the real can be distinguished: The "real" may refer, for example to what is objectively real, or to what is given in experience, or to concrete things.[2] Depending on the conception of the real, the task of reduction takes on a completely different form. Furthermore, it does not seem that any one of these ~~forms~~ *alternatives* can achieve the desired reduction in a satisfactory way.

One has to mention in this connection especially the efforts of the school of logical empiricism towards a "unified language" for science. It is noteworthy that recently ~~one has deliberately distanced oneself from~~ the attempt to reduce all statements to those about ~~concrete things~~ *the concrete* [Konkretes] *has been abandoned.* ~~It seems that this was suggested~~ *This was prompted* in particular by the requirements in the field of semantics (an analysis of meaning of the syntactical forms of language).

~~Our discussion of the question whether the introduction of ideal objects can, in principle, be avoided in the language of science will not be based on a presupposition.~~ *We will not base our discussion on any presuppositions regarding the the possibility of avoiding the introduction, in principle, of ideal entitites in the language of science.* In any event, the existing situation is ~~as follows: In~~ *that in* areas of research (and even in ~~our~~ *the* manner of thinking ~~about~~ *of* everyday life) we are constantly dealing with ideal objects; and *we adopt* this familiar attitude ~~will be adopted~~ here.

As yet this attitude in no way includes an assumption about an inde-

[2]The "real" may refer, e.g., to the objectively real, or to the experientially given, or to the concretely material. [Maybe better?—RZ]

pendent existence of ideal objects. ~~But~~ It is understandable *though* that such an assumption has, in fact, often been connected with ideal objects—particularly if we agree with Ferdinand Gonseth, according to whom the more general concept of an object arises from a more primitive, cruder ~~idea~~ *representation* of an object [Gegenstandsvorstellung] that is expressed in a "physique de l'object quelconque." As regards the cruder things,[3] their character of objectivity is most intimately tied to ~~existence—an~~ existence independent of our perception and imagination [Verstellung—representation?]. ~~And~~ Thus it is easy to understand that in the case of general objects we are ~~also~~ inclined to attribute their objective character to an independent existence. ~~But~~ It is not at all necessary to do so, *however*. Here it is especially significant that refraining from an assumption of ideal existence does not prevent us from using existence statements about ideal objects that ~~permit interpretation~~ *can be interpreted* without that particular assumption. Let us bring to mind some crucial cases of such interpretations:

a) ~~By~~ existence of an ideal object ~~one may understand~~ *may mean* the distinct and complete imaginability [Vorstellbarkeit—representability?] of the object.

b) Existence of an ideal object of a particular kind may mean that this object is realized in something given objectively in nature. Thus, for instance, the observation that a certain word has different meanings in a language tells us that in the use of this language, ~~applications~~ *tokens* [Verwendungen] of the word occur with different meanings.

[3] Muret-Sanders entry: Dinglichkeit: 1. philos. reality. [How about 'materiality'? Meaning here is medium-sized physical object!? —RZ]

c) An existence claim concerning ideal objects can be made with reference to a structured ~~form~~ *entity* [Gebilde] ~~into~~ *of* which that object ~~passes as~~ *is* a constituent part. Examples of this are statements about constituent parts of a figure, as when we say, "the configuration of ~~the die~~ *a cube* contains 12 edges," or statements about something that occurs in a ~~certain~~ *partiucular* play, or about provisions that are part of Roman law. We are going to call existence in this sense, i.e., existence within a comprehensive structure, "relative ~~[dependent]~~ *[bezogene]* existence."

d) Existence of ideal objects may mean that one is led to such objects in the course of certain reflections. The statement that there are judgments in which relations appear as subjects, for example, expresses *the fact that* that we are ~~also~~ led to such "second order" judgments (as they are called) when forming judgments.

In case a) the existence of the ideal object is nothing but the ~~imagined entity [image entity]~~ *representational objectivity [Vorstellungs-Gegenständlichkeit]* (in the sense of ~~simple imagining~~ *representation proper*); in case b) existence amounts to a natural reality; case c) is concerned with an immanent fact of a total structure ~~that is~~ under consideration.

