

Bernays Project: Text No. 15

**Theses and remarks on the philosophical  
questions and on the situation of the  
logico-mathematical foundational research  
(1937)**

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(Thesen und Bemerkungen zu den philosophischen Fragen und zur  
Situation der logisch-mathematischen Grundlagenforschung, 1937.)

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*Volker's changes in **bold**, Richard's in underlined; incorporated  
uncontentious/stylistic changes.*

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*I. Philosophy and syntax*

1. Scientific philosophy [*wissenschaftliche Philosophie*] consists of the fundamental considerations of the **organization** resp. reorganization of the language of science [*Wissenschaftssprache*] and the considerations which ~~refer to~~ concern the possible fundamental interpretations and points of view [*Auffassungen*] of the scientific approaches [*Ansätze*].

2. The syntax, as it is developed in Carnap's book *Logische Syntax der Sprache* following [*in Anlehnung an*] Hilbert's meta-mathematics, the ~~studies~~

investigations [*Untersuchungen*] of the Polish logicians, and those of Gödel on formalized languages, considers [*betrachtet*] the mathematical properties of formalized languages of science [*Wissenschaftssprachen*].

3. If the syntax should contain ~~ascertainments~~ assertions [*Feststellungen*], it must take place in an interpreted [*gedeuteten*] language.

If a formal definition is to be used to make ~~precise~~ a philosophical concept formation [*Begriffsbildung*] precise, then either the formal definition has to be provided [*versehen*] with an interpretation or ~~that~~ ~~[?]~~ ~~precision~~ this precisification is achieved indirectly [*jene Präzisierung erfolgt indirekt*] by demanding a syntactic property of the formal definition which itself has then to be determined in a way that can be interpreted [*in deutbarer Weise*].

4. That a formal language functions as a syntax-language [*Syntax-Sprache*] using, for instance, Gödel's method of arithmetization [*Arithmetisierungsmethode*], is based on the intuitive-concrete validity [*anschaulich-konkreten Gültigkeit*] of arithmetic.

## *II. Logic and mathematics*

1. Instead of the Kantian “analytic–synthetic” distinction, which in its general formulation suffers from ~~encounters~~ fundamental problems ~~in its general version~~, the introduction of a different kind of distinction is recommended itself, a distinction between “*formally*” and “*objectively*” [*gegenständlich*] *motivated elements* of a theory, i.e., between elements (terms [*Termini*], axioms, **modes of [technical term in syllogistics]** )inferences [*Schlußweisen*]) that are introduced for the sake of the elegance, the simplicity, and the

rounding off [*Abrundung*] of **the** system, and those that are introduced with regard to the matters of fact [*Sachverhalte*] of the domain in question [*des zu behandelnden Gegenstandsgebietes*].

Remark: This distinction ~~surely~~ admittedly does not yield a sharp classification [*Einteilung*], since formal and objective [*gegenständliche*] motives can ~~superpose~~ overlap [*? superponieren*] ~~each other~~.

2. Systematic logic forms a domain of application [*Anwendungsgebiet*] for mathematical considerations [*Betrachtung*]. The connection between logic and mathematics in the systems of logistic [*Logistik*] corresponds [*ist eine entsprechende*] to that of physics and mathematics in the systems of theoretical physics.

3. What is mathematical [*das Mathematische*] can not only be found ~~only~~ in connection with the ~~logical formalism of statements~~ sentential formalism of logic [*? Satzformalismus*], rather we find mathematical relations also in intuitable objects [*? anschaulicher Gegenständlichkeit*] [**I would prefer: intuitive objectivity: me too—R**]; in particular, we ~~meet~~ find [*treffen wir*] mathematical relationships [*Verhältnisse*] in all domains **of the physical and the biological** [*in allen Gebieten des Physikalischen und Biologischen*].— The independence of the mathematicals from language has been emphasized in particular by Brouwer.

