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The Basic Notions of Pure Geometry in Their Relation to Intuition (1925)

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Dictionary:

Elementarbegriff: elementary concept; Grundbegriff: basic concept; Grundsatz: principle; Gerade: straight line; Linie: line; Körper: solid; Fläche: surface; berühren: touch; Grenze: boundary; eigentliche Axiome: proper axioms; Satz: proposition; Lage: position; Ort: location

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A discussion of the relation between axiomatic geometry and intuition can be carried out under very different aspects and on the basis of different epistemological preconditions.

The present book, written by R. STROHAL with the essential collaboration of FRANZ HILLEBRAND, sets out to emphasize a certain methodological and epistemological view of geometry. It is stated in the introduction that the "psychological prehistory" of geometrical concepts and principles [*Grundsätze*] is the subject of the investigation. In fact, however, already the more specific elaboration of the program shows that it does by no means concern questions of genetic psychology, but questions such as: In what way do we have to revert to intuition when introducing geometrical concepts; what role does intuition play for the formation of the basic concepts and the complex concepts as well as for setting up the principles of geometry; and how do we have to evaluate the epistemological character of these principles? |[197r]

In this connection the author does not at all intend to make geometry appear as being determined to the greatest possible extent by intuition.

On the one hand Strohal, as he mentions in the beginning, wants to leave the question of an application to "our space" aside (he does not, in fact, go to such an extreme); he is concerned with the foundations of *pure* geometry. A foundation of geometry by spatial experience does not come into consideration for him. But he also excludes a rational foundation of appeal to an aprioristic evidence of geometrical intuition, because he does not accept any aprioristic evidence other than the analytic one and does not recognize any rational character in intuition. He does not undertake a closer discussion of the concept of "intuition," but starts from the view, which he takes as self-evident, as it were, and which is admittedly also common among exact researchers, according to which intuition is not capable of giving us perfectly clear objects, and also not of presenting us with a relation as necessary, so that all idealizations and all insights of strict generality come about only by way of conceptual abstraction.

In considering his epistemological position one should now think that Strohal should welcome the standpoint of Hilbert's formal axiomatics as |[198l] in accordance with his views and his intention. In fact, however, he by no means agrees with this present-day axiomatics, but explicitly opposes it, in particular he objects to Hilbert's foundation of geometry.

It is difficult to explain comprehensibly and in few words how Strohal intends to deal with geometry because in his conception different intentions are at play. In any case, this present attempt to diverge in principle from the current standpoint of axiomatics and to go back to older tendencies may seem appealing to some at first sight, but it is, on closer inspection, only suited to bring our current standpoint into brighter light, and to make clear the justification of the motives from which it arose in a particularly precise way. But especially from this point of view it seems to be not useless to present the main points of Strohal's views and to discuss his presentation critically.

In particular, Strohal deals exhaustively with the *formation of concepts*. First of all, the role of intuition, according to Strohal, consists in the following:

1. Elementary concepts are obtained from intuition by processes of abstraction.

2. Intuition serves as a cause (causa occasionalis) for the formation of complex concepts (for "synthetical definitions") by suggesting the formation of certain conceptual syntheses. This is done by setting up sharp definitions by combining elementary concepts which replace intuitive concepts, i. e., concepts directly taken from intuition (like the

intuitive concept of a straight line or of the circle); the extension of a concept formed this way does not have to coincide completely with the corresponding intuitive concept.

For one thing, we have to take into account here that the intuition under consideration by no means always has to be spatial intuition, e. g., according to Strohal, the elementary concept of *congruence*, which he identifies, in the style of Bolyai, with "indistinguishability except for location," is obtained in the way that first of all "the intuitive givenness of indistinguishable qualities, colors, sounds, odors etc." leads to a vague concept of indistinguishability (sameness); from this we get the rigorous concept of indistinguishability as a limit concept by a process of abstraction (pp. 71–72).

It is above all essential, however, that we are not free, according to Strohal, to introduce just any concept obtained from intuition by abstraction as an elementary concept. He rather claims that: a concept may be regarded as an elementary concept only "if an entity falling under the extension of the respective concept cannot also be given by conceptual marks," or in a more succinct formulation: "Where it is *possible* at all to define a concept explicitly, there one *has to* define it."