In these three cases the interpretation of the existence statement provides a kind of immediate ~~reduction in content~~ contentual reduction. Case d) is different in that "being led" to objects is not to be understood as a mere psychological fact but as something objectively appropriate. Here reference is made to the development of intellectual situations ~~in which~~ *with the* factors of freedom and ~~commitment are~~ *obligation* operative *therein*,—freedom in the sense in which Gonseth speaks of a "charte de nos libertes" (for example, the freedom to add in thought a further element to a totality of elements ~~which is imagined~~ *represented* as surveyable) and, on the other hand, ~~commitment~~ *obligation* which consists, e.g., in the fact that the means we use for the description and ~~intellectual mastery~~ *mental control* of entities yield, on their part, new and possibly even more complex entities.

Yet even this interpretation of existence statements introduces no assumption of independent existence of ideal objects. The existence statement is kept within the particular conceptual context, and ~~a~~ *no* philosophical (ontological) question of modality which goes beyond this context is ~~not~~ entered into. Whether such a question is meaningful at all is left open.

These ~~reflections refer~~ *considerations apply* to ideal objects in general. But what of the specific case of mathematical objects, which, as has been noted, ~~belong to the~~ *are* ideal objects? By applying the above considerations, ~~to them (the mathematical objects)~~ we notice that we already have a kind of answer to the question of what existence may mean in mathematics. However, the thesis under discussion, namely that existence with respect to mathematical objects is ~~identical in meaning~~ synonymous with consistency is intended to offer a simpler answer. For the discussion of this claim we have by now gained several clarifying points. Let us then turn to this discussion.

First let us ~~substitute~~ *replace* the obviously somewhat abbreviated formulation of the claim with a more detailed one. What is meant is surely this: Existence of an object (of a ~~form~~ *entity*, a structure) with certain required properties ~~means nothing other~~ in the mathematical sense ~~than~~ *means nothing but* the consistency of those required properties. The following simple example may ~~elucididate the point~~ *illustrate this*. There is an even prime number, but there is no prime number divisible by 6. Indeed, the properties "prime number" and "even~~-numbered~~" are ~~consistent~~ *compatible*, but the properties "prime number" and "divisible by 6" ~~are inconsistent~~ *contradict each other*. Examples like this ~~one~~ give the impression that the explanation of mathematical existence in terms of consistency is entirely satisfactory. It must be noted, however, that ~~this explanation shows no real accomplishment either~~ *these examples do not show the power of the explanation.* ~~That is to say, all that is demonstrated is how, from the existence of an example, one infers consistency and, on the other hand, how, from an inconsistency one infers non-existence. But it is not shown~~ *They only demonstrate how one deduces consistency from the existence of an example and, on the other hand, non-existence from inconsistency, but not* how, from an already established consistency, one infers existence. And that, after all, would be the decisive case.

This remark alone is enough to make us ~~suspicious~~ *hesitate*. ~~For we notice~~ *It draws attention to the fact* that in mathematics existential claims are not usually deduced from proofs of consistency but, conversely, that proofs of consistency are given by ~~presentations of~~ *exhibiting* models which verify the satisfaction of the required properties in each particular case in the sense of a positive ~~finding~~ *assertion.* In other words, the usual proofs of consistency are ~~evidence~~ *proofs* [Nachweise] of the *satisfiability* of ~~reqirements~~ *conditions*, or more precisely: of the satisfaction of ~~demands~~ *conditions* on an ideal ~~object~~ *entity.*