4. We must acknowledge that numerical relations [*Beziehungen*] express actual [**“real”, or “matters of fact” or omit**] facts [*? Tatsächlichkeiten*]. This becomes particularly clear ~~by means of the~~ in syntax: e.g., if a formula  $A$  is derivable in a formalism  $F$ , then this is a fact [*Tatsache*] which as such can be exhibited [*vorweisen*] and verified [*nachgeprüft*] explicitly. On the other

hand, this derivability [*Ableitbarkeit*] is represented in the language of syntax [*Syntaxsprache*] by a numerical relation.

We also have a way of **reviewing** [**“verifying” is too restricted; perhaps, but the analogy with physical laws suggests “confirm”.**] [*Nachprüfung*] arithmetical statements [*Sätze*] of ~~the form of generality~~ general form [*Allgemeinheit*], e.g., the statement that every whole number can be represented as the sum of four or ~~less quadratic numbers~~ fewer squares [*Quadratzahlen*] can be confirmed in a sense analogous to physical laws, ~~only~~ except that in the former case one is confronted with a computational arrangement [*? Rechenanordnung*] and in the latter case with an experimental arrangement [*Anordnung*]; in both cases a particular result to be obtained is predicted by the law.

5. In both the logic of ordinary language [*Umgangssprache*] and symbolic logic we have formally and objectively [*gegenständlich*] motivated elements side by side. An objective [*gegenständliche*] motivation is present in so far as the logical terms [*Termini*] and principles ~~bear reference~~ refer in part to ~~particular~~ certain very general characteristics [*Charakteristika*] of actuality [*Wirklichkeit*] better: reality?. In particular, Paul Hertz has pointed out this objective [*gegenständlich*] side of logic. F. Gonseth also speaks of logic as a general “théorie de l’objet”.

On the other ~~side~~ hand, the fact remains that the ~~extension~~ scope [*Umkreis*] and the problems [*Problemstellung*] of logic are oriented according to certain **basic** [or **“fundamental”**] features [*Grundzügen*] of the structure of language [*Sprachstruktur*].

### III. On the question of mathematical intuition [Anschauung]

1. In Kant's doctrine of pure intuition [*Lehre von der reinen Anschauung*] the assumption of a mathematical intuition is afflicted [*behaftet*] with various questionable [*bedenklichen*] **additional aspects** [*? Zusatzmomenten*]. We can leave aside all these additions [*zusätzlichen Momente*], ~~like such as~~ the claim of ~~the obligation~~ [*Verbindlichkeit*] of that the intuition of space and time is required for physics **and** the distinction between "sensuous [*sinnlich*]" and "pure" intuition, **but** still acknowledge, **however**, that ~~there is an intuitive mathematical idea~~ [*anschauliche mathematische Vorstellung*] of spacial relationships [*Verhältnissen*], on the basis of which, at least to a certain extent, ~~we can quasi read off properties of configurations by means of their intuitive representation.~~ spacial relationships can be represented in an intuitive mathematical way, and we can, at least to a certain extent, read off the properties of configurations, as it were, from their intuitive representation. The kind of **phantasy** [**"imagination" is "Einbildungskraft"**] [*Phantasie*] involved does not have to be fundamentally different from that which a ~~composing musician~~ composer uses in the domain of **sounds** [*Töne*] when he ~~predetermines~~ envisages [*vorausbestimmt*] combinations of tones [*Klangkombinationen*] in his imagination [*Vorstellung*].

2. It is ~~suggested not~~ advisable to distinguish between "arithmetical" and "geometrical" intuition not according to spatial or temporal moments [*Momente des Räumlichen und Zeitlichen*], but with regard to the distinction ~~of~~ what is between discrete and ~~what is~~ continuous [*dem Diskreten und dem Kontinuierlichen*]. ~~Thereafter~~ Accordingly, the **idea** [*Vorstellung*] of a figure that

is composed of discrete parts, in which the parts themselves are considered either only in their relation to the whole figure or according to certain coarser distinctive features [*Unterscheidungsmerkmalen*] that have been specially singled out, is arithmetical; **furthermore**, the **idea** [*Vorstellung*] of a formal process that is performed with such a figure and that is considered only with regard to the change that it causes [**two words omitted. I'd put them back in as the sentence is incomprehensible otherwise**] is likewise arithmetical. By contrast, the representations [*Vorstellungen*] of continuous change, of continuously variable [*variierbar*] magnitudes, moreover topological representations [*Vorstellungen*], like those of the **shapes** of lines and plains [*Linien- und Flächengestalten*], are geometrical. Why is “Vorstellung translated both as “idea” and as “representation” in this para?