Sure enough, this "criterion" is completely undetermined; since the possibility of defining a concept explicitly depends essentially on the choice of geometrical principles, and the selection of principles depends on the choice of elementary concepts.|[198r]

The motivation for the criterion is also quite unsatisfactory. Strohal asserts that the explanation of a concept has to make it possible "to decide whether an object which is given in some way falls under the extension of the respective concept or not" (p. 18). For instance, we have to be able to decide whether the geometrical location of all points equidistant from two fixed points A, B falls under the extension of the concept of a straight line; such a task would be hopeless, he thinks, if one would regard the concept of a straight line as a basic concept (p. 19). Again STROHAL does not consider that the extensional relations between geometrical concepts are determined only by the principles of geometry and that on the other hand they can also make it possible to prove a complex concept to be extensionally equivalent to an elementary concept. Lacking a more immediate justification, he says "obviously."

On the other hand, despite the indeterminacy of the criterion, the aim pursued with it can be recognized: Geometry should—like a philosophical science—advance from the highest generality to the particular by way of conceptual synthesis. It must therefore not be founded on the concepts of particular geometrical entities taken as elementary concepts, but only on those of an entirely general character.

Because of this methodological demand, STROHAL is forced to depart completely from the well-known elementary construction of geometry as it can be found in EUCLID and in similar form also in HILBERT's foundations. He finds a formation of geometrical concepts analogous to his principle in LOBACHEFSKY and BOLYAI. He follows these two, especially LOBACHEFSKY, in introducing the elementary concepts. On the basis of an exhaustive discussion he arrives at the following system of elementary concepts:

1. the spatial (spatial formations);

2. the contact (the adjoining);

3. the “having-it-inside” (the relation of whole to part);

4. the congruence (indistinguishability except for location).

Obviously we are here dealing with a construction of geometry according to which the *topological* properties of space are prior and their introduction is followed by the introduction of the *metric*. This method of constructing geometry and its systematic advantages are familiar to the mathematician—especially since the investigations of RIEMANN and HELMHOLTZ[1] on the foundations of geometry. He will not be satisfied, however, with having only this kind of foundation available. In particular, the usual elementary approach to foundation has the great methodological advantage that here geometry, like elementary number theory, starts from considering certain simple, easily comprehensible objects, and that one does not need to introduce the concept of continuity and limit processes at the outset. In any event |[199l] one will insist on the freedom to choose the basic concepts relative to the viewpoint according to which geometry is carried out.

STROHAL concedes, however, that it is possible in principle that systems other than the one he gives “connect with intuition immediately in a different manner, i. e., are based on other elementary concepts” (p. 63). In fact, however, he rejects almost all other ways of giving foundations.

In his opinion, e. g., the concept of a straight line should not be taken

[1]Helmholtz’s group-theoretic conception, which was carried further by LIE and HILBERT, is however not in line with STROHAL’s intention (as will be seen from the following). The “derivation of the elementary spatial concepts from that of equality” sketched by WEYL (in the first paragraph of his book “Raum–Zeit–Materie”) is more in accordance with it.

as a basic concept.[2] He also deliberately avoids introducing the point as a basic element. In his system the point is defined as the common boundary of two lines [*Linie*] which touch each other, the line results accordingly from two touching surfaces and the surface from two touching solids.

He completely rejects the idea of taking the *concept of direction* as an elementary concept at the outset. He declares that if one intends to use the concept of direction for defining the straight line, this would "only be possible by considering the concept 'equidirected' as an elementary concept which is not further reducible, and with this to connect to the intuition of 'straightness' itself. That is to say, since no intuition can yield this elementary concept other than that of an intuitive straight line, this amounts to regarding the straight line itself as an elementary concept." (p. 56). By contrast, one should remark that one can obtain the distinction of directions starting at a point intuitively independent of the idea [*Vorstellung*] of straightness by considering different parts of the visual field and by the imaginations of directions connected to our impulses of motion. And moreover, as far as comparison of directions starting from *different* points is concerned, STROHAL, according to his methodological principles would have to accept their synthetic introduction by linking the concept of direction with the concept of "indistinguishability," since he arrives at the comparison of lengths of segments in different locations in a very similar way. In particular the pure closeness geometry [*reine Nahegeometrie*] discovered by WEYL has recently