An unusual development was brought about by Hilbert's proof theory in that it demanded consistency proofs in the sense of showing the impossibility of arriving deductively at an inconsistency. ~~As a~~ *A* preconditon *of* such a proof ~~requires~~ *is* that the methods of deduction to be considered can be clearly delimited. The methods of symbolic logic provide the technique for making the process of logical inference more precise. We are thus in a position to delimit the methods of inference used in mathematical theories, especially in number theory and the theory of functions, by an exactly specified system of rules. This is, however, ~~merely~~ *only* a delimitation of the inferences used *de facto* in the theories. In general ~~however,~~ this does not lead to making an unrestricted concept of consistency more precise, ~~On the contrary, that is achieved~~ *but* only *consistency* in a certain relatively elementary domain of ~~the formation of~~ logico-mathematical ~~concepts~~ *concept-formations*. In this domain the concept of mathematical proof can be delimited in such a way that one can show: each requirement that does not lead deductively to an inconsistency can (in a more precisely specified sense) be satisfied. This completeness theorem of Gödel makes particularly clear that the ~~agreement~~ *claimed coincidence* [konstatiertes Zusammenfallen] of consistency with satisfiability ~~which is verified here,~~ is nothing less than a triviality, but is ~~conditioned essentially by~~ *substantially contingent on* the structure of the ~~above mentioned~~ domain of statements and inferences *considered*. If one goes beyond this domain, making the methods of proof ~~more~~ precise no longer ~~leads to the agreement~~ *yields the coincidence* of consistency ~~with~~ *and* satisfiability. This agreement—as shown again by Gödel—cannot be achieved in general (~~given the natural demands made on~~ *if certain natural requirements are imposed on* the concept of provability).

There is, of course, the possibility of extending the concept of proof by means of a more general ~~notion~~ *concept* of "consequence," following a method developed by Carnap and Tarski, so that for the resulting concepts of logical validity and contradictoriness (leading to a ~~inconsistency~~ *contradiction*)~~, which are tied to the notion of "consequence",~~ *we have* the alternative that ~~results is that~~ every purely mathematical ~~statement~~ *proposition* is either logically valid or contradictory. Consequently also, every demand on a mathematical object is either inconsistent or ~~there is an object that satisfies it~~ *satisifed by an object.*

~~Here, then,~~ *Thus* the identification of existence with consistency appears to receive exact confirmation. On closer inspection, however, one notices that the decisive factor is anticipated, so to speak, by the definition. For, on the basis of the definition, a mathematical demand on a mathematical object is ~~always~~ *already* contradictory ~~the moment there is no object to satisfy it~~ *if it is not satisfied by any object.* Accordingly, in the field of mathematics the ~~agreement between~~ *coincidence of* consistency of a demand and satisfaction by an object says no more than that an object of a species S that satisfies a ~~demand D~~ *condition* C exists if and only if not every object of the species S violates the ~~demand D~~ *condition* C.

Of course, from the standpoint of classical mathematics and logic this is a valid equivalence~~ relation~~. But using this equivalence to interpret existence statements is surely unsatisfactory: ~~If with respect to a general statement the assertion that it incurs an exception is thought to require, as an existential statement, an explanation of content, then, surely, the negation of that general statement~~ *If the claim that there is an exception to a universal proposition is considered to be in need of a contentual explanation, since it is an* *existential statement, then the negation of that universal proposition certainly* is no clearer as to its content. The equivalence between the negation of a ~~general statement to~~ *universal proposition and* an ~~existence statement has, among other things, precisely the role (in classical mathematics) that by means of this equivalence the sense of the negation of a general statement is more clearly explicated~~ *existential proposition serves, among other things, to explicate of the sense of the negation of a universal proposition more clearly (in classical mathematics.* ~~Characteristic of this circumstance is furthermore that Brouwer' s intuitionism, which does not recognize that equivalence, at the same time also denies any sense whatsoever to the straightforward negation of a general mathematical statement and in its stead introduces a sharpened negation, that of absurdity, which, once aqain, contains~~ *This is also indicated by Brouwer's intuitionism, which does not recognize this equivalence. At the same time, it denies that simple negation of a universal mathematical proposition has any sense at all, and introduces a stronger [verschärfte] negation—absurdity—which includes* an existential factor (since "absurdity" is to be understood as an effective possibility of a refutation).