3. The boundaries [*Grenzen*] of what is intuitively representable [*der anschaulichen Vorstellbarkeit*] are ~~blurred~~ vague [*unscharf*]. This is the reason that has led to the systematical sharpening of the arithmetical and geometrical concepts that are obtained by intuition, as it has been done in part by the axiomatic method [*axiomatische Verfahren*], in part by the introduction of formally motivated kinds of judgments and rules of inference [*Urteils- und Schlußweisen*]. What is methodically special in this case is that the formally motivated elements that were to be introduced had already been provided largely by logic, like the principle of “tertium non datur”, which is synonymous [*gleichbedeutend*] with the assumption that every **statement** can be negated [*der Negationsfähigkeit eines jeden Satzes*] in the sense of a strict contradictory opposite [*strikt kontradiktorischen Gegenteils*]; in addition also the objectification [*Vergegenständlichung*] of the concepts (predicates, relations) and extensions of

concepts [*Begriffsumfänge*].

*Remark.* [*Anmerkung*] It is noteworthy historically, that in Aristotelian logic the tertium non datur is nowhere required in the well-known 19 **modes** [**there are only 4 figures, but 19 modes (Barbara, Celarent, etc.)**] of inference [*Schlußfiguren*], because the general affirmative judgment is interpreted in such a way that it asserts the existence of objects that fall under the concept of subject [*Subjektbegriff*]. (Note the rule “ex mere negativis nihil sequitur” [insert parenthetical translation!] from this point of view.)

#### *IV. On the problematic of the foundations* [*Grundlagen-Problematik*]

1. The method of sharpening mathematics by abstract means [*Methode der abstrakten Verschärfung*] as it is applied [*zur Auswirkung kommt*] in analysis and set theory has been opposed by some mathematicians, as is well known, from the very beginning ~~found opposition from a part of the mathematicians~~. In its most distinctive form [*ausgeprägtesten Form*] this opposition has the goal to replace the usual method [*Verfahren*] of introducing formally motivated elements by one that is performed completely within the framework of arithmetical evidence; geometric intuitiveness [*Anschaulichkeit*] is to be eliminated [*ausgeschaltet werden*] and, on the other hand, all abstract concept formations [*Begriffsbildungen*] and **modes** of inference [*Schlußweisen*] that do not possess arithmetical intuitiveness [*Anschaulichkeit*] are to be avoided.

2. The grounding [*Begründung*] of a substantial part of existing mathematics that was begun by Kronecker and has been carried out by Brouwer according to the goal **(of a mathematics orientated at aiming at arith-**

**metical evidence**) mentioned in 1. has not converted the mathematicians to accept the standpoint of ~~the~~ arithmetical evidence. The reasons for this may be the following:

a) Those who are looking for intuitiveness in mathematics will feel the complete [*restlos*] elimination of geometrical intuition to be unsatisfying and artificial. In fact, the reduction of the continuous to the discrete succeeds only in an approximate sense. On the other hand, those who are striving for sharp concepts [*Begrifflichkeit*] will prefer those methods that are most beneficial [*am günstigsten*] from the systematic standpoint [*Standpunkt der Systematik*].

b) In the Brouwer's method, distinctions are introduced into the language of mathematics and play **an essential** role, whose **importance** [*Bedeutbarkeit*] is only apparent from the standpoint of the syntax of this language. That the “tertium non datur” is invalid, as Brouwer claims, can only be stated [*konstatiert werden*] as a *syntactic* matter of fact [*Sachverhalt*], but not as one of mathematical objectiveness [*Gegenständlichkeit*] itself.