[2]Incidentially, STROHAL considers a straight line only as a spatial object, or straightness as a property of a line. He does not consider at all the possibility of introducing collinearity as a relation between *three* points.

clarified that, indeed, the a priori comparability of separate segments is by no means more easily comprehensible than the comparability of directions starting from distinct points. Here Strohal only repeats an old prejudice.

Strohal also rejects the characterization of the relation of congruence by the concept of *rigid motion* as a circular procedure. "The concept of a rigid solid which occurs in this connection can again be explained in no other way than by presupposing the congruence of the different positions of this solid. If one wants to understand the rigid solid as an elementary concept, however, one will find that to obtain it no other intuitions will help than those which give us the concept of congruence itself, so that the detour through the concept of a rigid solid becomes pointless" (pp. 17–18). This argumentation would be justified only if the concept of a rigid solid would have to be formed as an ordinary generic concept [*Gattungsbegriff*] |[199r], e. g., in such a way that starting from an empirical intuition of the rigid solid one arrives by abstraction at the concept of the perfectly rigid solid. In fact, however, it is possible to carry out a completely different process instead, which consists in sharpening by abstraction the intuitive matters of fact about rigid bodies concerning freedom of motion and coincidence into a strict lawfulness, and then forming the geometrical concept of a rigid solid with respect to this *lawfulness*. In its mathematical formulation, this kind of concept formation emerges by considering rigid motions from the outset not individually, but by considering *the group of rigid motions*.

This thought, which originates in Helmholtz, and which was groundbreaking for an entire line of geometrical research and in the face of relativity theory is increasingly topical, is not mentioned by Strohal at all.

Now, if so many approaches adopted by mathematics in order to erect geometry are rejected [*ausgeschaltet*], one would expect that the way of justification so decisively preferred by STROHAL would be presented as a paragon of methodology. In fact, however, the considerations with the help of which STROHAL explains the method, following LOBACHEFSKY, which leads from the elementary concepts of the spatial, contact and of the having-it-inside to the distinction of dimensions and to the concepts of surface, line and point, are far from the precision we are now used to in dealing with such topological questions; on the basis of these considerations one cannot even determine whether those three elementary concepts are sufficient for the topological characterization of space.—

Up to now we have only regarded that part of STROHAL's considerations dealing with geometrical *concept formation*. STROHAL's standpoint, however, becomes really clear only in the way in which he conceives of the *principles* of geometry.

It is essential to this view that STROHAL sticks to the separation of the κοιναὶ ἔννοιαι (communes animi conceptiones) and the αἰτήματα (postulata) as it is found in EUCLID's "Elements." STROHAL regards this distinction as fundamentally significant, and sees an essential shortcoming of recent foundations of geometry in their deviation from this distinction.

To this it has to be remarked first of all that deviating from EUCLID in this point is not a result of mere sloppiness but is completely intentional. EUCLID puts the propositions of the *theory of magnitude*, which are gathered under the title κοιναὶ ἔννοιαι, before the *specifically geometrical* postulates, as propositions of greater than geometrical generality and which are to be *applied* to geometry.

The kind of application, however, leads to fundamental objections since the subordination of geometrical relations under the concepts occurring in the κοιναὶ ἔννοιαι is tacitly presupposed in several cases where the possibility of such a subordination represents a geometrical law which is by no means self-evident.

Hilbert in particular has criticized Euclid's application of the principle that the whole is greater than the part in the theory of the areas of plane figures in this way—an application which would only be justified, if |[200l] one could presuppose without a second thought that one could assign to every rectilinear plane figure [*geradlinig begrenzten ebenen Figur*] a positive quantity as its area (in such a way that congruent figures have the same area and that by joining surfaces the areas add up).[3]

While considering such a case one recognizes that the essential point in applying the κοιναὶ ἔννοιαι always lies in the conditions of applicability. If these conditions are recognized as satisfied, the application of the respective principle in most cases becomes entirely superfluous, and sometimes the proposition to be proved by applying the general principle belongs itself to these conditions of applicability.