The difficulties to which we have been led here ultimately arise from the fact that the concept of consistency itself is not at all unproblematical. The ~~approval so widely given to explaining~~ *common acceptance of the explanation of* mathematical existence in terms of consistency is no doubt due in considerable part to the circumstance that~~, on the basis of straightforward examples that come to mind, one has formed an unduly simple concept of what constitutes~~ *the simple cases one has in mind produce an unduly simplistic idea of what* consistency (~~reconcilability~~ *compatibility*) of conditions *is.*

One thinks of the compatibility of conditions as something ~~directly inherent~~ *the complex of conditions wears on its sleeve [etwas gleichsam direkt Anhaftendes]*, as it were, ~~in the complex of conditions~~ such that one need only ~~analyze~~ *sort out* the content of the conditions clearly in order to see whether they are in agreement or not. In fact, however, ~~the role of the conditions is that of being effective in functional application and through combination with one another~~ *the conditions affect each other in functional use and by combination*. ~~What emerges from this is not contained as a constituent part in what is given by the conditions~~ *The result is not contained as a consitutent part of what is given in the conditions*. It is probably the erroneous idea of such ~~inherency~~ *inherence* that gave rise to the view of the tautological character of mathematical ~~statements~~ *propositions*.

~~But apart from~~ *Leaving* the difficulties connected with the concept of consistency and the relation between consistency and satisfiability *aside*, there is ~~yet quite~~ another aspect ~~It draws our attention~~ *which points* to the fact that it is not *always* appropriate ~~for mathematics, at least not without exception,~~ to interpret existence as consistency *in mathematics*. Let us consider the case of existence axioms of ~~a mathematical theory constructed by the axiomatic method~~ *an axiomatic mathematical theory*. Interpreting the existence statement as an assertion of consistency in this case, yields confusion insofar as in an axiomatic theory consistency relates to the system of axioms as a whole. ~~A condition that concerns consistency can indeed function as a preceding postulate for the formation of an axiomatic system.~~ *The condition of consistency may well be a prior postulate for the design of any axiom system.* ~~But postulated axioms always have the purpose at least in the usual form~~

~~of axiomatic theory of generating consequences [Bindungen: commitments, bonds].~~ *The axioms themselves, however, are intended to to generate commitments [Bindungen zu stiften], at least in the usual form of axiomatics. An existence axiom does not say that we may postulate an entity in certain circumstances, but that we are committed [sind gebunden] to postulate it under these circumstances.*

On the other hand, we also have an appropriate understanding of axiomatic existence statements ~~in readiness~~ *available* on the basis of our initial reflections. That is to say, if we consider that an axiomatic system as a whole may be regarded as a description of a certain structural formation—for example, an axiomatic system of Euclidean geometry [may be regarded] as describing the structure of a Euclidean manifold—then we recognize that the existence claims within an axiomatic theory can be understood as statements about *relative existence*: Just as ~~three edges start from each corner in the configuration of a die~~ *each corner is incident to three edges in the configuration of a cube*, so also ~~a straight line passes~~ *there is a line* through ~~each of~~ *any* two ~~different~~ *distinct* points in the manifold of Euclidean space; and the theorem of Euclidean geometry which states that for any two points there exists a straight line ~~that passes~~ through both expresses this fact of related existence.[4]

[4]~~The unproblematic nature, so to speak, of related existence has been pointed out by~~ Bruno von Freytag-Löringhoff *has emphasized what is unproblematic, so to speak, about relative existence* in his article, "The Ontological Foundations of Mathematics," (Halle 1937) to which the present investigation owes a number of suggestions. In this connection the author speaks of the "~~little~~ *small* existence problem". His point of view, however, differs from the one presented here in that he regards the identification of existence with

It must be admitted, to be sure, that the viewpoint of relative existence, as appropriate as it is for the practical application of the existence concept in mathematics, only postpones, as it were, the philosophical question of mathematical existence. For related existence is scientifically significant only ~~if~~ *insofar as* the particular total structure, on which the relation [Bezogenheit] is based, is to be regarded as mathematically existent. The question thus arises: what is the status of the existence of those total structures; for example, the existence of the number series, the existence of the continuum, the existence of the Euclidean space-structure and also of other space-structures?