*Comment:* [*Bemerkung*] ~~The Brouwerian idea to characterize~~ Brouwer's idea of characterizing the continuum as a set of choice sequences is ~~by~~ in itself [*an sich*] independent of the rejection of the “tertium non datur”. ~~For sure,~~ Certainly no “tertium non datur” can hold with regard to indefinite predicates of choice sequences. But one could **nevertheless** choose a standpoint such that the “tertium non datur” is retained for number theoretic properties of ~~lawful serieses~~ lawlike sequences [*gesetzlicher Folgen*]. In this manner one would obtain an extension of Weyl's theory of the continuum of 1918.

3. The standpoint that Hilbert adopts in his proof theory is ~~thereby~~ characterized ~~that it~~ by meeting both the requirements [*gerecht werden*] of the



formal systematic [*formalen Systematik*] and those of arithmetical evidence. As a means to unify these goals he employs the distinction [*dient ihm die Sonderung*] between mathematics and meta-mathematics, which is modeled after [*nachgebildet ist*] the Kantian partitioning of philosophy into “critique” and “system”. As is well known, the main task that Hilbert assigns to meta-mathematics as a critique of proof [*Beweiskritik*] is to show the consistency [*Widerspruchsfreiheit*] of the usual practice [*Verfahren*] of mathematics. The problem is ~~thought~~ intended to be tackled in stages.

~~During the accomplishment of the~~ In the course of accomplishing this task, however, considerable difficulties arise, which are in part unexpected. **An essential** reason for difficulties which have not yet been overcome is that the ~~distance~~ difference [*Abstand*] between a formalism of intuitive arithmetic and that of usual mathematics is greater than Hilbert **had** presumed [*vermutet*].

In the formalism of number theory the “tertium non datur” can be eliminated in a certain sense. The proofs of the consistency of the number theoretic formalism by Gödel and Gentzen are based on this fact. But as soon as one **passes over to** number-*functions* [*Zahlfunktionen*] such an elimination is no longer possible [*ist nicht mehr die Rede*]. This ~~result~~ follows in particular from a theorem which has been proved by S. C. Kleene after the concept of a “computable” function had been made more precise; it says that there are number-functions which are definable with the symbols of the number theoretic formalism (including a symbol for “the smallest number  $x$  that has the property  $\mathfrak{P}(x)$ ”), but which are not computable.

*Comment.* — The concept of a computable function was made more

precise in two independent ways: using the concept of a “generally recursive” [*allgemein-rekursiven*] function due to Herbrand and Gödel and by Church’s concept of a “ $\lambda$ -definable” function; both concepts have been shown to be co-extensional [*umfangsgleich*] by A. Church and Kleene.

4. While the task of a consistency proof for analysis is still an unsolved problem, in a different direction, namely in the domain [*Gebiet*] of ~~stage-free~~ untyped [or: type-free] [*stufenfreien*] formalisms of ~~combinatorial~~ combinatory logic, proofs of ~~the~~ consistency have succeeded. The theory of “combinators” which has been formulated by H. B. Curry following Schönfinkel, is such an untyped [*stufenfreier*] calculus; ~~moreover~~ and so is the theory of “conversions” ~~which was founded~~ established by Church. Both these formal theories, whose close connection has been shown by J. B. Rosser, yield a far-reaching [*weittragenden*] and logically satisfying formalism for definitions [*Definitionsformalismus*]. The consistency of operating with combinators (in the sense of unambigu-ousness) has been proved a while ago by Curry, that of the formalism of conversions **recently** by Church and Rosser.

The ~~stage free~~ untyped [*stufenfreien*] ~~combinatorial~~ combinatory formalisms also yield a new ~~stimulation~~ suggestion for the formation [*Gestaltung*] of how systems of logistic may be constructed. An integration of these domains may ~~possibly~~ perhaps lead to a **reform** of the whole of logistic ~~on the whole~~. Sure enough, an adequate approach for such an integration is not available yet.