Putting the κοιναὶ ἔννοιαι at the beginning therefore appears to be a continuous temptation to commit logical mistakes and to be more suited to obscure the true geometrical state of affairs than to make it clear, and this is the reason why this method has been completely abandoned.

[3] Hilbert has shown that this presupposition in fact need not always be satisfied by constructing a special "non-Archimedian" and "non-Pythagorean" geometry.

Strohal seems to be ignorant of these considerations; in any case he does not mention Hilbert's criticism with even a syllable. He aims at setting out the distinction between the two kinds of principles anew. In particular, this appears to him to be necessary already because, in his opinion, the κοιναὶ ἔννοιαι is of a completely different epistemological character than the postulates, namely that of evident analytic propositions, whereas postulates are not expressions of knowledge at all; they are only *suggested* to us by certain experiences.

Strohal therefore calls the κοιναὶ ἔννοιαι the "proper axioms." He considers it a particular success of his theory of geometrical concept formation that it makes the analytical nature of the κοιναὶ ἔννοιαι comprehensible. He locates this comprehensibilty in that these axioms, as propositions about a single elementary relation each, have the sense of an *instruction* [*Anweisung*], specifying from which relational intuitions [*Relationsanschauungen*] one has to abstract the elementary concept "in order to turn the axiom concerned into an identical proposition" (p. 70). This characterization amounts to the claim that the axioms in question constitute logical identities based on the contentual view of the elementary concepts.

It seems curious that such geometrically empty [*nichtssagend*] propositions should be regarded as "proper axioms" of geometry, and one furthermore wonders to what end one needs to specifically posit these propositions as principles at all, since the elementary concepts are introduced contentually anyway.

For instance, one of these axioms is the proposition that if a is indistinguishable from b and b from c, then a is indistinguishable from c. This proposition is, because of the meaning of "indistinguishability," a consequence of the purely logical proposition: if two things a, b behave the same with respect to the applicability or non-applicability of a predicate P and also b, c behave in this respect the same, then a and c also behave in this respect the same.

We now have the following alternative: Either the concept "indistinguishable" is used in its contentual meaning, then we have before us a proposition which can be understood [*einsehen*] purely logically, and there is no reason to list such a proposition as an axiom, since in geometry we regard the laws of logic $|^{200r}$ as an obvious basis anyway. Or else the concept "indistinguishable" and also the other elementary concepts will not be applied contentually at all; rather, only concept *names* are introduced initially, and the axioms give certain *instructions* about their meaning. Then we are on the standpoint of formal axiomatics, and the κοιναὶ ἔννοιαι are nothing other than what are called *implicit definitions* following HILBERT.

Those places where STROHAL stresses that the κοιναὶ ἔννοιαι do not provide a "proper definitions" or an "explicit definitions" of elementary relations (pp. 68 and 72) indicate that this is indeed STROHAL's view—who, sure enough, carefully avoids using the term "implicit definition" anywhere.

¿From this standpoint it is not suitable, however, to ascribe to the axioms in question the character of *being evident.* They then simply consitute *formal conditions* for certain initially undetermined relations, and then there is also no principal requirement of separating these axioms from the "postulates."

So either setting up the axioms, which according to STROHAL have the role of κοιναὶ ἔννοιαι, is altogether superfluous, or the separation of these axioms as analytically evident propositions from the other principles is not justified.

Furthermore, however, we find the same ills that discredited EUCLID's κοιναὶ ἔννοιαι again in the application of these axioms in STROHAL: the formulation of these propositions, which can easily be confused with geometrically contentful propositions, leads to logical mistakes, and these are in fact committed.