Here we encounter examples in which the identification of existence with consistency is justified. Thus we are justified when we say that the existence of non-Euclidean (Bolyai-Lobachevsky) geometry lies in its consistency. But even in such a case, the situation surely is that the proof of consistency follows from ~~a demonstration~~ *the exhibition of a model [Aufweisung]* and that thereby the assertion of consistency is ~~sharpened to the point of establishing the existence of~~ *strengthened to the assertion that* a model that satisfies the axioms *exists*—"~~existence~~ *exists*" referring here to the domain of the arithmetic of real and complex numbers. In analogous ways manifold proofs of consistency in the sense of satisfiability can be produced; for example, the proof of the consistency of a non-Archimedean geometry (i.e., a geometry with infinitely small segments); further, the consistency of calculating with imaginary magnitudes, taking the theory of real numbers as a basis.

consistency as appropriate to the ~~little~~ *small* existence problem, whereas in this presentation the viewpoint of relative existence is offered as ~~a corrective interpretation in direct opposition to that identification of~~ *correction set against the view that identifies* existence with consistency.

Most of such model constructions occur in the domain of the theory of the mathematical continuum (the theory of real numbers). ~~As to the axioms of the continuum themselves, starting with the number series, satisfiability can again be recognized by a substantial addition of set-theoretical construction processes.~~ *The satisfiability of the axioms of the continuum itself can be seen starting from the number series, by essential use of set-theoretical construction processes.*

But where do all these reductions lead? We finally reach the point at which we make reference to an ideal framework. It is a thought-system involving a kind of methodological attitude to which, in the final analysis, the mathematical existence ~~claims~~ *posits* [Existenz-Setzungen] relate.

Descriptively, we can state that the mathematician moves with confidence in this ideal framework and that here he has at his disposal a kind of acquired evidence (for which constructions, even of a complicated nature, such as infinite series of numbers, present themselves as something objective). The consistency of this ~~method~~ *methodology* has been *so* well tested in the most diversely combined forms of application ~~so~~ that there is *de facto* no doubt about ~~its de facto existence~~ *it*; it is, of course, the precondition for the validity of the existence ~~claims~~ *posits* made within the ideal framework. But here again we notice that we cannot simply identify existence with consistency, for consistency applies to the structure as a whole, not to the individual ~~thing posited as existing~~ *posit*.

Let us consider the ~~facts~~ *situation* more closely, using the example of the number series. The postulation of the number series is included in the framework of our mathematical operations [Operierens]. But what does consistency of the number series mean? If, ~~in response to this question, we are satisfied with the contention that an infinite progression of the counting process is to be understood by idealizing the way we imagine it,~~ *we are content in answering this question by appeal to an unbounded continuation of the process of counting (represented as an idealization)* then we view existence as an entity. We view it thus whether we regard the number series merely as a sphere of ideal objects or, in accordance with a stronger idealization, as a structural formation in itself. And only from this entity do we deduce [infer] consistency.[5] If, however, consistency is to be seen from the point of logic, then, on the one hand, the conditions contained in the ~~idea~~ *representation* [Vorstellung] of the number series must be understood conceptually and, on the other hand, we must base [that which constitutes] logical consequence on a more precise notion.

In this connection we also come to realize that the concept of logical consequence ~~suggests a similar kind of manifoldness as~~ *gives rise to an unbounded variety similar to* that of the number series ~~This is due to the choice of structure of the deduction-processes.~~ *due to the possibilities of combining inference processes.* Furthermore, it becomes apparent that the domain of logic can be understood in a narrower or a wider sense and ~~is therefore problematic with respect to its appropriate delimitation~~ *that therefore its appropriate delimitation is problematic.*

At this point we come to the field of inquiry of foundational research in mathematical logic. Its controversial nature stands in sharp contrast to the aforementioned confidence in performing mathematical operations within the

[5] Consider translating "Gegenständlichkeit" in this passage as "objectivity"??–RZ

framework of the usual methods.

The difficulties we are facing here are as follows: The usual framework for operating mathematically~~, it is true,~~ is adequately determined for use in classical theories; at the same time, however, certain indeterminacies with regard to the delimitation and method of foundation ~~[reduction]~~ [Fundamentierung] remain. If one endeavors to eliminate these, one faces several alternatives, and the views about deciding among these are divided. The differences of opinion are reflected in particular in the effort to obtain the foundation ~~[reduction]~~ of mathematics from a standpoint of unconditionality[6] [vom Standpunkt der Voraussetzungslosigkeit], such that one relies solely on the absolutely trivial or the absolutely evident. It becomes apparent here that there is no unanimity whatsoever on the question of what is to be considered as trivial or completely evident.

