Bernays Project: Text No. 11

Methods for demonstrating consistency and their limitations (1932)

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Comments:

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||[201] The methods that were used to prove the consistency of formalized theories from the finitist standpoint can be surveyed according to the following classification.

||[202]1. *Method of valuation.* It has obtained its essential development by Hilbert's procedure of trial valuation. Using this procedure Ackermann and v. Neumann demonstrated the consistency of number theory—admittedly, under the restrictive condition that the application of the inference from n to $(n + 1)$ is only allowed to formulas with just free variables.

2. *Method of integration.* This can only be applied to such domains that are completely mastered mathematically. For those it allows one to give a completely positive answer not only to the question of consistency, but also to those of completeness and decidability. Such domains are in particular:

a) the monadic function calculus, which was treated conclusively by Löwenheim, Skolem, and Behmann.

b) Fragments of number theory. To such [formalisms] Herbrand and Presburger have applied the method. Thereby it becomes obvious that the Peano axioms, using the function calculus of "first order" (with the axioms for equality) as a foundation, do not yet suffice for the development of number theory. Only by adding the recursive equations for addition and multiplication do we arrive at full number theory[1].

3. *Method of elimination.* Its idea can already be found in Russell and Whitehead, in particular in the application to the concept "that which." However, the actual implementation of the idea is tedious. A significant simplification is brought about by Hilbert's approach, which ties in with the introduction of the "ε-symbol." First, this approach yields again—as has been shown by Ackermann—the result of the method of valuation in a simpler way.

From here, moreover, one arrives at a new proof of a theorem, which was discovered and proved for the first time by Herbrand. It is a converse to Löwenheim's famous theorem about satisfiability in countable domains and it also yields a general procedure for the treatment of questions about consistency.

The limitation of the results at hand presents itself, despite the insights obtained in multiple ways, as a fundamental one; this is because of Gödel's

[1] The situation is different if one, like Dedekind, takes the standpoint of the logic of classes as basic from the outset; this standpoint, however, contains stronger assumptions than are needed for number theory.

new theorem—and a conjecture by v. Neumann connected to it—on the limits of decidability in formal systems.