Two cases are especially characteristic. 1. As an example of a proper axiom the proposition is given[4] that in a "cut," i. e., when two adjoining parts of a solid (spatial entity) touch, one always has to distinguish *two sides of the cut* (p. 64). This proposition is tautological, however, since as the two adjoining parts are called "sides" of the cut (p. 23), it says nothing but that if two parts of a solid touch each other (adjoin), two adjoining parts have to be distinguished. This proposition, moreover, is completely irrelevant for geometry. However, it seems to state something geometrically important, since given the wording one thinks of another proposition which expresses a topological property of space.

The following mistake shows that STROHAL himself is not immune to confusions of a similar kind. He raises the following question (on the occasion of a discussion of the concept of congruence): "Is it possible to find two solids connected by a continuous series of such solids which have one and the same surface in common, i. e., which all touch in *one* surface?" "We have to answer this question in the negative," he continues, "because it follows from the explanation of a surface that only *two* solids are able to touch each other in one and the same surface" (pp. 42–43).

[4]In this example STROHAL follows some considerations of LOBACHEFSKY.

2. The famous axiom: “The whole is greater than the part,” which became, as mentioned, the source of a mistake for Euclid, is interpreted by Strohal in the following way: The axiom hints at an elementary concept “greater,” “which can be obtained by abstraction from a $|^{201l}$ divided solid.” The procedure of abstraction is characterized “by examining that relation which obtains between the totality of all part-solids [*Teilkörper*] (the *whole*) and one of them (the *part*). For the concept “greater” obtained this way, the proposition “Totum parte maius est” is an identity” (p. 77). At this place we should disregard that in this interpretation the “whole” is wrongly identified with the totality of all part-solids. In any case, it follows from this interpretation that the proposition “a is greater than b” is only another expression for b being a part of a. So we have again a perfect tautology, from which one can infer nothing for geometry; in particular it is impossible to derive from this the proposition that a body cannot be congruent with one of its parts—which also follows from the fact that this proposition is generally valid only under certain restrictions, anyway. (For instance, a half line can turn into a part by a congruent translation, and equally a spatial octant into a partial octant by a congruent translation.)

In fact, however, Strohal would have to have some formulation of this proposition at his disposal for the theory of congruence—which he, however, does not develop in this respect; for otherwise it would not be certain that this “indistiguishability irrespective the location” does not just mean *topological equality*. Indeed, in the conceptual system which Strohal takes as a basis the first three elementary concepts: spatial object, adjoining, having-it-inside, all belong to the domain of topological determinations, and only by the concept of congruence the *metric* is introduced into geometry. Therefore, the concept of congruence must contain a *new distinguishing property* besides the element of correspondence. In the concept of indistinguishability irrespective the location[5] such a distinguishing property, however, is not given in itself; for this, one also needs a principle according to which certain objects which are at the outset only determined as different with respect to the position but not as topologically different, can also be recognized as *distinguishable irrespective the location.* In other words: the introduction of the *difference in size* is what is important. The principle that the whole is greater than the part should actually help us achieve this. This will be impossible, however, if we interpret the proposition in the way Strohal does; because from this interpretation it cannot be derived that an object a which is greater than b is also *distinguishable* from it, even with respect to location.

This circumstance perhaps escaped Strohal; for otherwise he would have realized the fact that his concept of indistinguishability irrespective the location does not yet yield geometrical congruence. Thus, we find here a gap very similar to that in Euclid's doctrine of the area.

The result of this consideration is that the method of putting the κοιναὶ ἔννοιαι first becomes even more objectionable through the modified interpretation given to it by Strohal; in any case, it does not appear to be an example that should be followed.

At the same time Strohal's characterization of these axioms has led us

[5]The "location" of a solid is, according to the definition Strohal took from Lobachevsky with a certain revision (pp. 24 and 93), synonymous with the boundary of the solid.

to assume that he does not keep the contentual view of elementary concepts even within geo|^{201r}metry itself, or as the case may be he does not make use of it for geometrical proofs. This assumption is confirmed by STROHAL's discussion of the *postulates* of geometry.

According to STROHAL we are not *forced* to posit the postulates either by intuition or by logical reasons, "but caused [to do so] by certain experiences" (p. 97). For pure geometry they have the meaning of stipulations [*Festsetzungen*]; they are "tools for definitions for geometrical space, their totality forms the definition of geometrical space" (p. 103). Contentually they are characterized as "exclusions of certain combinations of elementary concepts which are a priori possible" (p. 103).