To be sure, this difference of opinion is less irritating if one dismisses the idea ~~of the necessity for~~ *that* an unconditional foundation, obtained from a starting point determined entirely *a priori, is necessary.* Instead, one could adopt the epistemological viewpoint of Gonseth's philosophy which does not restrict the character of duality—a duality due to the combination of rational and empirical factors—to knowledge in the natural sciences, but finds it in all areas of knowledge. For the abstract fields of mathematics and logic this means specifically that thought-formations are not determined purely *a priori.* They rather grow out of a kind of ~~intellectual~~ *mental* experimentation. This view is confirmed when we ~~reflect on~~ *consider* the founda-

[6]other suggestions: without assumptions, without presuppositions, absolute foundation, from first principles—RZ

tional research in mathematics. Indeed, it becomes apparent here that one is ~~obliged~~ *forced* to adapt the methodological framework to the requirements of the task [at hand] by trial and error. Such experimenting, which must be judged as an expression of failure according to the traditional view, seems entirely ~~realistic~~ *appropriate* [sachgemäß] from the viewpoint of ~~intellectual~~ *mental* experience. In particular, from this standpoint experiments that ~~have shown themselves~~ *turned out* to be unfeasible cannot *eo ipso* be ~~addressed as~~ *considered methodological* mistakes ~~in method~~. Instead, they can merit appreciation as stages in ~~intellectual~~ *mental* experimentation (if they are set up according to the general intent and are ~~performed consequentially~~ *carried out consistently*). Seen thus, the ~~majority~~ *variety* of competing foundational undertakings ~~involves nothing~~ *is not* objectionable ~~Instead, it~~ *but* appears analogous to the multiplicity of competing theories ~~as we encounter them~~ *encountered* in ~~a number of~~ *several* stages of development of research in the natural sciences.

If we now examine more closely the—at least partial—methodological analogy ~~presented here of~~ *between* the foundational speculations ~~to the~~ *and* theoretical research in the natural sciences, we are lead to think that with each more precise delimitation of a methodological framework for mathematics (or for an area of mathematics) a certain domain of mathematical reality [Tatsächlichkeit] is intended, and that this reality is independent of the particular ~~formation~~ *configuration/structure* of that framework to a certain degree. This can be made clear by the axiomatic theory of geometry. As we know, the theory of Euclidean geometry can be developed in various axiomatic ways. The resulting structural laws [Struktur-Gesetzlichkeit] of Euclidean geometry, hwoever, are independent of the particular way in which this is done. In a similar sense the relations in the theory of the mathematical continuum and the disciplines associated with it are independent of the particular way in which the real numbers are introduced, and even more so of the particular method of ~~foundational reduction~~ theoretical foundation [grundlagentheoretische Fundierung]. In a foundational investigation those relations, ~~to which we are led forcibly~~ *which are forced on us*, as it were, as soon as we ~~agree to certain methods~~ *settle on certain versions* of the calculus and of operating mathematically, have the role of the given [the factors that are given], the more precise theoretical ~~fixation~~ *definition* [Fixierung] of which is the ~~problem here~~ *task at hand*. The method of this fixation can contain problematic elements which do not affect the ~~factors that are~~ given [das Gegebene], so to speak.

The viewpoint gained in this way places a mathematical reality face to face with a ~~methodic~~ *methodological* framework constructed for the ~~fixation~~ *definition* of this reality. This is also quite compatible with the results of the descriptive analysis to which Rolin Wavre has subjected the relationship of invention and discovery in mathematical research. ~~What is pointed out here is the intertwining of two elements [factors]:~~ *He points out that two elements are interwoven,* ~~on the one hand~~ the invention of concept formations, ~~on the other hand~~ *and* the discovery of ~~lawful~~ *lawlike* relations between the conceived entities, and furthermore ~~the circumstance~~ that the conceptual invention is ~~directed toward~~ *aimed at* discovery.