The point of this characterization emerges from STROHAL's view of the deductive development of geometry. According to STROHAL, this development proceeds by a continued combination of properties, i. e., by forming synthetical definitions. In forming the first syntheses one is only bound by those restrictions resulting from the κοιναὶ ἔννοιαι. "Incidentially, one can proceed completely arbitrarily in combining elementary concepts," i. e., the decision "whether one wants to unite certain elementary concepts in a synthesis or to exclude such a union," is caused by motives, "which lie outside of pure geometry." "However, in arbitrarily excluding the existence [*Bestehen*] of a certain combination, one introduces a proposition into pure geometry which has to serve as a norm for further syntheses. propositions of this kind are called requirements [*Forderungen*], αἰτήματα, postulates." "In forming higher syntheses" one has to show that these "do not contradict the postulates already set up. One must, as we say concisely, prove the *possibility*, the *existence*, of the defined object. Here, existence and possibility mean the same, and amount to nothing but *consistency* with the postulates" (pp. 98–99 and p. 102).

What is most striking in this description of the geometrical method [*Verfahren*] is that here, contrary to all familiar kinds of geometrical axiomatics, only a *negative* content is ascribed to the postulates, namely that of exclusion of possibilities, whereas all existential propositions in geometry [*geometrische Existenzsätze*] are only interpreted as statements [*Aussagen*] about consistency.

Strohal's view is in accordance with the direction of his philosophical school which includes Brentano's theory of judgement as an essential element. According to this theory, all general judgements are negative existential judgements whose content is that the matter of a judgement (a combination of the contents of ideas [*Vorstellungsinhalte*]) is rejected (excluded).

In fact every general judgement can be brought into this logical form. By producing such a normal form, however, the existential moment is not removed, but only transferred into the formation of the matters of judgements.

One thus also does not succeed in geometry in excluding existential claims completely or rather in reducing them to consistency claims. One can only hide an existential claim by a double application of negation. Strohal proceeds in this way for instance when he speaks of an αἴτημα which excludes the assumption "that when dividing a geometrical solid no parts can ever be congruent" (p. 93). We find another such example in his discussion of Dedekind's continuity |^{202l} axioms. After having spoken of the divisions of a line segment AB which has the cut property, and furthermore of the construction of a cut in the point C, he continues: "In *excluding* the possibility of such a division of some line segment AB on which such a point C is not found, I assert the αἴτημα of continuity for the line segment" (p. 113). Talk of "occurring", "being found", or "existence" all amount to the same. And in any case here, where the setting up of of postulates is concerned, the interpretation of existence in the sense of consistency with postulates is not permissible [*angängig*].

The identification of existence and consistency is justifiable in two cases: first, with respect to geometrical space whose existence indeed only consists in the consistency of the postulates defining it; and second also with respect to geometrical objects, but only under the condition of the *completeness of the systems of postulates*.

If the system of postulates is complete, i. e., if, the postulates already decide, for every combination (every synthesis) of elementary concepts whether they are permitted [*zugelassen*] or excluded, then indeed the possibility (consistency) of an object coincides with its existence.

However, as long as one is in the process of obtaining a system of postulates, i. e., of the stepwise determination of geometrical space, one has to distinguish between existence and consistency. From the proof of the consistency of a synthesis it only follows that it agrees with the postulates *already set up*; it may nevertheless be possible to exclude this synthesis by a further postulate. By contrast, an *existence proof* says that already by the prior postulates one is logically *forced* to accept the respective synthesis.

Let us take as an example "absolute geometry," which results from ordinary geometry by excluding the parallel axiom. In this geometry one can assume, without contradiction with the postulates, a triangle with an angular sum of a right angle; if we would identify consistency with existence in this context, we would get the proposition: "In absolute geometry there is a triangle with the angular sum of one right angle." Then the following proposition would equally hold: "In absolute geometry there exists a triangle with an angular sum of two right angles." Hence, in absolute geometry both a triangle with an angular sum of a right angle and one with an angular sum of two right angles would have to exist. This consequence contradicts, however, a theorem proved by LEGENDRE according to which in absolute geometry the existence of a triangle with an angular sum of two right angles implies that *every* triangle has this angular sum.