With respect to the latter, it is frequently the case that the invention is guided by a discovery already more or less clearly available and that it serves the purpose of ~~bringing the discovery to conceptual determination~~ *making it conceptually definite*, thereby *also* making it accessible to communication. The necessity of adapting the concepts to the demands of giving expression to something objective exists in this situation as much as it does in similar situations in the theoretical natural sciences. Thus the concepts of the differential quotient and of ~~the domain of rationality~~ *a field* [Rationalitätsbereich] have been introduced with a view to giving expression to something objective in the same way as were the concepts of entropy and the electrical field.

For the constitution of a framework of mathematical deduction we assume a case of the same methodological type when we speak of a mathematical reality that is to be explicated by that framework.

If we now apply ~~these reflections~~ *this viewpoint* to our question of mathematical existence, we obtain ~~a substantial augmentation~~ *an essential completion* of our earlier observation that in the final analysis the existence statements in our mathematical theories are related [bezogen] to a system of thought that functions as a ~~methodic~~ *methodological* framework. This ~~relatedness~~ *relativity* [Bezogenheit] of the existence statements ~~henceforth~~ *now* seems compensated to a large extent in that the essential properties of the reality [Tatsächlichkeit] intended by the ~~methodic~~ *methodological* framework are ~~virtually~~ invariant*, so to speak,* ~~as compared with~~ *with respect to* the particulars (the invented aspects) of that framework.

Furthermore, it must be noted here that the mathematical reality [Tatsächlichkeit] also stands out from ~~each~~ *any* delimited ~~methodic~~ *methodological* framework insofar as it is never fully exhausted by it. On the contrary, ~~from~~ the conception of a deductive framework *always results in* further mathematical relations ~~follow each time~~ which go beyond that framework.

Do we not—so one may ask—return with such a view of mathematical reality [Tatsächlichkeit] to the assumption of an ideal existence of mathematical objects which we rejected as unmotivated at the outset of our reflections? To ~~answer~~ *respond to* this question we must recall the limits of the analogy between mathematical and physical reality. Our concern here is with something very elementary.

It is inherent in the purpose of concept formation in the natural sciences that it seeks to provide us with an orientational [??] interpretation of the environment. Therefore, in the natural sciences the modality [mode of being] of the factually real plays ~~an eminent~~ *distinguished* role, and in comparison with this [concrete] reality [Wirklichkeit] all other existence that can come into question appears as mere improper existence, as when we speak of the existence of the relations of natural laws. This holds true, in fact, even though the statements concerning the existence of natural laws in substance [contentually] go beyond what is ascertainable in the domain of the factual.

In mathematics we do not have such a ~~precise~~ *marked* difference in modality [the mode of being]. For the mathematician's mode of reflection, the individual mathematical entity [Gegenständliche] does not present itself as something that exists in a more eminent sense than the ~~lawful~~ *lawlike* relations. Indeed, one might say that there is no *clear* difference at all ~~in a clear sense~~ between a direct entity and a system of laws to which it is subject, since a number of laws present themselves by means of formal developments which, on their part, possess the character of the direct entity [Gegenständliche].[7]

[7]In this passage again, "das direkt Gegenständliche" might better be translated as "the

Even systems of axioms may be considered as structural ~~formations~~ *entity* [Gebilde]. In mathematics, therefore, we have no reason to assume existence in a sense fundamentally different from that in which we assume the existence of ~~lawful~~ *lawlike* relations.

This eliminates the various doubts that seem to conflict with our interpretation [view] of the ~~relatedness~~ *relativity* [Bezogenheit] of mathematical existence statements to a system of conceptualization [Begrifflichkeit] (to a deductive framework): Irrespective of the various possibilities of the ~~structure~~*construction* [Anlage] of such a system of conceptualization this interpretation [view] ~~is not the same as~~ *does not amount to* relativism. On the contrary, we can form the idea of a mathematical reality [Tatsächlichkeit] that is independent in each case of the particulars of the ~~structure~~ *construction* of the deductive framework. The thought of such a mathematical reality [Tatsächlichkeit] on the other hand, does not mean a return to the view of an independent existence of mathematical objects. It is not a question of ~~existence~~ *being* [Dasein] but of relational, structural ~~bonds~~ *connections* [Bindngen] and of the emergence (being induced) of ideal objects from other such objects.