In order, therefore, to characterize the existence of geometrical objects with the help of their consistency with the postulates, as STROHAL intends to do, one has to have a *complete* system of postulates for which no decision concerning the admission of a synthesis remains open. This prerequisite of completeness is not mentioned by STROHAL anywhere, and furthermore, it does not follow from his description of the progressive method of forming and excluding syntheses whether this way ever comes to a conclusion. |[202r]

Disregarding all these objections, however, which concern the special kind of characterization of the postulates and of the progressive method of obtaining them, it has to be remarked above all that, according to the description of geometry which STROHAL gives here in the section on the postulates, geometry turns out to be pure conceptual combinatorics,—such as it could not be performed in a more extreme way in formal axiomatics: Combinations of elementary concepts are tried out; in doing so the content of these concepts is not taken into account, but only certain axioms representing this content which act as initial rules of the game. Moreover, certain combinations are excluded by arbitrary stipulations, and now one stands back and sees what remains as possible.

Here, the detachment [*Loslösung*] from the contentual formation of concepts is executed to the same degree as in HILBERT's axiomatics; the initial contentual introduction of elementary concepts does not show in this development; they are, so to speak, eliminated with the help of the κοιναὶ ἔννοιαι
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Thus we have here—similar to EUCLID's foundation of geometry—the state of affairs that the contentual determination of the elementary concepts is completely idle [*leerläuft*], i. e., precisely that state of affairs for the sake of which one refrains from a contentual formulation [*Fassung*] of the elementary concepts in the new axiomatics.

In EUCLID's foundation, however, the state of affairs is different insofar as here the postulates are still given in an entirely intuitive way. In the first three postulates the close analogy with [*Anlehnung an*] geometrical drawing is especially apparent. The constructions required here are nothing but idealizations of graphical procedures. This contentual formation of the postulates permits the interpretation according to which the postulates are positive existential claims concerning intuitively evident [*ersichtlich*] possibilities which receive their verification based on the intuitive content of the elementary concepts. For STROHAL, such a standpoint of *contentual axiomatics* is out of the question, since he considers an intuitively evident verification of the postulates to be impossible and therefore he can admit [*zuerkennen*] only the character of stipulations for the postulates.

So STROHAL's sketch of the geometrical axiomatics ends in a conflict between the intuitive introduction of concepts and the completely non-intuitive way in which the geometrical system of doctrines [*Lehrgebäude*] is to be developed as a purely conceptual science starting from the definition of geometrical space given by the postulates,—a discrepancy which is only scantily veiled by the twofold role of the the κοιναὶ ἔννοιαι, [which function] on the one hand as analytically evident [*einsichtig*] propositions, on the other hand as initial restrictive conditions for conceptual syntheses.

In the light of these unsatisfying results one wonders on what grounds STROHAL rejects the simple and systematic [*konsequent*] standpoint of HILBERT's axiomatics. This question is even more appropriate as STROHAL knows full well the reasons leading to HILBERT's standpoint. Thus he himself says: "The intuitions representing the causa occasionalis for forming the syntheses, do not enter ... into geometry in the sense that one could immediately prove a proposition correct by referring to intuition;" moreover, shortly thereafter: "As soon as the axioms"—STROHAL is here only referring to the κοιναὶ ἔννοιαι—"are formulated, |[203l] the specific nature of elementary concepts has no further influence on the development of geometrical deduction" (pp. 132–133).

Indeed, there are also no conclusive objections in STROHAL's polemic against HILBERT's foundation of geometry, which can be found in the final section of his book.

Here his main argument is that in HILBERT's conception of axiomatics the contentual element is only *pushed back* to the formal properties of the basic relations, i. e., to relations of higher order. The formal requirements on the basic relations which are expressed in the axioms would themselves have to be regarded contentually [so STROHAL] and the contentual representations [*inhaltliche Vorstellungen*] necessary for this could again be obtained only by abstraction from the respective relational intuitions. Thus, concerning the higher relations which constitute the required properties of the basic geometrical relations, one "has arrived at the reference to intuition which axiomatics precisely wants to avoid" (p. 129).

This argument misses the essential point. What is to be avoided by HILBERT's axiomatics is the reference to *spatial intuition*.

The point of this method is that of intuitive contents only that is retained which *essentially enters into* geometrical proofs. By satisfying this demand we free ourselves from the special sphere of ideas [*Vorstellungsbereich*] in the area [*Sachgebiet*] of the spatial, and the only contentual representation we use is the primitive kind of intuition which concerns the elementary forms of the combination of discrete, bounded objects, and which is the common precondition for all exact scientific thinking—which was stressed in particular by HILBERT in his recent investigations on the foundations of mathematics.[6]

This methodological detachment from spatial intuition is not to be identified with ignoring the spatial-intuitive starting point of geometry. It is also not connected with the intention—as STROHAL insinuates—"to act as if these and exactly these axioms had found together in the system of geometry due to some inner necessity" (p. 131). On the contrary, the names of spatial objects and of spatial connections of the respective objects and relations are maintained deliberately in order to make the correlation with

[6]Cf. especially the treatise: "Neubegründung der Mathematik". Hamburg 1922.

spatial intuitions and facts evident, and to keep it continuously in mind.

The inadequacy of STROHAL's polemic becomes especially apparent when he goes on to artificially create an opportunity for an objection. While reporting on the procedure of proving the consistency of the geometrical axioms, he states: "For this purpose one chooses as an interpretation, e. g., the concepts of ordinary geometry; by this HILBERT's axioms transform into certain |[203r] propositions of ordinary geometry whose compatibility, i. e., consistency is already established independently. Or one interprets the symbols by numbers or functions; then the axioms fade into certain relations of numbers whose compatibility can be ascertained according to the laws of arithmetic" (p. 127).

STROHAL added the first kind of interpretation himself; in HILBERT there is not a single syllable about an interpretation by "ordinary geometry." STROHAL nevertheless has the nerve to connect an objection to HILBERT's method with this arbitrarily added explanation: "If one, say, proves the consistency of HILBERT's axioms by interpreting its "points", "lines", "planes" as points, lines, planes of EUCLIDean geometry whose consistency is established, then one presupposes ... that these objects are already defined elsewhere" (p. 130).

In sum one gets the impression that STROHAL, out of a resistance against the methodological innovation which is given by the formal standpoint of axiomatics compared with the contentual-conceptual opinion, rejects the acceptance of HILBERT's standpoint instinctively.

STROHAL exhibits this behaviour, however, not only against HILBERT's axiomatics, but also against most of the independent and important thoughts that recent science has contributed to the present topic. This spirit of hostility is expressed in the book under review not only by how it divides praise and criticism, but even more in the fact that essential achievements, considerations and results are simply ignored. For instance (as already mentioned earlier), STROHAL passes over the famous investigation of HELMHOLTZ, which concerns the present topic in the closest sense, in complete silence, and likewise over KANT's doctrine of spatial intuition. And as to the strict mathematical proof of the independence of the parallel axiom from the other geometrical axioms, STROHAL presents this as if it were still an unsolved problem: "This question will finally be clarified only if one shows that no consequence of the other postulates *can* ever collide with a denial of the parallel postulate" (p. 101). And this statement cannot be explained by ignorance for, as can be seen from other passages, STROHAL knows of KLEIN's projective determination of measure [*Maßbestimmung*], and is also familiar with POINCARÉ's interpretation of non-Euclidean geometry by spherical geometry within EUCLIDean space (from a review by WELLSTEIN). The explanation instead is to be found in STROHAL's oppositional emotional attitude, who refuses to appreciate the significance of the great achievements of recent mathematics.

A novice reader can thus only receive a distorted picture of the development of geometrical science from STROHAL's book. Those who are informed about the present state of our science might take STROHAL's failed enterprise, in view of the various methodological tendencies that work together in it, as an opportunity to think through the principal questions of axiomatics anew.

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