In order not to be one-sided, however, our thoughts on mathematical existence still demand a supplementary view. We have carried out this reflection in accordance with the attitude of the mathematician who directs his attention purely toward the entity ~~[these entities]~~ [die Gegenständlichkeit, maybe "objecthood" in this case?]. If, however, we bear in mind our methodological comparison between the mathematical (foundational) starting points and

directly objective" or "the directly objectual".

those of physics, then we ~~may take notice that that~~ *might realize that this* analogy also applies to a point we have not yet taken into account: Just as the theoretical language and the theoretical attitude of physics is substantially supplemented by the attitude and language of the experimentalist, so also the theoretical attitude in mathematics is supplemented by a mode of reflection that is directed toward the procedural aspect of ~~the~~ mathematical ~~endeavor~~ *activity*. Here we are concerned with existence statements that do not refer to abstract entities but to arithmetical expressions, to formal ~~developments~~ *expansions* [as in "Reihenentwicklung"?], operations, definitions, ~~procedures of solution~~ *methods for finding solutions*, etc. The significance of such a constructive mode of reflection and expression, which finds application especially in Brouwer's ~~kind of~~ intuitionism and in the method of Hilbert's proof theory, will also be acknowledged by ~~that mathematician who is~~ *mathematicians who are* not willing to be content with an exclusively constructive mathematics and, therefore, just as little with an action language [Tätigkeits-Sprache] of mathematics as the only form of mathematical expression.

In this ~~sense~~ *context* it should also be emphasized, with respect to Hilbert's undertaking of a proof theory proceeding from an operative (constructive) standpoint, that the ~~scientific theoretical~~ *philosophical*[8] interest in this undertaking is not at all tied to those philosophical ~~teachings~~ *conceptions* of "formalism" that arose from the original version of the formulation of the ~~problem~~ *aim* [Aufgabestellung] of proof theory. In order to appreciate the methodological fruitfulness of ~~the~~ proof theory, there is in particular no need

[8] "wissenschaftstheoretisch" is "with respect to philosophy of science" but that would be somewhat awkward. —RZ

to take the position that the theories which are subjected to symbolic formalization (for proof theoretical purposes) are to be ~~equated henceforth and quite simply~~ *simply identified from then on* with the schema of their symbolic formalism and, consequently, are to be considered merely as a technical apparatus.

We must also bear in mind that the kind of motivation of the conceptual system of contemporary mathematics which results from connection with the problems that gave rise (in several stages) to that conceptual system does not lose its significance through the proof-theoretical investigation of consistency. On the contrary, such a motivation is assumed to have already been ~~accomplished [completed] on starting~~ *been given before* the proof-theoretical investigation *begins.*[9]

Finally it should be remembered—as regards the methods of the constructive proof theory and also those of Brouwer's ~~kind of~~ intuitionism—that with these methods one does not remain in the domain of ~~properly imagined objective entities~~ *representationally objective, properly so called* [im Bereich des eigentlich Vorstellungs-Gegenständlichen]. The concept of the effective is idealized and extended here in the sense of an adaptation to the theoretical demands—of course in a ~~basically more elementary~~ way *which is in principle more elementary* than it is done in ordinary mathematics. The methodological standpoint in this case is thus also not one of unconditionality, but,

[9] As regards the task of a *systematic* motivation of the concept formations of classical mathematics, we ~~come~~ *are led* to the problem already mentioned regarding ~~the acquisition of~~ *of obtaining* a deductive framework that is as appropriate and as satisfactory as possible. This problem constitutes a major topic of contemporary foundational research in mathematics.

once again, we are concerned with an ideal framework that includes general kinds of ~~claims [premises]~~ *positing* [Setzungen]. Our preceding reflections, therefore, are also applicable to this constructive mathematics.

On the whole our ~~thoughts [reflections]~~ *considerations* point to the fact that it is not indicated either to ~~carry too far~~ exaggerate the methodological difference between mathematics and the factual sciences, which is undeniable there, nor to underestimate the philosophical problems associated with mathematics.